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The Positive Tropical Grassmannian, the Hypersimplex, and the $m = 2$ Amplituhedron

Tomasz Łukowski[1,*], Matteo Parisi[2,3] and Lauren K. Williams[4]

[1]Department of Physics, Astronomy and Mathematics, University of Hertfordshire, College Lane, AL10 9AB Hatfield, UK, [2]Center of Mathematical Sciences and Applications, Harvard University, 20 Garden St., Cambridge, MA 02138, USA, [3]School of Natural Sciences, Institute for Advanced Study, 1 Einstein Dr., Princeton, NJ 08540, USA, and [4]Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138, USA

**Correspondence to be sent to: e-mail: t.lukowski@herts.ac.uk*

The positive Grassmannian $Gr_{k,n}^{\geq 0}$ is a cell complex consisting of all points in the real Grassmannian whose Plücker coordinates are non-negative. In this paper we consider the image of the positive Grassmannian and its positroid cells under two different maps: the *moment map* μ onto the *hypersimplex* [31] and the *amplituhedron map* $\tilde{Z}$ onto the *amplituhedron* [6]. For either map, we define a *positroid dissection* to be a collection of images of positroid cells that are disjoint and cover a dense subset of the image. Positroid dissections of the hypersimplex are of interest because they include many matroid subdivisions; meanwhile, positroid dissections of the amplituhedron can be used to calculate the amplituhedron's 'volume', which in turn computes scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills. We define a map we call *T-duality* from cells of $Gr_{k+1,n}^{\geq 0}$ to cells of $Gr_{k,n}^{\geq 0}$ and conjecture that it induces a bijection from positroid dissections of the hypersimplex $\Delta_{k+1,n}$ to positroid dissections of the amplituhedron $\mathcal{A}_{n,k,2}$; we prove this conjecture for the (infinite) class of *BCFW dissections*. We note that T-duality is particularly striking because the hypersimplex is an $(n-1)$-dimensional polytope while the amplituhedron $\mathcal{A}_{n,k,2}$ is a $2k$-dimensional non-polytopal subset of

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the Grassmannian $Gr_{k,k+2}$. Moreover, we prove that the *positive tropical Grassmannian* is the secondary fan for the regular positroid subdivisions of the hypersimplex, and prove that a matroid polytope is a positroid polytope if and only if all 2D faces are positroid polytopes. Finally, toward the goal of generalizing T-duality for higher m, we define the *momentum amplituhedron* for any even m.

1 Introduction

In 1987, the foundational work of Gelfand–Goresky–MacPherson–Serganova [31] initiated the study of the Grassmannian and torus orbits in the Grassmannian via the *moment map* and *matroid polytopes*, which arise as moment map images of (closures of) torus orbits. Classifying points of the Grassmannian based on the moment map images of the corresponding torus orbits leads naturally to the *matroid stratification* of the Grassmannian. The moment map image of the entire Grassmannian $Gr_{k+1,n}$ is the $(n-1)$-dimensional hypersimplex $\Delta_{k+1,n} \subseteq \mathbb{R}^n$, the convex hull of the indicator vectors $e_I \in \mathbb{R}^n$ where $I \in \binom{[n]}{k+1}$. Over the last decades there has been a great deal of work on matroid subdivisions of the hypersimplex [39, 45, 61]; these are closely connected to the *tropical Grassmannian* [36, 61, 63] and the *Dressian* [36], which parametrizes regular matroidal subdivisions of the hypersimplex.

The matroid stratification of the real Grassmannian is notoriously complicated: Mnev's universality theorem says that the topology of the matroid strata can be as bad as that of any algebraic variety. However, there is a subset of the Grassmannian called the *totally non-negative Grassmannian* or (informally) the *positive Grassmannian* [49, 54], where these difficulties disappear: the restriction of the matroid stratification to the positive Grassmannian gives a cell complex [54, 56, 57], whose cells S_π are called *positroid cells* and labelled by (among other things) *decorated permutations*. Since the work of Postnikov [54], there has been an extensive study of *positroids* [9, 10, 51]—the matroids associated to the positroid cells. The moment map images of positroid cells are precisely the positroid polytopes [68], and as we will discuss in this paper, the *positive tropical Grassmannian* [65] (which equals the *positive Dressian* [66]) parametrizes the regular positroid subdivisions of the hypersimplex.

