

Conformal Sigma-Models on Supercoset Targets

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ABSTRACT

We investigate the quantum behaviour of sigma models on coset superspaces G/H defined by \mathbb{Z}_{2n} gradings of G . We find that, whenever G has vanishing Killing form, there is a choice of WZ term which renders the model quantum conformal, at least to one loop. The choice coincides with that for which the model is known to be classically integrable. This generalizes results for models associated to \mathbb{Z}_4 gradings, including IIB superstrings in $AdS_5 \times S^5$.

1 Introduction and overview

Non-linear sigma models with supermanifolds as targets are of importance both in string theory and condensed matter physics. In condensed matter they are relevant in a variety of applications [1], notably the theory of the quantum Hall effect [2]. There are interesting connections to Lagrangian formulations of logarithmic conformal field theories [3].

In string theory, the proposal of the AdS/CFT correspondence [4] prompted renewed interest in superstrings on curved Ramond-Ramond backgrounds¹. The Green-Schwarz action for IIB strings in $AdS_5 \times S^5$, given by Metsaev and Tseytlin in [6] (see also [7]), takes the form of a sigma model on the coset superspace $PSU(2, 2|4)/SO(1, 4) \times SO(5)$. To gain insight into how this theory might be quantized, sigma models on simpler targets, the supergroups $PSL(n|n)$, were considered in [8, 9] (see also [10]). As we recall below, these supergroups have vanishing Killing form and are therefore

¹For more on the technical issues of formulating string theories on supermanifold targets see [5].

Ricci-flat, which guarantees that sigma models on them are conformal to one-loop; the result of Bershadsky, Zhukov and Vaintrob in [8] is that they are exactly conformal. (The representation theory of the relevant superalgebras [11] thus becomes important.) This remarkable fact should be contrasted with the more familiar sigma models on bosonic groups, where a correctly normalized WZ term (which may be thought of as a parallelizing torsion [12]) must be added to the action if the theory is to be quantum conformal [13].

It was subsequently shown by Berkovits, Bershadsky, Hauer, Zhukov and Zwiebach [14] that sigma models on certain *quotients* G/H of Ricci-flat supergroups G by bosonic subgroups are again conformal to one loop — and can be expected to be so exactly — given a suitable WZ term. These targets include the coset superspace $PSU(2, 2|4)/SO(1, 4) \times SO(5)$, as well as $PSU(1, 1|2)/U(1) \times U(1)$, whose bosonic geometry is $AdS_2 \times S^2$. The vital property in each case is that the isotropy subgroup H is the fixed point set of a \mathbb{Z}_4 grading of G . This, in particular, allows the addition of the required WZ term.

What is striking is that the \mathbb{Z}_4 grading was also a key element in the demonstration by Bena, Polchinski and Roiban [15] that the sigma model on $PSU(2, 2|4)/SO(1, 4) \times SO(5)$ is classically integrable. (See also [25], which contains references to the extensive recent work on this subject.) It was shown by one of the present authors [16] that a similar construction holds, more generally, for sigma models on spaces G/H defined by \mathbb{Z}_m gradings: for any m , the grading permits the addition of a certain preferred WZ term, and with this WZ term the equations of motion may be put into Lax form, ensuring integrability, just as in the \mathbb{Z}_4 case.

Given this result, and the dual role played by the \mathbb{Z}_4 -grading, it is natural to hope that conformal invariance is also present in sigma models whose targets are quotients of (again, Ricci-flat) supergroups G by subgroups H defined by gradings of order greater than 4. In this paper, we show that this is indeed the case, at least at one loop.

The structure of this paper is as follows: in section 2 we introduce some notation and discuss supergroups with vanishing Killing form and related coset superspaces. We also write down the sigma model action with WZ term, and recall the one-loop beta-functions. In section 3 we compute the Ricci curvature of a torsion connection on a reductive homogeneous superspace, and then in section 4 we demonstrate that, for the connection whose torsion is given by the preferred WZ term, this curvature vanishes, completing our argument. Some conclusions, and comments about the consequences of this result, are given in section 5.

2 Sigma models on graded coset superspaces

We shall consider non-linear sigma models on certain homogeneous superspaces. We begin by summarizing the relevant facts about Lie supergroups and establishing some notation. For full

details see [17].

2.1 Supergroups

Let G be a Lie supergroup and let \mathfrak{g} be the complexification of its Lie superalgebra. Write

$$\mathfrak{g} = \mathfrak{g}^{(0)} + \mathfrak{g}^{(1)} \quad (2.1)$$

for the decomposition of \mathfrak{g} into its even and odd parts, and pick a basis $\{t_a\}$ of \mathfrak{g} consisting of the disjoint union of bases of $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$. For every $t_a \in \mathfrak{g}^{(0)}$, let $|a|$ be an even number; for every $t_a \in \mathfrak{g}^{(1)}$, let $|a|$ be an odd number.²

