Non-local charges and quantum integrability of sigma models on the symmetric spaces

$SO(2n)/SO(n){\times}SO(n)$ **and** $Sp(2n)/Sp(n){\times}Sp(n)$

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ABSTRACT

Non-local conserved charges in two-dimensional sigma models with target spaces $SO(2n)/SO(n){\times}SO(n)$ and $Sp(2n)/Sp(n){\times}Sp(n)$ are shown to survive quantization, unspoiled by anomalies; these theories are therefore integrable at the quantum level. Local, higher-spin, conserved charges are also shown to survive quantization in the $SO(2n)/SO(n){\times}SO(n)$ models.

Classical, two-dimensional sigma models on compact symmetric spaces G/H are integrable by virtue of conserved quantities which can arise as integrals of local or non-local functions of the underlying fields (the accounts in [1]-[5] contain references to the extensive literature). Since these models are asymptotically free and strongly coupled in the infrared, their quantum properties are not straightforward to determine. Nevertheless, following Lüscher [6], Abdalla, Forger and Gomes showed [7] that, in a G/H sigma model with H simple[1], the first conserved non-local charge survives quantization (after an appropriate renormalization [6, 7, 8]), which suffices to ensure quantum integrability of the theory. By contrast, calculations using the $1/N$ expansion reveal anomalies that spoil the conservation of the quantum non-local charges in the $CP^{N-1} = SU(N)/SU(N-1){\times}U(1)$ models for $N > 2$, and in the wider class of theories based on the complex Grassmannians $SU(N)/SU(n){\times}SU(N-n){\times}U(1)$ for $N > n > 1$ [9].

It was long suspected, therefore, that the G/H sigma models were quantum integrable *only* for H simple. So it was something of a surprise when exact S-Matrices were proposed

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[1]Here, and throughout this paper, we shall use 'simple' to mean that the corresponding Lie algebra has no non-trivial ideals. Hence $U(1)$ is simple in our terminology, in addition to the usual non-abelian simple groups of the Cartan-Killing classification [13].

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for the family of models based on $SO(2n)/SO(n){\times}SO(n)$, which were then shown to pass stringent tests using the Thermodynamic Bethe Ansatz (TBA) [10, 11]. This was followed by the construction of S-matrices for the models with target spaces $Sp(2n)/Sp(n){\times}Sp(n)$ [12], which were again shown to be consistent with TBA calculations.

In this letter we reconcile these recent S-matrix results with the previous, well-known approach of [7], by showing that the latter techniques can, in fact, be used to show that the first non-local charge *does* survive quantization, unspoiled by anomalies, in the sigma-models with target spaces $SO(2n)/SO(n){\times}SO(n)$ (for $n \geq 3$) and $Sp(2n)/Sp(n){\times}Sp(n)$ (for $n \geq 1$). We also argue that these techniques cannot be extended to any other new classes of models, at least in any obvious way: the non-local charge is protected from anomalies only if H is simple or if the target space belongs to one of these two additional families of Grassmannians. As a supplement to our discussion, we will show at the end of the paper how the quantum integrability of the $SO(2n)/SO(n){\times}SO(n)$ models can also be established using a *local* conservation law.

We begin by summarizing the construction of the G/H sigma model [1, 2]. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{1}$$

be the decomposition of the Lie algebra $\mathfrak{g}$ of the compact group G into the Lie algebra $\mathfrak{h}$ of H and its orthogonal complement $\mathfrak{m}$; the condition for G/H to be a symmetric space is

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{2}$$

The sigma model can be formulated using fields $g(x^\mu) \in G$ and $A_\mu(x^\mu) \in \mathfrak{h}$ which are subject to gauge transformations

$$g(x^\mu) \mapsto g(x^\mu)h(x^\mu), \qquad A_\mu \mapsto h^{-1}A_\mu h + h^{-1}\partial_\mu h \tag{3}$$

for any $h(x^\mu) \in H$, thus ensuring that the physical degrees of freedom belong to G/H. The fields also transform under a global G symmetry

$$g(x^\mu) \mapsto Ug(x^\mu), \qquad A_\mu \mapsto A_\mu \tag{4}$$

for any $U \in G$. The lagrangian for the theory, which is invariant under each of these symmetries, is

$$\mathcal{L} = -\frac{1}{2\lambda}\text{Tr}\,(k_\mu k^\mu) = -\frac{1}{2\lambda}\text{Tr}\,(j_\mu j^\mu) \tag{5}$$