Besides the moment map, there is another interesting map on the positive Grassmannian, which was recently introduced by Arkani–Hamed and Trnka [6] in the context of *scattering amplitudes* in $\mathcal{N}=4$ SYM. In particular, any $n \times (k+m)$ matrix Z with maximal minors positive induces a map $\tilde{Z}$ from $Gr_{k,n}^{\geq 0}$ to the Grassmannian $Gr_{k,k+m}$, whose image has full dimension mk and is called the *amplituhedron* $A_{n,k,m}$ [6]. The case

$m = 4$ is most relevant to physics: in this case, the *BCFW recurrence* (named for Britto, Cachazo, Feng, and Witten [13]) gives rise to collections of $4k$-dimensional cells in $Gr_{k,n}^{\geq 0}$ whose images *tile* or *triangulate* the amplituhedron.

Given that the hypersimplex and the amplituhedron are images of the positive Grassmannian, which has a decomposition into positroid cells, one can ask the following questions. When does a collection of positroid cells give – via the moment map – a *positroid dissection* of the hypersimplex? By *dissection*, we mean that the images of these cells are disjoint and cover a dense subset of the hypersimplex (but we do not put any constraints on how their boundaries match up). When does a collection of positroid cells give – via the $\widetilde{Z}$-map – a dissection of the amplituhedron? We can also ask about *positroid tilings*, which are dissections coming from cells on which the moment map (respectively, the $\widetilde{Z}$-map) is injective.

The combinatorics of positroid tilings for both the hypersimplex and the amplituhedron is very interesting: Speyer's f-vector theorem [61, 62] gives an upper bound on the number of matroid polytopes of each dimension in a matroidal subdivision coming from the tropical Grassmannian. In particular, it says that the number of top-dimensional matroid polytopes in such a subdivision of $\Delta_{k+1,n}$ is at most $\binom{n-2}{k}$. This number is in particular achieved by finest positroid subdivisions [66]. Meanwhile, the third author together with Karp and Zhang [44] conjectured that the number of cells in a tiling of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ for even m is precisely $M(k, n-k-m, \frac{m}{2})$, where

$$M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

is the number of plane partitions contained in an $a \times b \times c$ box. Note that when $m = 2$, this conjecture says that the number of cells in a tiling of $\mathcal{A}_{n,k,2}(Z)$ equals $\binom{n-2}{k}$.

What we show in this paper is that the appearance of the number $\binom{n-2}{k}$ in the context of both the hypersimplex $\Delta_{k+1,n}$ and the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ is not a coincidence! Indeed, we can obtain tilings of the amplituhedron from tilings of the hypersimplex, by applying a *T-duality map*. This T-duality map sends loopless positroid cells S_π of $Gr_{k+1,n}^{\geq 0}$ to coloopless positroid cells $S_{\hat{\pi}}$ of $Gr_{k,n}^{\geq 0}$ via a simple operation on the decorated permutations, see Section 5. T-duality sends *tiles* for the hypersimplex (cells where the moment map is injective) to tiles for the amplituhedron (cells where $\widetilde{Z}$ is injective), see Proposition 6.6, and moreover it sends dissections of the hypersimplex to dissections of the amplituhedron, see Theorem 6.5 and Conjecture 6.9. This explains the two appearances of the number $\binom{n-2}{k}$ on the two sides of the story.

The fact that dissections of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ are in bijection is a rather surprising statement. Should there be a map from $\Delta_{k+1,n}$ to $\mathcal{A}_{n,k,2}(Z)$ or vice-versa? We have $\dim \Delta_{k+1,n} = n-1$ and $\dim \mathcal{A}_{n,k,2}(Z) = 2k$, with no relation between $n-1$ and $2k$ (apart from $k \leq n$) so it is not obvious that a nice map between them should exist. Nevertheless we do show that T-duality descends from a certain map that can be defined directly on positroid cells of $Gr^{\geq 0}_{k+1,n}$.