We assume for simplicity that a suitable representation has been chosen and the t_a are concrete matrices. Elements of G near the identity are then of the form

$$1 + X^a t_a + \dots \quad (2.2)$$

where, for each a , the parameter X^a is Grassmann even (i.e. a c-number) if $|a|$ is even and Grassmann odd if $|a|$ is odd.³ Since G is a group, the commutator

$$[X^a t_a, Y^b t_b] = X^a t_a Y^b t_b - Y^b t_b X^a t_a \quad (2.3)$$

closes onto some $Z^c t_c$, and, given the statistics of the X^a and Y^b , this implies that \mathfrak{g} closes under the graded commutator:

$$[t_a, t_b]_{\pm} = t_a t_b - (-)^{|a||b|} t_b t_a = f^c{}_{ab} t_c, \quad (2.4)$$

where we define also the structure constants $f^a{}_{bc}$. It follows from

$$0 = [Z, [X, Y]] + [X, [Y, Z]] + [Y, [Z, X]] \quad (2.5)$$

for $X = X^a t_a$, $Y = Y^a t_a$ and $Z = Z^a t_a$ that the $f^a{}_{bc}$ satisfy the graded Jacobi identity:

$$0 = (-)^{|a||c|} f^e{}_{ad} f^d{}_{bc} + (-)^{|b||a|} f^e{}_{bd} f^d{}_{ca} + (-)^{|c||b|} f^e{}_{cd} f^d{}_{ab}. \quad (2.6)$$

The Killing form $K(X, Y) = K_{ab} X^a Y^b$ on \mathfrak{g} is defined by

$$K_{ab} = (-)^{|d|} f^d{}_{ac} f^c{}_{bd} \quad (2.7)$$

and is an invariant (i.e. $K([X, Z], Y) + K(X, [Y, Z]) = 0$), graded-symmetric, bilinear form. The unusual feature of Lie superalgebras, in comparison with Lie algebras, is that there exist simple Lie superalgebras whose Killing forms vanish identically. As we recall in section 3 below, the vanishing

² $|a|$ will be defined fully in (2.12) below; only $|a|$ modulo 2 is relevant here.

³These parameters will also satisfy some reality conditions, which depend on the choice of real form of \mathfrak{g} . (For example, if the real form is the real span of the t_a , the conditions are simply that the X^a are real.)

of the Killing form implies that G is Ricci-flat. This will be crucial for conformal invariance, so we will focus on these cases. However, even if K_{ab} is zero, \mathfrak{g} may possess a non-degenerate, invariant, graded-symmetric bilinear form. We assume that such a form exists and denote it by

$$\langle \cdot, \cdot \rangle \tag{2.8}$$

(In the examples in appendix A, $\langle \cdot, \cdot \rangle$ is given by the supertrace in the defining representation, rather than the adjoint.)

2.2 Graded coset superspaces

We are concerned with models whose targets are coset superspaces G/H with the special property that \mathfrak{g} is \mathbb{Z}_{2n} -graded and \mathfrak{h} , the complexified Lie algebra of the isotropy subgroup H , is the subspace of grade zero. That is, we suppose that \mathfrak{g} may be written as a direct sum

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots + \mathfrak{g}_{2n-1} \tag{2.9}$$

of vector subspaces (where $\mathfrak{g}_0 = \mathfrak{h}$), and that this decomposition respects the graded Lie bracket (2.4):

$$[\mathfrak{g}_r, \mathfrak{g}_s]_{\pm} \subset \mathfrak{g}_{r+s \pmod{2n}}. \tag{2.10}$$

We suppose further that this grading is compatible with the splitting (2.1) of \mathfrak{g} into its even and odd graded subspaces, in the sense that

$$\mathfrak{g}_{2s} \subset \mathfrak{g}^{(0)}, \quad \mathfrak{g}_{2s+1} \subset \mathfrak{g}^{(1)}, \quad \text{for } s = 0, 1, \dots, n-1. \tag{2.11}$$

The basis $\{t_a\}$ can be chosen to be a disjoint union of bases of the \mathfrak{g}_k , and one can then consistently define

$$|a| = s \iff t_a \in \mathfrak{g}_s. \tag{2.12}$$

It is also useful to adopt the following conventions for the naming of indices:

$$\begin{aligned} a, b, \dots & \quad \text{generators of } \mathfrak{g} \\ \alpha, \beta, \dots & \quad \text{generators of } \mathfrak{h} \\ i, j, \dots & \quad \text{generators of } \mathfrak{g} \setminus \mathfrak{h}. \end{aligned} \tag{2.13}$$

Finally, we assume that the inner product $\langle \cdot, \cdot \rangle$ respects the grading, in the sense that if $X \in \mathfrak{g}_r$ and $Y \in \mathfrak{g}_s$ then $\langle X, Y \rangle = 0$ unless $r + s = 0 \pmod{2n}$. Non-degeneracy implies that if $X \in \mathfrak{g}_r$ and $\langle X, Y \rangle = 0$ for all $Y \in \mathfrak{g}_{2n-r}$ then $X = 0$.

One family of examples of such coset superspaces is given in appendix A.