where we use the covariant derivative $D_\mu g \equiv \partial_\mu g - gA_\mu$ to define the related, $\mathfrak{g}$-valued currents

$$k_\mu \;\equiv\; g^{-1}D_\mu g = g^{-1}\partial_\mu g - A_\mu \tag{6}$$

$$j_\mu \;\equiv\; -(D_\mu g)g^{-1} = -gk_\mu g^{-1}. \tag{7}$$

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Note that k_μ is gauge-covariant, transforming as $k_\mu \mapsto h^{-1}k_\mu h$ under (3), but it is invariant under (4); its covariant derivative is $D_\mu k_\nu \equiv \partial_\mu k_\nu + [A_\mu, k_\nu]$. In contrast, j_μ is gauge-invariant, but transforms in the adjoint representation of G; it is the Noether current for the global symmetry (4).

The gauge field A_μ is non-dynamical and the effect of varying it in the lagrangian is to impose the constraint $k_\mu \in \mathfrak{m}$. The equation of motion obtained by varying g can be written in terms of either current:

$$D_\mu k^\mu = \partial_\mu k^\mu + [A_\mu, k^\mu] = 0 \qquad \Longleftrightarrow \qquad \partial_\mu j^\mu = 0. \tag{8}$$

It is now that the symmetric space condition (2) enters crucially for the first time, because it implies, in conjunction with $k_\mu \in \mathfrak{m}$, the identities

$$0 \;=\; D_\mu k_\nu - D_\nu k_\mu \;\in\; \mathfrak{m} \tag{9}$$

$$F_{\mu\nu} \;\equiv\; \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = -[k_\mu, k_\nu] \;\in\; \mathfrak{h} \tag{10}$$

Equivalently, we have the zero-curvature condition[2] for the gauge-invariant current:

$$\partial_\mu j_\nu - \partial_\nu j_\mu + 2\,[j_\mu, j_\nu] = 0. \tag{11}$$

This, together with the conservation of j_μ, is sufficient to show that the $\mathfrak{g}$-valued non-local charge

$$Q(t) = \int dx\, j_1(t,x) \;+\; \int\!\!\int dxdy\, \theta(x{-}y)\, [\, j_0(t,x),\, j_0(t,y)\,] \tag{12}$$

is conserved, which guarantees the integrability of the model at the classical level.

The crucial question to be settled in the quantized theory is whether the definition and conservation of the non-local charge, and hence the integrability of the theory, can be maintained. A potential problem arises from the second term in (12): it contains products of operators at the same spacetime point, and therefore entails a careful regularization and renormalization of Q. The approach of [6, 7, 8] is to use point-splitting regularization and consider the short-distance behaviour of the bracket expressed as an operator product expansion (OPE)

$$[\, j_\mu(t,x{+}\epsilon),\, j_\nu(t,x{-}\epsilon)\,] \;\sim\; \sum_k C_{\mu\nu}^{(k)}(\epsilon)\, Y^{(k)}(t,x)\,. \tag{13}$$

Here $\{Y^{(k)}(t,x)\}$ is a complete set of local operators of canonical dimension at most two and $C_{\mu\nu}^{(k)}(\epsilon)$ are c-number-valued functions which can be singular as $\epsilon \to 0$. We include in the OPE all terms which are divergent or non-zero in the limit $\epsilon \to 0$.

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[2]This terminology is standard but potentially confusing. The curvature it refers to is that of j_μ regarded as a connection, and not the curvature $F_{\mu\nu}$ of the non-dynamical gauge field A_μ.

The operator product expansion must, however, transform correctly under all of the symmetries of the theory. The left-hand side transforms under the adjoint action of the global symmetry G in (4), and is invariant under gauge transformations (3). Thus each operator Y in the expansion on the right-hand side must also transform in this way. But any such operator can be written $Y = gXg^{-1}$, where X is invariant under the global G symmetry and instead transforms covariantly as $X \mapsto h^{-1}Xh$ under gauge transformations. The task is therefore to determine all operators X of this type with mass dimension two or less.