The T-duality map provides a handy tool for studying the amplituhedron $\mathcal{A}_{n,k,2}(Z)$: we can try to understand properties of the amplituhedron (and its dissections) by studying the hypersimplex and applying T-duality. For example, we show in Section 7 that the rather mysterious *parity duality*, which relates dissections of $\mathcal{A}_{n,k,2}(Z)$ with dissections of $\mathcal{A}_{n,n-k-2,2}$, can be obtained by composing the hypersimplex duality $\Delta_{k+1,n} \simeq \Delta_{n-k-1,n}$ (which comes from the Grassmannian duality $Gr_{k+1,n} \simeq Gr_{n-k-1,n}$) with T-duality on both sides. As another example, we can try to obtain "nice" dissections of the amplituhedron from correspondingly nice dissections of the hypersimplex. In general, dissections of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ may have unpleasant properties, with images of cells intersecting badly at their boundaries, see Section 8. However, the *regular subdivisions* of $\Delta_{k+1,n}$ are very nice polyhedral subdivisions. By Proposition 9.12, the regular positroid dissections of $\Delta_{k+1,n}$ come precisely from the positive Dressian $Dr^{+}_{k+1,n}$ (which equals the positive tropical Grassmannian $\mathrm{Trop}^{+} Gr_{k+1,n}$). And moreover the images of these subdivisions under the T-duality map are very nice subdivisions of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$, see Section 10. We speculate that $\mathrm{Trop}^{+} Gr_{k+1,n}$ plays the role of secondary fan for the regular positroid subdivisions of $\mathcal{A}_{n,k,2}(Z)$, see Conjecture 10.7.

One step in proving Proposition 9.12 is the following new characterization of positroid polytopes (see Theorem 3.9): a matroid polytope is a positroid polytope if and only if all of its two-dimensional faces are positroid polytopes.

Let us now explain how the various geometric objects in our story are related to scattering amplitudes in supersymmetric fields theories. The main emphasis so far has been on the so-called "planar limit" of $\mathcal{N} = 4$ super Yang-Mills. In 2009, the works of Arkani-Hamed–Cachazo–Cheung–Kaplan [3] and Bullimore–Mason–Skinner [18] introduced beautiful Grassmannian formulations for scattering amplitudes in this theory. Remarkably, this led to the discovery that the positive Grassmannian encodes most of the physical properties of amplitudes [1]. Building on these developments and on Hodges' idea that scattering amplitudes might be 'volumes' of some geometric object [37], Arkani-Hamed and Trnka arrived at the definition of the *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ [6] in 2013.

The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$ is the object most relevant to physics: it encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM. However, the amplituhedron is a well-defined and interesting mathematical object for any m. For example, the $m = 1$ amplituhedron $\mathcal{A}_{n,k,1}$ can be identified with the complex of bounded faces of a cyclic hyperplane arrangement [43]. The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}(Z)$, which is a main subject of this paper, also has a beautiful combinatorial structure, and has been recently studied e.g. in [7, 16, 44, 46, 47]. From the point of view of physics, $\mathcal{A}_{n,k,2}(Z)$ is often considered as a toy-model for the $m = 4$ case. However it has applications to physics as well: $\mathcal{A}_{n,2,2}$ governs the geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at the subleading order in perturbation theory for the so-called 'MHV' sector of the theory, and remarkably, the $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}(Z)$ is also relevant for the 'next to MHV' sector, enhancing its connection with the geometries of loop amplitudes [41].

Meanwhile, in recent years physicists have been increasingly interested in understanding how *cluster algebras* encode the analytic properties of scattering amplitudes, both at tree- and loop- level [31]. This led them to explore the connection between cluster algebras and the *positive tropical Grassmannian* which was observed in [65]. In particular, the positive tropical Grassmannian has been increasingly playing a role in different areas of scattering amplitudes: from bootstrapping loop amplitudes in $\mathcal{N} = 4$ SYM [4, 22, 38] to computing scattering amplitudes in certain scalar theories [20].