2.3 The action

Sigma models on G/H are most conveniently expressed in terms of a dynamical field $g(x^\mu) \in G$ (where x^μ are worldsheet coordinates, and $\eta_{\mu\nu}$ will be the worldsheet metric). We write

$$j_\mu = g^{-1} \partial_\mu g \in \mathfrak{g} \quad (2.14)$$

for the current invariant under the global left action

$$g \mapsto Ug, \quad U \in G \quad (2.15)$$

of G on itself. This decomposes into currents of definite grade:

$$j_\mu = j_\mu^\alpha t_\alpha + j_\mu^i t_i \quad (2.16)$$

and under the local right action

$$g \mapsto gh \quad \text{with} \quad h(x^\mu) \in H, \quad (2.17)$$

corresponding to the redundancy in the choice of representative g from the coset gH ,

$$j_\mu^\alpha \mapsto (h^{-1} j_\mu h)^\alpha + (h^{-1} \partial_\mu h)^\alpha, \quad j_\mu^i \mapsto (h^{-1} j_\mu h)^i. \quad (2.18)$$

It is now possible to write down the most general local action for a sigma model with B -field constructed using $\langle \cdot, \cdot \rangle$ and with the required symmetries. Global G symmetry is ensured by writing the action in terms of j ; local H symmetry requires that it takes the form

$$S = \int d^2x (p_{|i|} \eta^{\mu\nu} + q_{|i|} \varepsilon^{\mu\nu}) j_\mu^i j_\nu^j \langle t_i, t_j \rangle \quad (2.19)$$

where $p_s = p_{2n-s}$ and $q_s = -q_{2n-s}$ are arbitrary real coefficients. (In this expression the summation convention applies as usual over the indices i, j . The point is that since we require only H -invariance we can rescale each pair $(s, 2n-s)$ of subspaces separately, both in the kinetic and WZ terms.)

However, a certain choice of the coefficients q_s is privileged. It was shown in [16] that the sigma model becomes classically integrable if⁴

$$q_s = 1 - \frac{s}{n}. \quad (2.20)$$

The goal of the present paper is to show that, for supergroups G with vanishing Killing form, the same choice of q_s *also* ensures conformal invariance, at least at one loop order. To see this, we first recall some well-known [18] facts about the renormalization-group behaviour of sigma models.

⁴Note that, as in that paper, we shall set $p_1 = \dots = p_{n-1} = 1, p_n = \frac{1}{2}$ throughout, which is the natural choice in the sense that the kinetic term is then simply $\frac{1}{2}(j-A)_\mu(j-A)^\mu$. In the \mathbb{Z}_4 case, this gives the sigma-model for the Berkovits, rather than Green-Schwarz, superstring. The latter has $p_1 = 0, p_2 = \frac{1}{2}$, which corresponds to a degenerate metric on G/H , and quantization requires kappa gauge-fixing.

2.4 Sigma model β -functions to one-loop

We begin by quoting the β -functions for the general non-linear sigma model with B -field. Let m, n, \dots denote coordinate-induced tangent-vector indices on G/H . Then the classical action is

$$S = \frac{1}{2} \int d^2x (\eta_{\mu\nu} g_{mn} + \varepsilon_{\mu\nu} B_{mn}) \partial^\mu \phi^m \partial^\nu \phi^n \quad (2.21)$$

where $\phi^m(x^\mu)$ is the sigma-model field, g_{mn} is the metric on the target space and $B_{mn} = -B_{nm}$ can be regarded as the potential for the torsion 3-form $H = dB$. Under changes in the energy scale μ , these quantities are renormalized according to $\mu \frac{\partial}{\partial \mu} g_{mn} = \beta_{(mn)}$ and $\mu \frac{\partial}{\partial \mu} B_{mn} = \beta_{[mn]}$, with, to one-loop order [18],

$$\beta_{mn} = -\frac{\hbar}{2\pi} \hat{R}_{mn} + O(\hbar^2) \quad (2.22)$$

where \hat{R} is the generalized Ricci curvature of the metric connection for g with torsion H . To connect the notation in (2.19) with the general sigma model action (2.21), define

$$e_m^a = (g^{-1} \partial_m g)^a, \quad (2.23)$$

so that $j_\mu^a = e_m^a \partial_\mu \phi^m$ and the metric and B -field in (2.19) are $g_{mn} = e_m^i e_n^j g_{ij}$, $B_{mn} = e_m^i e_n^j B_{ij}$ with

$$g_{ij} = \langle t_i, t_j \rangle p_{|i|}, \quad B_{ij} = \langle t_i, t_j \rangle q_{|i|}. \quad (2.24)$$

The task is now purely geometrical: we must show that the choice of Wess-Zumino (i.e B -field) term given by the coefficients q_s in (2.20) renders G/H Ricci-flat. In what follows it is easiest to work in the non-coordinate-induced basis (indices i, j, \dots) of tangent vectors on G/H defined by the vielbeins e_m^i

3 Ricci curvature of G/H

3.1 The torsion-free connection

The identity $de + e \wedge e = 0$ satisfied by the (super-)Maurer-Cartan form

$$g^{-1} dg = e = e^a t_a = e^\alpha t_\alpha + e^i t_i. \quad (3.1)$$

yields the structure equations

$$de^i = -\frac{1}{2} f^i_{jk} e^j \wedge e^k - f^i_{j\alpha} e^j \wedge e^\alpha \quad (3.2)$$

$$de^\alpha = -\frac{1}{2} f^\alpha_{ij} e^i \wedge e^j - \frac{1}{2} f^\alpha_{\beta\gamma} e^\beta \wedge e^\gamma. \quad (3.3)$$