There is a unique gauge-covariant operator of dimension one, namely the current k_μ, and there are two obvious candidates with the correct transformation properties and dimension two, $D_\mu k_\nu$ and the curvature $F_{\mu\nu}$ of the connection A_μ. Let us assume for the moment that these are the only operators that appear. Then, since $F_{\mu\nu}$ is antisymmetric and $D_\mu k_\nu$ is symmetric, the OPE takes the form

$$[\, j_\mu(t, x+\epsilon),\, j_\nu(t, x-\epsilon)\,] \quad \sim \quad C_{\mu\nu}^\rho(\epsilon)\, g k_\rho g^{-1} + C_{\mu\nu}^{\rho\sigma}(\epsilon)\, g\,(D_\rho k_\sigma + F_{\rho\sigma})\, g^{-1} \qquad (14)$$

$$= \quad -C_{\mu\nu}^\rho(\epsilon)\, j_\rho - C_{\mu\nu}^{\rho\sigma}(\epsilon)\, \partial_\sigma j_\rho, \qquad (15)$$

where the coefficient functions $C_{\mu\nu}^\rho(\epsilon)$ and $C_{\mu\nu}^{\rho\sigma}(\epsilon)$ are respectively linearly and logarithmically divergent as $\epsilon \to 0$ (we have suppressed the common spacetime argument (t, x) for all the operators on the right-hand side). The resulting expression (15) depends only on j_μ and its derivatives, and this is sufficient [6, 7, 8] to show that the charge Q can be properly defined in the quantum theory and that it is conserved. For completeness, we give a sketch of the arguments in an appendix.

The key assumption above, that k_μ, $D_\mu k_\nu$ and $F_{\mu\nu}$ are the only gauge-covariant terms that can appear in the OPE, is certainly valid if each of these operators transforms in an irreducible representation of H, because there are no other local, gauge-covariant quantities of the correct dimensions that can be constructed from the constituent fields. Both k_μ and $D_\mu k_\nu$ take values in $\mathfrak{m}$, which *always* carries an irreducible representation of H for a compact symmetric space G/H with G simple [13]. But $F_{\mu\nu}$ is valued in $\mathfrak{h}$, which carries the adjoint representation of H, and this is irreducible if and only if H is simple[3]. If $H = H_1 \times H_2 \times \ldots \times H_r$, with a corresponding decomposition of the Lie algebra $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \ldots \oplus \mathfrak{h}_r$, the total curvature can be decomposed into irreducible components $F_{\mu\nu} = F_{\mu\nu}^{(1)} + F_{\mu\nu}^{(2)} + \ldots + F_{\mu\nu}^{(r)}$ where each $F_{\mu\nu}^{(i)} \in \mathfrak{h}_i$ transforms non-trivially only under H_i. In these circumstances our key assumption breaks down because the term $C_{\mu\nu}^{\rho\sigma} g F_{\rho\sigma} g^{-1}$ in the OPE (14) must then be replaced by

$$C_{\mu\nu}^{(1)\,\rho\sigma} g F_{\rho\sigma}^{(1)} g^{-1} + C_{\mu\nu}^{(2)\,\rho\sigma} g F_{\rho\sigma}^{(2)} g^{-1} + \ldots + C_{\mu\nu}^{(r)\,\rho\sigma} g F_{\rho\sigma}^{(r)} g^{-1}\,. \qquad (16)$$

[3]See footnote 1.

The coefficient functions $C^{(i)\,\rho\sigma}_{\mu\nu}$ are unrelated to one another in general, and so it will not generally be possible to re-express this OPE solely in terms of j_μ and its derivatives as in (15). The conclusion of [7] was thus that one should expect anomalies to spoil the conservation of Q whenever H is not simple.

But consider now the target spaces $SO(2n)/SO(n){\times}SO(n)$. This family is clearly rather special in that, while $\mathfrak{h}$ is not simple, it is the direct sum

$$\mathfrak{h} = \mathfrak{so}(n)_1 \oplus \mathfrak{so}(n)_2 \tag{17}$$

of two identical subalgebras, and these subalgebras are simple, provided $n \neq 4$. Since neither of the subalgebras is in any way preferred, it is then natural to expect that $gF^{(1)}_{\mu\nu}g^{-1}$ and $gF^{(2)}_{\mu\nu}g^{-1}$ should have the *same* coefficient in the OPE, in which case the usual argument for quantum conservation of the non-local charge would still hold. The way to formulate this idea precisely is to show that there is a discrete symmetry τ of the target space $SO(2n)/SO(n){\times}SO(n)$ which exchanges the roles of the two factors in the denominator.