From the first of these one reads off the components of the torsion-free ($de^i + \omega^i_j \wedge e^j = 0$) connection 1-form

$$\omega^i_j = -\frac{1}{2} (-)^{|j||k|} f^i_{jk} e^k - f^i_{j\alpha} e^\alpha, \quad (3.4)$$

whose curvature 2-form is

$$\mathcal{R}^i_j = d\omega^i_j + \omega^i_p \wedge \omega^p_j = \frac{1}{4} \left((-)^{|j||p|} f^i_{jp} f^p_{kl} + (-)^{|k||p|+|j||l|} f^i_{pk} f^p_{jl} + 2f^i_{j\alpha} f^\alpha_{kl} \right) e^k \wedge e^l \quad (3.5)$$

(The graded Jacobi identity (2.6) ensures that the components proportional to $e^k \wedge e^\alpha$ and $e^\beta \wedge e^\gamma$ vanish, as they must to maintain H -gauge-covariance.) The curvature tensor is then

$$R^i_{jkl} = -\frac{1}{2} (-)^{|j||p|+|k||l|} f^i_{jp} f^p_{lk} + \frac{1}{4} (-)^{|k||p|+|j||l|} f^i_{pk} f^p_{jl} + \frac{1}{4} (-)^{|j||k|+|k||l|} f^i_{lp} f^p_{jk} + f^i_{j\alpha} f^\alpha_{kl} \quad (3.6)$$

and the Ricci tensor, which is graded-symmetric in $i \leftrightarrow j$ given the Jacobi identity, is⁵

$$R_{ij} = (-)^{|k|} R^k_{jki} = -\frac{1}{4} (-)^{|k|} f^k_{ip} f^p_{jk} - f^\alpha_{ip} f^p_{j\alpha}. \quad (3.7)$$

Note in particular that the Ricci curvature of G itself is proportional to the Killing form (to see this, suppose H were the trivial subgroup). Now, from the definition (2.7),

$$K_{ij} = 2f^\alpha_{ip} f^p_{j\alpha} + (-)^k f^k_{ip} f^p_{jk} \quad (3.8)$$

and hence

$$R_{ij} = \frac{1}{4} (-)^{|k|} f^k_{ip} f^p_{jk} - \frac{1}{2} K_{ij}, \quad (3.9)$$

so that, finally, if the Killing form vanishes then

$$R_{ij} = \frac{1}{4} (-)^{|k|} f^k_{ip} f^p_{jk}. \quad (3.10)$$

3.2 The torsion connection

We now turn to the generalized Ricci tensor in (2.22). The connection with torsion $H = dB$ is

$$\hat{\omega}^i_j = \omega^i_j + \frac{1}{2} H^i_j, \quad (3.11)$$

where $H^i_j = H^i_{kj} e^k$, for then

$$de^i + \hat{\omega}^i_j \wedge e^j = H^i, \quad (3.12)$$

where $H^i = \frac{1}{2} H^i_{jk} e^j \wedge e^k$ is the torsion 2-form. The curvature 2-form is

$$\begin{aligned} \hat{\mathcal{R}}^i_j &= d\hat{\omega}^i_j + \hat{\omega}^i_p \wedge \hat{\omega}^p_j \\ &= \mathcal{R}^i_j + \frac{1}{2} dH^i_j + \frac{1}{2} H^i_p \wedge \omega^p_j + \frac{1}{2} \omega^i_p \wedge H^p_j + \frac{1}{4} H^i_p \wedge H^p_j. \\ &= \mathcal{R}^i_j + \frac{1}{4} H^i_p \wedge H^p_j + \left(\partial_k H^i_{lj} - H^i_{pj} \omega^p_{lk} - (-)^{|k||l|} H^i_{lp} \omega^p_{jk} + \omega^i_{pk} H^p_{lj} \right) e^k \wedge e^l. \end{aligned} \quad (3.13)$$

and we write the expression in parentheses here as $\nabla_k H^i_{lj}$. Then

$$\hat{R}^i_{jkl} = R^i_{jkl} + \frac{1}{4} H^i_{kp} H^p_{lj} - \frac{1}{4} (-)^{|k||l|} H^i_{lp} H^p_{kj} + \frac{1}{2} \nabla_k H^i_{lj} - \frac{1}{2} (-)^{|k||l|} \nabla_l H^i_{kj}, \quad (3.14)$$

⁵Note that $f^k_{kp} = 0$ by virtue of the gradation of \mathfrak{g} .

and the generalized Ricci tensor is

$$\hat{R}_{ij} = (-)^{|k|} \hat{R}^k_{jki} = R_{ij} + (-)^{|k|} \frac{1}{4} H^k_{ip} H^p_{jk} + (-)^{|k|} \frac{1}{2} \nabla_k H^k_{ij}. \quad (3.15)$$

It now remains to show that this quantity vanishes, implying one-loop conformal invariance of the sigma model.

4 Vanishing of the β -functions

We now return to the particular B -field specified by (2.20). As noted in [16], this choice of q_s leads to a particularly simple form for the torsion $H = dB$. Using the structure equations (3.3) one finds that

$$H_{ijk} = \begin{cases} +f_{ijk} & |i| + |j| + |k| = 2n \\ -f_{ijk} & |i| + |j| + |k| = 4n \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

(where the third index on f is lowered using the metric $\langle \cdot, \cdot \rangle$) and hence that the torsion tensor H^i_{jk} , with indices in their natural positions, is

$$H^i_{jk} = \begin{cases} +f^i_{jk} & |j| + |k| = |i| \\ -f^i_{jk} & |j| + |k| = 2n + |i| \\ 0 & \text{otherwise} \end{cases} \quad (4.2)$$

Now since the components H^i_{jk} are constant ($\partial_l H^i_{jk} = 0$) and invariant under the action of the isotropy subgroup H , we have, from the definition of ∇ in (3.13),

$$\begin{aligned} \nabla_l H^k_{ij} &= H^p_{ij} \omega^k_{pl} - H^k_{pj} \omega^p_{il} - (-)^{|i||l|} H^k_{ip} \omega^p_{jl} \\ &= H^p_{ij} f^k_{lp} - H^k_{pj} f^p_{li} - (-)^{|i||l|} H^k_{ip} f^p_{lj} \end{aligned} \quad (4.3)$$

and hence

$$(-)^{|k|} \nabla_k H^k_{ij} = -(-)^{|k|} H^k_{pj} f^p_{ki} - (-)^{|k||i|+|k|} H^k_{ip} f^p_{kj} \quad (4.4)$$

$$= (-)^{|k|} \left(H^k_{ip} f^p_{jk} - (-)^{|i||j|} H^k_{jp} f^p_{ik} \right). \quad (4.5)$$

(To reach the final line note that f^i_{jk} and H^i_{jk} are non-zero only when $|i| = |j| + |k| \pmod{2n}$.)

The generalized Ricci curvature is therefore

$$\hat{R}_{ij} = \frac{1}{4} (-)^{|k|} \left(f^k_{ip} f^p_{jk} + H^k_{ip} H^p_{jk} \right) + \frac{1}{2} (-)^{|k|} \left(H^k_{ip} f^p_{jk} - (-)^{|i||j|} H^k_{jp} f^p_{ik} \right). \quad (4.6)$$

It follows almost immediately that the metric β -function vanishes: the graded-symmetric part of (4.6) is proportional to

$$(-)^{|k|} \left(f^k_{ip} f^p_{jk} + H^k_{ip} H^p_{jk} \right) \quad (4.7)$$

and if, in any given term in the summation on the right, $|j| + |k| = |p|$ then, since $|i| = 2n - |j|$, $|i| + |p| = 2n + |k|$; conversely if $|j| + |k| = 2n + |p|$ then $|i| + |p| = |k|$. So in each term precisely one minus sign appears, and the HH and ff summations cancel perfectly.

We now turn to demonstrating that the B -field β -function vanishes. This argument is considerably more subtle. The graded-antisymmetric part of (4.6) is

$$\hat{R}_{[ij]} = \frac{1}{2}(-)^{|k|} \left(H^k{}_{ip} f^p{}_{jk} - (-)^{|i||j|} H^k{}_{jp} f^p{}_{ik} \right). \quad (4.8)$$

We shall treat this grade by grade. Suppose $|i| = r$ and $|j| = 2n - r$ with $r \in \{1, 2, \dots, 2n - 1\}$. Then, writing k_t for an index of the grade t subspace, we find

$$\begin{aligned} \hat{R}_{[i_r j_{2n-r}]} &= -H^{p_1}{}_{i_r k_{2n-r+1}} f^{k_{2n-r+1}}{}_{j_{2n-r} p_1} + H^{p_2}{}_{i_r k_{2n-r+2}} f^{k_{2n-r+2}}{}_{j_{2n-r} p_2} - \dots \\ &\dots - (-)^r H^{p_{r-1}}{}_{i_r k_{2n-1}} f^{k_{2n-1}}{}_{j_{2n-r} p_{r-1}} - (-)^r H^{p_{r+1}}{}_{i_r k_1} f^{k_1}{}_{j_{2n-r} p_{r+1}} + \dots \\ &\dots - H^{p_{2n-1}}{}_{i_r k_{2n-r-1}} f^{k_{2n-r-1}}{}_{j_{2n-r} p_{2n-1}} \\ &= f^{p_1}{}_{i_r k_{2n-r+1}} f^{k_{2n-r+1}}{}_{j_{2n-r} p_1} - f^{p_2}{}_{i_r k_{2n-r+2}} f^{k_{2n-r+2}}{}_{j_{2n-r} p_2} - \dots \\ &\dots + (-)^r f^{p_{r-1}}{}_{i_r k_{2n-1}} f^{k_{2n-1}}{}_{j_{2n-r} p_{r-1}} - (-)^r f^{p_{r+1}}{}_{i_r k_1} f^{k_1}{}_{j_{2n-r} p_{r+1}} + \dots \\ &\dots - f^{p_{2n-1}}{}_{i_r k_{2n-r-1}} f^{k_{2n-r-1}}{}_{j_{2n-r} p_{2n-1}}, \end{aligned} \quad (4.9)$$

where the factor $\frac{1}{2}$ disappears because the definition of H ensured that the two terms in (4.8) simply reinforced one another (or, equivalently, that the first term automatically had the necessary graded-antisymmetry).