The existence of the discrete symmetry τ is perhaps most easily understood by recalling that points on the Grassmannian $SO(N)/SO(n){\times}SO(N{-}n)$ can be identified with n-dimensional subspaces in N-dimensional Euclidean space. The factors in the denominator are the linear isometry groups of such an n-dimensional subspace and its orthogonal complement. The special feature which arises when $N = 2n$ is simply that the orthogonal complement to an n-dimensional subspace is itself n-dimensional, and so τ can be defined as the map which exchanges these subspaces. This is an isometry of the Grassmannian and, therefore, a symmetry of the sigma-model.

To express τ in more concrete terms, consider the following block forms for general elements of $\mathfrak{so}(2n)$ and its subalgebra $\mathfrak{so}(n)_1 \oplus \mathfrak{so}(n)_2$:

$$\begin{pmatrix} P & R \\ -\widetilde{R} & Q \end{pmatrix} \in \mathfrak{so}(2n) \,, \qquad \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{so}(n)_1 \,, \qquad \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \in \mathfrak{so}(n)_2 \tag{18}$$

(P, Q and R are $n{\times}n$ real matrices with P and Q both antisymmetric and a tilde denotes a transpose). Let

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SO(2n) \tag{19}$$

and consider the inner automorphism of $\mathfrak{so}(2n)$ defined by

$$\tau : \quad \begin{pmatrix} P & R \\ -\widetilde{R} & Q \end{pmatrix} \;\mapsto\; T^{-1} \begin{pmatrix} P & R \\ -\widetilde{R} & Q \end{pmatrix} T = \begin{pmatrix} Q & \widetilde{R} \\ -R & P \end{pmatrix} \tag{20}$$

which evidently maps $\mathfrak{m} \to \mathfrak{m}$ and $\mathfrak{h} \to \mathfrak{h}$ in such a way that the entries of the $\mathfrak{so}(n)_1$ and $\mathfrak{so}(n)_2$ subalgebras are interchanged. From this, we define a transformation on sigma

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model fields

$$\tau : \quad g \mapsto gT , \quad A_\mu \mapsto T^{-1} A_\mu T \tag{21}$$

which leaves the Lagrangian (5) invariant; note also the behaviour of the currents and field strength[4]

$$\tau : \quad j_\mu \mapsto j_\mu , \quad F_{\mu\nu} \mapsto T^{-1} F_{\mu\nu} T . \tag{22}$$

Now, terms in the current commutator OPE must be invariant under τ, because the currents themselves are. The two irreducible components of the curvature (for $n \neq 4$) have the block forms

$$F^{(1)}_{\mu\nu} = \begin{pmatrix} P_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{so}(n)_1 , \qquad F^{(2)}_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & Q_{\mu\nu} \end{pmatrix} \in \mathfrak{so}(n)_2 \tag{23}$$

and it follows from (20) and (22) that the action of τ on $F^{(1)}_{\mu\nu}$ and $F^{(2)}_{\mu\nu}$ is to exchange $P_{\mu\nu} \leftrightarrow Q_{\mu\nu}$. The combinations $gF_{\mu\nu}g^{-1} = g(F^{(1)}_{\mu\nu} + F^{(2)}_{\mu\nu})g^{-1}$ and $g(F^{(1)}_{\mu\nu} - F^{(2)}_{\mu\nu})g^{-1}$ are clearly even and odd, respectively, under τ and the OPE must therefore take the form (14), as claimed. In essence, the symmetry which constrains the OPE here is actually the semi-direct product

$$\mathbb{Z}_2^{(\tau)} \ltimes SO(n)_1 \times SO(n)_2 . \tag{24}$$

Although $F_{\mu\nu} \in \mathfrak{so}(n)_1 \oplus \mathfrak{so}(n)_2$ carries a reducible representation of $SO(n)_1 \times SO(n)_2$, it carries an *irreducible* representation of this larger group and so no decomposition of $F_{\mu\nu}$ is allowed in the OPE.

The existence of the discrete symmetry τ has thus enabled us to extend the approach of [7, 6] and deduce that the quantum $SO(2n)/SO(n) \times SO(n)$ sigma models possess conserved non-local charges, ensuring quantum integrability, for $n \neq 4$. The model with $n = 4$ is also quantum integrable, and for exactly similar reasons, but this deserves some additional explanation.