The plan is to show that (4.9) can be re-written as a certain sum of terms involving $f^{k_t}{}_{k_t \alpha}$. To do this we use the vanishing of the ij th component of the Killing form, which it is useful to write as a sum of separate summations restricted to subspaces of definite grade:

$$\begin{aligned} 0 &= f^\alpha{}_{i_r k_{2n-r}} f^{k_{2n-r}}{}_{j_{2n-r} \alpha} - f^{p_1}{}_{i_r k_{2n-r+1}} f^{k_{2n-r+1}}{}_{j_{2n-r} p_1} + f^{p_2}{}_{i_r k_{2n-r+2}} f^{k_{2n-r+2}}{}_{j_{2n-r} p_2} - \dots \\ &\dots - (-)^r f^{p_{r-1}}{}_{i_r k_{2n-1}} f^{k_{2n-1}}{}_{j_{2n-r} p_{r-1}} + (-)^r f^{p_r}{}_{i_r k_\alpha} f^{k_\alpha}{}_{j_{2n-r} p_r} - (-)^r f^{p_{r+1}}{}_{i_r k_1} f^{k_1}{}_{j_{2n-r} p_{r+1}} + \dots \\ &\dots - f^{p_{2n-1}}{}_{i_r k_{2n-r-1}} f^{k_{2n-r-1}}{}_{j_{2n-r} p_{2n-1}}. \end{aligned} \quad (4.10)$$

We require also the graded Jacobi identity (2.6), which, for the components of interest, takes the form

$$0 = (-)^{r|e|} f^a{}_{i_r b} f^b{}_{j_{2n-r} e} + (-)^r f^a{}_{j_{2n-r} b} f^b{}_{e i_r} + (-)^{r|e|} f^a{}_{e b} f^b{}_{i_r j_{2n-r}}. \quad (4.11)$$

The key idea is that this, since it holds for all indices a and e , holds in particular when $a = e$. One can sum the resulting identities over any set of indices a one chooses. The sums relevant here are

those over the (bases of the) individual subspaces of grades $r = 1, 2, \dots, 2n - 2, 2n - 1$:

$$\begin{aligned}
0 &= f^{p_1}_{i_r k_{2n-r+1}} f^{k_{2n-r+1}}_{j_{2n-r} p_1} + f^{p_1}_{j_{2n-r} k_{r+1}} f^{k_{r+1}}_{p_1 i_r} + f^{p_1}_{p_1 \alpha} f^{\alpha}_{i_r j_{2n-r}} \\
0 &= f^{p_2}_{i_r k_{2n-r+2}} f^{k_{2n-r+2}}_{j_{2n-r} p_2} + (-)^r f^{p_2}_{j_{2n-r} k_{r+2}} f^{k_{r+2}}_{p_2 i_r} + f^{p_2}_{p_2 \alpha} f^{\alpha}_{i_r j_{2n-r}} \\
&\vdots \\
0 &= f^{p_{2n-2}}_{i_r k_{2n-r-2}} f^{k_{2n-r-2}}_{j_{2n-r} p_{2n-2}} + (-)^r f^{p_{2n-2}}_{j_{2n-r} k_{r-2}} f^{k_{r-2}}_{p_{2n-2} i_r} + f^{p_{2n-2}}_{p_{2n-2} \alpha} f^{\alpha}_{i_r j_{2n-r}} \\
0 &= f^{p_{2n-1}}_{i_r k_{2n-r-1}} f^{k_{2n-r-1}}_{j_{2n-r} p_{2n-1}} + f^{p_{2n-1}}_{j_{2n-r} k_{r-1}} f^{k_{r-1}}_{p_{2n-1} i_r} + f^{p_{2n-1}}_{p_{2n-1} \alpha} f^{\alpha}_{i_r j_{2n-r}}.
\end{aligned} \tag{4.12}$$

Consider now subtracting from equation (4.9) a linear combination (with coefficients, say, $a_1, a_2, \dots, a_{2n-2}, a_{2n-1}$) of these identities and a multiple b of the identity (4.10). The goal is to reduce the right-hand side of (4.9) to an expression involving *only* terms of the type $f^{p_t}_{p_t \alpha} f^{\alpha}_{i_r j_{2n-r}}$. This demand produces a set of linear equations for the coefficients a_r and b . First, the terms $f^{\alpha}_{i_r k_{2n-r}} f^{k_{2n-r}}_{j_{2n-r} \alpha}$ and $f^{p_r}_{i_r k_{\alpha}} f^{k_{\alpha}}_{j_{2n-r} p_r}$ do not appear if and only if

$$0 = (-)^r b + a_r, \quad 0 = b - (-)^r a_{2n-r} \tag{4.13}$$

respectively. And the remaining terms — the terms which are present in (4.9) as it stands — are cancelled if and only if

$$\begin{aligned}
1 &= a_1 - (-)^r a_{2n-r+1} - b, \\
-1 &= a_2 - (-)^r a_{2n-r+2} + b, \\
1 &= a_3 - (-)^r a_{2n-r+3} - b, \\
&\vdots \\
(-)^r 1 &= a_{r-1} - (-)^r a_{2n-1} - (-)^r b, \\
-(-)^r 1 &= a_{r+1} - (-)^r a_{2n+1} - (-)^r b, \\
&\vdots \\
-1 &= a_{2n-3} - (-)^r a_{2n-r-3} - b, \\
1 &= a_{2n-2} - (-)^r a_{2n-r-2} + b, \\
-1 &= a_{2n-1} - (-)^r a_{2n-r-1} - b.
\end{aligned} \tag{4.14}$$

These equations are soluble: a⁶ solution is

$$a_t = -(-)^t \left(1 - \frac{t}{n}\right), \quad b = 1 - \frac{r}{n}, \tag{4.15}$$

and one is left, after these subtractions, with

$$f^{\alpha}_{i_r j_{2n-r}} \sum_{t=1}^{2n-1} (-)^t \left(1 - \frac{t}{n}\right) f^{k_t}_{k_t \alpha}. \tag{4.16}$$

⁶The solution is not always unique. In particular when $r = n$ the situation is simplified since $b = 0$ and there is some freedom in the choice of the a_i .

Since the summation here does not depend on r , we have shown that

$$\hat{R}_{[ij]} = f^\alpha{}_{ij} \sum_{t=1}^{2n-1} (-)^t \left(1 - \frac{t}{n}\right) f_{k_t\alpha}^{k_t}. \quad (4.17)$$

But now, just as for the \mathbb{Z}_4 case in [14], the two-form with these components is exact, and in fact zero if H is semisimple. The argument for exactness is as follows: call the sum on the right V_α , and note that the graded Jacobi identity implies that $V_\alpha f^\alpha{}_{\beta\gamma}$ vanishes.⁷ It then follows from the structure equations (3.3) that

$$\hat{R}_{[ij]} e^i \wedge e^j \sim dV_\alpha e^\alpha \quad (4.19)$$

and hence that the B -field is renormalized only by a total derivative, which has no physical effect since it does not modify the torsion $H = dB$.

Furthermore, if the isotropy subgroup H is semisimple then every generator T_α of H can be expressed in terms of commutators of other generators and so the vanishing of $V_\alpha f^\alpha{}_{\beta\gamma}$ actually implies the vanishing of V_α . The two-form β^B is then not merely exact but actually zero.

5 Conclusions and outlook

In this paper we have calculated the generalized Ricci curvature of coset superspaces G/H defined by \mathbb{Z}_{2n} gradings of Ricci-flat supergroups. We have shown that these supercosets admit a Ricci-flat torsion connection, and that, consequently, sigma models with such spaces as targets are quantum conformal to one loop for an appropriate choice of WZ term. As in the \mathbb{Z}_4 case in [14] we expect that conformal invariance is exact, but further work is needed to establish this.

This result provides strong evidence to support the suggestion, made in [8], that quotienting Ricci-flat supergroups — specifically products of $PSL(n|n)$ — by suitable subgroups might be a fruitful way to find new lagrangian conformal field theories. Theories of this type have been vital in the study of the gauge/string correspondence, and we hope that a more general class of such theories might be used to shed further light on this remarkable subject [29, 30].

Another reason for interest in these backgrounds is their possible relevance to mirror symmetry. Some years ago, it was argued [19, 20, 21] that certain Ricci-flat supergroups could profitably be regarded as Calabi-Yau supermanifolds, and that they actually arise as mirrors of rigid Calabi-Yau manifolds. This prompted work on Ricci-flat supermanifolds [22, 23, 24]. Our targets are Ricci-flat only in the generalized sense, but one can speculate that there are links to be made.

Finally, it is interesting that the same WZ term needed for classical integrability, as evidenced by the existence of one-parameter families of flat currents, is also necessary for generalized Ricci

⁷For each grade t , we have

$$f^{k_t}{}_{k_t\alpha} f^\alpha{}_{\beta\gamma} + f^{k_t}{}_{\beta p_t} f^{p_t}{}_{\gamma k_t} + f^{k_t}{}_{\gamma p_t} f^{p_t}{}_{k_t\beta} = 0 \quad (4.18)$$

and, on renaming the indices, it is clear that the last two terms cancel.

flatness of the target. One can hope that there are deeper connections to be made between Ricci-flat targets and classical integrability.