The denominator of the symmetric space $SO(8)/SO(4) \times SO(4)$ involves four simple factors rather than two:

$$\mathfrak{h} = \mathfrak{so}(4)_1 \oplus \mathfrak{so}(4)_2 \cong \left(\mathfrak{su}(2) \oplus \mathfrak{su}(2) \right) \oplus \left(\mathfrak{su}(2) \oplus \mathfrak{su}(2) \right) . \tag{25}$$

There are then four irreducible curvature components appearing in (16), but discrete symmetries of the sigma model again force all of the OPE coefficient functions to be equal, as required. This is actually a consequence of our original symmetry τ, which exchanges $\mathfrak{so}(4)_1$ and $\mathfrak{so}(4)_2$, and just one additional symmetry τ', constructed so as to interchange

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[4]In the definition (21) we have chosen to combine the automorphism acting on g with left multiplication by T so as to ensure that the current j_μ is invariant, rather than covariant, under the symmetry. This is helpful, but not essential, for the arguments that follow.

the $\mathfrak{su}(2)$ subalgebras *within* each copy of $\mathfrak{so}(4)$. For a single copy of $\mathfrak{so}(4)$, the $\mathfrak{su}(2)$ subalgebras can be exchanged by conjugating by a 4×4 matrix such as $L = \mathrm{diag}(1, -1, -1, -1)$ (there are many possible choices for L; any two differ by an element of $SO(4)$). We can therefore define the desired symmetry τ' by replacing T with T' in (20) and (21), where

$$T' = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \in SO(8) . \tag{26}$$

There are other, more exotic discrete symmetries of the $SO(8)/SO(4) \times SO(4)$ model which arise from outer automorphisms of $\mathfrak{so}(8)$ [13] and which permute the four $\mathfrak{su}(2)$ subalgebras in (25) in any desired way, but these need not concern us here.

Turning now to the $Sp(2n)/Sp(n) \times Sp(n)$ sigma models with $n \geq 1$, we can apply almost identical arguments to those used above for the real Grassmannians. Block forms for general elements of $\mathfrak{sp}(2n)$ and its subalgebra $\mathfrak{sp}(n)_1 \oplus \mathfrak{sp}(n)_2$ can be obtained from (18) by taking P, Q, R to be $2n \times 2n$ complex matrices, with P and Q antihermitian and the tilde in (18) denoting hermitian conjugation; these matrices must also satisfy

$$PJ - JP^* = QJ - JQ^* = RJ - JR^* = 0 \tag{27}$$

where J is a $2n \times 2n$ symplectic structure (a real, antisymmetric matrix with $J^2 = -1$). The block form for T in (19) and the definition of τ in (21) are unchanged, and the reasoning which restricts the form of the OPE and hence implies the quantum conservation of the non-local charge proceeds just as before.

Similar results cannot be expected for other compact symmetric spaces G/H with H non-simple, however. Our arguments require that H consists of a product of identical simple subgroups, *and* that there is a group of discrete symmetries which acts transitively on these factors. The first condition holds for the families $SO(2n)/SO(n) \times SO(n)$ and $Sp(2n)/Sp(n) \times Sp(n)$ and for just one other case, namely $G_2/SU(2) \times SU(2)$ [13]. (Note that the complex Grassmannians $SU(2n)/SU(n) \times SU(n) \times U(1)$ are ruled out because of the extra $U(1)$ factor in the denominator.) For the second condition to hold, it must be possible to introduce τ as an automorphism of $\mathfrak{g}$ which commutes with the involutive automorphism defining G/H, and which permutes the simple factors in $\mathfrak{h}$.[5] This is not possible for $G_2/SU(2) \times SU(2)$ because the two $SU(2)$ factors can be distinguished: they are embedded inequivalently in G_2 and they act in different representations on $\mathfrak{m}$ [13].

To conclude our discussion, we will give an alternative demonstration of the quantum

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[5]It is interesting to note that for $\mathfrak{g} = \mathfrak{so}(2n)$ and $\mathfrak{g} = \mathfrak{sp}(2n)$ the involutive automorphism τ defines the symmetric spaces $SO(2n)/U(n)$ and $Sp(2n)/U(2n)$ respectively. Commuting, involutive automorphisms have recently proved useful in classifying integrable boundary conditions for symmetric space sigma models on the half line [15].

integrability of the sigma models on $SO(2n)/SO(n)\times SO(n)$, quite independent of the non-local charges that have been the subject of the paper so far.

The classical integrability of the G/H symmetric space sigma models can also be understood in terms of higher-spin conserved currents that are *local* in the fields and are related to H-invariant symmetric tensors on $\mathfrak{m}$ [5]. It is usually difficult to draw conclusions about the survival of such conservation laws at the quantum level, but there are some notable exceptions which can be analysed very simply using an approach due to Goldschmidt and Witten [14]. Their method entails enumerating all possible terms which could violate a given classical conservation equation, and comparing with the number of such terms which can be written as total derivatives, and whose appearance would therefore constitute a modification of the conservation equation, rather than a violation of it. Global symmetries in general, and discrete symmetries in particular, again play a crucial role.

To carry out such an analysis for the $SO(2n)/SO(n)\times SO(n)$ sigma model it is convenient to reformulate it using a field $\Phi^{ab}(x^{\mu})$ which is a real, symmetric, traceless $2n\times 2n$ matrix constrained to satisfy $\Phi^2 = 1$ (this is used in [10]). The lagrangian for Φ is free except for the constraint, and the equations of motion are easily found (using a Lagrange multiplier) to be

$$\partial^{\mu}\partial_{\mu}\Phi + \Phi(\partial^{\mu}\Phi)(\partial_{\mu}\Phi) = 0 . \tag{28}$$

There are no gauge fields in this formulation and the $SO(2n)$ global symmetry (4) acts by $\Phi \mapsto U\Phi U^{T}$. Actually, the symmetry extends to $O(2n)$ by including a transformation

$$\mu : \quad \Phi \mapsto M\Phi M^{T}, \qquad MM^{T} = 1 , \qquad \det M = -1 . \tag{29}$$

The discrete symmetry (21) is now simply

$$\tau : \quad \Phi \mapsto -\Phi . \tag{30}$$

(The relation of this new formulation to our previous description of the model is revealed by writing $\Phi = gNg^{T}$, where $N = \mathrm{diag}(1,-1)$ in the basis (18) and $g \in SO(2n)$ with a redundancy $g \mapsto gh$, $h \in SO(n)\times SO(n)$.)

In this new notation, the Noether currents (7) are antisymmetric matrices j_{μ}^{ab}, whose definition and conservation may be written

$$j_{\mu} = \tfrac{1}{2}\Phi\,\partial_{\mu}\Phi , \qquad \partial_{\mp}j_{\pm} = \pm[j_{+}, j_{-}] \tag{31}$$

(light-cone components for vectors in Minkowski space are defined by $u_{\pm} = u_0 \pm u_1$). A local, classically-conserved quantity can be constructed from j_{μ} using any symmetric invariant tensor, but we shall concentrate here on the Pfaffian for $SO(2n)$ which yields the

conservation law[6]

$$\partial_{-}\left(\varepsilon_{a_1 b_1 a_2 b_2 \ldots a_n b_n} \, j_{+}^{a_1 b_1} j_{+}^{a_2 b_2} \ldots j_{+}^{a_n b_n} \right) = 0 \tag{32}$$

This higher-spin current is clearly even under τ, but it is odd under μ, since the Pfaffian transforms with a factor $\det M = -1$, and it is this which proves particularly useful in restricting the possible quantum corrections.

We now consider all local operators constructed from Φ and its derivatives whose symmetry properties allow them to appear as quantum corrections on the right-hand side of (32). To form an $SO(2n)$ invariant, the indices on all fields Φ^{ab}, $\partial_{\pm}\Phi^{ab}$, and higher derivatives, must be contracted with each other (using δ_{ab}) or with $\varepsilon_{a_1 a_2 \ldots a_{2n}}$. An ε-tensor is essential here, however, because without one we can construct only traces of products of matrices, which will all be even under μ. The antisymmetry of the ε-tensor then severely limits which products of matrices can be contracted with it, and we have the freedom to move matrices around within a product by using identities such as $\Phi(\partial_{\mu}\Phi) = -(\partial_{\mu}\Phi)\Phi$ which are consequences of the constraint $\Phi^2 = 1$. Finally, the symmetry τ is important in restricting the total number of fields Φ (including derivatives) to be even. Taking all these facts into account, we find that, up to terms which vanish on using the equations of motion, any quantum modification of (32) is proportional to

$$\partial_{+}\left(\varepsilon_{a_1 b_1 a_2 b_2 \ldots a_n b_n} \, j_{-}^{a_1 b_1} j_{+}^{a_2 b_2} \ldots j_{+}^{a_n b_n} \right) . \tag{33}$$

Because this is a derivative, the conservation law is guaranteed to survive at the quantum level, albeit in a modified form.