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A Example of graded coset superspaces

There are two classical families of simple Lie superalgebras for which the Killing form vanishes identically[17]: $A(r-1|r-1) = \mathfrak{psl}(r|r)$ and $D(r+1, r) = \mathfrak{osp}(2r+2|2r)$. Consider the former. In its defining representation, $\mathfrak{sl}(r|r)$ consists of matrices (“even super-matrices”) of the block form⁸

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix} \tag{A.1}$$

where A, B are $r \times r$ matrices with bosonic entries and X, Y are $r \times r$ matrices with fermionic entries with the condition that the supertrace vanishes:

$$\text{str } M := \text{tr } A - \text{tr } B = 0. \tag{A.2}$$

The set of such matrices forms a Lie algebra under the matrix commutator, but this algebra is not simple because the element

$$\begin{pmatrix} 1_{r \times r} & \\ & 1_{r \times r} \end{pmatrix} \tag{A.3}$$

is central and so generates a one-dimensional ideal. One reaches $\mathfrak{psl}(r|r)$ by quotienting out by this ideal, so we can regard $\mathfrak{psl}(r|r)$ as the set of matrices M as above but with $\text{tr } A = \text{tr } B = 0$, and the Lie product as the matrix commutator composed with the obvious projection back onto this subspace.

Finding a \mathbb{Z}_{2n} -grading of $\mathfrak{g} = \mathfrak{psl}(r|r)$ is equivalent to finding an automorphism of order $2n$ ⁹ For, given such an automorphism, the grade k subspace is the eigenspace with eigenvalue $e^{\pi i k/n}$, and conversely given such a grading one can define an automorphism by specifying its action on the subspaces of definite grade. We thus seek automorphisms σ of $\mathfrak{psl}(r|r)$ of even order such that the “even” eigenspaces of σ are bosonic and the “odd” eigenspaces are fermionic. That is to

⁸More precisely, this is the form of a general linear combination of the superalgebra generators with coefficients of appropriate Grassmann grade.

⁹That is, a map $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $[\sigma M, \sigma L] = \sigma[M, L]$, $\sigma^{2n} = 1$ and $\sigma^k \neq 1$ for all $k < 2n$.

say: we want the eigenspace with eigenvalue $e^{p\pi i/n}$ to be block on-diagonal, and the eigenspace with eigenvalue $e^{(p-\frac{1}{2})\pi i/n}$ to be block off-diagonal, for every $p \in \{1, 2, \dots, n\}$. One class of such automorphisms is as follows. Let

$$\tau : A \mapsto N^{-1}AN \quad (\text{A.4})$$

be an (inner) automorphism of $\mathfrak{sl}(r)$ of order n (so $N^n = \pm 1$ and hence, up to a similarity transformation[26] and an irrelevant overall phase,

$$N = \begin{pmatrix} 1_{a \times a} & & & \\ & e^{\frac{2\pi i}{n}} 1_{b \times b} & & \\ & & \ddots & \\ & & & e^{\frac{2\pi i(n-1)}{n}} 1_{c \times c} \end{pmatrix} \quad (\text{A.5})$$

where $a + b + \dots + c = r$). Consider the order $2n$ automorphism σ of $\mathfrak{psl}(r|r)$ defined by

$$\sigma : M \rightarrow g^{-1}Mg \quad \text{with} \quad g = \begin{pmatrix} N & \\ & e^{\frac{\pi i}{n}} N \end{pmatrix}. \quad (\text{A.6})$$

This sends

$$A \mapsto N^{-1}AN, \quad B \mapsto N^{-1}BN \quad (\text{A.7})$$

$$X \mapsto e^{\pi i/n} N^{-1}XN, \quad Y \mapsto e^{-\pi i/n} N^{-1}YN; \quad (\text{A.8})$$

and it is clear that the ‘‘even’’ eigenspace of σ with eigenvalue $e^{2\pi ip/n}$ consists of supermatrices of the form

$$\begin{pmatrix} A & \\ & B \end{pmatrix} \quad (\text{A.9})$$

where A and B are both eigenvectors of τ with eigenvalue $e^{2\pi ip/n}$, while the ‘‘odd’’ eigenspace of σ with eigenvalue $e^{2\pi ip/n} e^{\pi i/n}$ consists of supermatrices of the form

$$\begin{pmatrix} & X \\ Y & \end{pmatrix} \quad (\text{A.10})$$

where X and Y are eigenvectors of τ with eigenvalues $e^{2\pi ip/n}$ and $e^{2\pi i(p+1)/n}$ respectively.

(If one specializes to $n = 2$ the resulting spaces G/H are $PSL(a+b|a+b)/(SL(a) \times SL(b) \times GL(1))^2$. When $a = b = 1$ this is $PSL(2|2)/(GL(1))^2$ and one real form is $PSU(1,1|2)/(U(1)^2)$, whose bosonic geometry is $AdS_2 \times S^2$.[14] Another possible class of automorphisms involve the outer automorphism

$$\begin{pmatrix} A & X \\ Y & B \end{pmatrix} \mapsto \begin{pmatrix} -A^\top & -Y^\top \\ X^\top & -B^\top \end{pmatrix} \quad (\text{A.11})$$

of $\mathfrak{psl}(n|n)$, where A^\top is the transpose of A . The automorphism of $PSL(4|4)$ which defines $AdS_5 \times S^5$ is of this type.)

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