Appendix: the quantum non-local charge and its conservation

The definition of the quantum non-local charge given in [6, 7, 8] is

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\begin{aligned}
Q_{\delta}(t) &= Z(\delta)\int dx\, j_1(t,x) \;+\; \int\!\!\int dx\,dy\,\theta(x-y-\delta)\,[\,j_0(t,x),\,j_0(t,y)\,] \\
&= \int_{-\infty}^{\infty} dx\left(Z(\delta)j_1(t,x) \;+\; \int_{\delta}^{\infty} d\epsilon\,[\,j_0(t,x),\,j_0(t,x-\epsilon)\,]\right),
\end{aligned}\tag{34}
$$

where we must show that the cut-off can be removed, $\delta \to 0$, after choosing the renormalization factor $Z(\delta)$ so as to cancel the divergence arising from the commutator term. Notice that terms in the integrand which are logarithmically-divergent or finite as $\epsilon \to 0$

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[6]It is important to check that this higher-spin current does not vanish *identically*. The analysis of [5] ensures this: there is a non-vanishing invariant on $SO(2n)/SO(n)\times SO(n)$ inherited from the Pfaffian for $SO(2n)$, but this is not true for other real Grassmannians. Note also that conserved quantities in [5] are written in terms of the gauge-covariant currents, k_{μ} rather than j_{μ} in our present notation.

will *not* give rise to divergences in the integral as $\delta \to 0$. But the form of the remaining, linearly-divergent term in the OPE is fixed by two-dimensional spacetime symmetries (Lorentz, parity and time-reversal invariance) to be of the form

$$[\, j_0(t,x),\, j_0(t,x{-}\epsilon)\,] \sim C(\epsilon)\, j_1(t,x) \ . \tag{35}$$

It follows that the charge is well-defined as $\delta \to 0$ provided $Z'(\delta) = C(\delta)$.

Consider now an expression for the time derivative of the charge:

$$\partial_0 Q_\delta(t) = \int_{-\infty}^{\infty} dx \left\{\, Z(\delta)\partial_0 j_1(t,x) \, + \, [\, j_0(t,x),\, j_1(t,x{-}\delta) + j_1(t,x{+}\delta)\,] \,\right\} \,, \tag{36}$$

which is obtained on using conservation of j_μ, a shift in an integration variable, and the relation

$$\int_{-\infty}^{\infty} dx\, \partial_0 \,[\, j_0(t,x),\, j_0(t,x{-}\delta)\,] = -\int_{-\infty}^{\infty} dx \frac{\partial}{\partial\delta} \,[\, j_0(t,x),\, j_1(t,x{-}\delta) + j_1(t,x{+}\delta)\,] \ . \tag{37}$$

The right-hand side of (36) must be finite as $\delta \to 0$ if $Z'(\delta) = C(\delta)$ (since we know Q itself is well-defined in this limit) and, indeed, the identity (37) can be used once more to relate the singular part of the OPE for the current commutator

$$[\, j_0(t,x),\, j_1(t,x{-}\delta) \, + \, j_1(t,x{+}\delta)\,] \tag{38}$$

to the OPE in (35). The conclusion is that, up to space derivatives, the singular part of (38) must be of the form $W(\delta)\, \partial_0 j_1(t,x)$, where $W'(\delta) = C(\delta)$. Thus (36) is well-defined for $Z(\delta) = W(\delta) + a$, with a any constant.

Finally, to determine if the charge is conserved we must examine the finite, δ-independent terms in the OPE for (38). If (15) holds, the contribution to (36) is proportional to the integral over space of $\partial_0 j_1$, and this can be cancelled by choosing the constant a appropriately. If (15) does not hold, however, then we will in general have contributions from the curvature components $F_{\mu\nu}^{(i)}$ which cannot be cancelled by any choice of $Z(\delta)$, and the non-local charge will not be conserved. The anomalous contributions found for the complex Grassmannians in [9] are exactly of this type.

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