Finally, physicists have already observed a duality between the formulations of scattering amplitudes $\mathcal{N} = 4$ SYM in momentum space[1] and in momentum twistor space. This is possible because of the so-called 'Amplitude/Wilson loop duality' [8], which was shown to arise from a more fundamental duality in String Theory called 'T-duality' [17]. The geometric counterpart of this fact is a duality between collections of $4k$-dimensional 'BCFW' cells of $Gr_{k,n}^{\geq 0}$ which tile the amplituhedron $\mathcal{A}_{n,k,4}$ [26], and corresponding collections of $(2n-4)$-dimensional cells of $Gr_{k+2,n}^{\geq 0}$ which (conjecturally) tile the *momentum amplituhedron* $\mathcal{M}_{n,k,4}$; the latter object was introduced very recently by the first two authors together with Damgaard and Ferro [23]. In this paper we see that this duality, which we have evocatively called *T-duality*, extends beyond $m = 4$. In particular, for $m = 2$, the *hypersimplex* $\Delta_{k+1,n}$ and the $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}(Z)$ are somehow dual to each other, a phenomenon that we explore and employ to study properties of both objects. We believe that this duality holds for any (even) m:

[1] More precisely, it is 'spinor helicity' space, or, equivalently (related by half-Fourier transform), in twistor space. See [1, Section 8].

in Section 11 we introduce a generalization $\mathcal{M}_{n,k,m}$ of the momentum amplituhedron $\mathcal{M}_{n,k,4}$, and a corresponding notion of *T-duality*.

2 The positive Grassmannian, the hypersimplex, and the amplituhedron

In this section we introduce the three main geometric objects in this paper: the positive Grassmannian, the hypersimplex, and the amplituhedron. The latter two objects are images of the positive Grassmannian under the *moment map* and the $\widetilde{Z}$*-map*.

Definition 2.1. The *(real) Grassmannian* $Gr_{k,n}$ (for $0 \leq k \leq n$) is the space of all k-dimensional subspaces of $\mathbb{R}^n$. An element of $Gr_{k,n}$ can be viewed as a $k \times n$ matrix of rank k modulo invertible row operations, whose rows give a basis for the k-dimensional subspace.

Let $[n]$ denote $\{1, \ldots, n\}$, and $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. Given $V \in Gr_{k,n}$ represented by a $k \times n$ matrix A, for $I \in \binom{[n]}{k}$ we let $p_I(V)$ be the $k \times k$ minor of A using the columns I. The $p_I(V)$ do not depend on our choice of matrix A (up to simultaneous rescaling by a nonzero constant), and are called the *Plücker coordinates* of V.

2.1 The positive Grassmannian and its cells

Definition 2.2 ([54, Section 3]). We say that $V \in Gr_{k,n}$ is *totally nonnegative* if $p_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$. The set of all totally nonnegative $V \in Gr_{k,n}$ is the *totally nonnegative Grassmannian* $Gr_{k,n}^{\geq 0}$; abusing notation, we will often refer to $Gr_{k,n}^{\geq 0}$ as the *positive Grassmannian*. For $M \subseteq \binom{[n]}{k}$, let S_M be the set of $V \in Gr_{k,n}^{\geq 0}$ with the prescribed collection of Plücker coordinates strictly positive (i.e. $p_I(V) > 0$ for all $I \in M$), and the remaining Plücker coordinates equal to zero (i.e. $p_J(V) = 0$ for all $J \in \binom{[n]}{k} \setminus M$). If $S_M \neq \emptyset$, we call M a *positroid* and S_M its *positroid cell*.

Each positroid cell S_M is indeed a topological cell [54, Theorem 6.5], and moreover, the positroid cells of $Gr_{k,n}^{\geq 0}$ glue together to form a CW complex [56].

As shown in [54], the cells of $Gr_{k,n}^{\geq 0}$ are in bijection with various combinatorial objects, including *decorated permutations* π on $[n]$ with k anti-excedances and equivalence classes of *reduced plabic graphs* G of type (k, n). In Section 12 we review these objects and give bijections between them. This gives a canonical way to label each positroid by a decorated permutation and an equivalence class of plabic graphs; we will correspondingly refer to positroid cells as S_π, S_G, etc.

2.2 The moment map and the hypersimplex

The *moment map* from the Grassmannian $Gr_{k,n}$ to $\mathbb{R}^n$ is defined as follows.

Definition 2.3. Let A be a $k \times n$ matrix representing a point of $Gr_{k,n}$. The *moment map*[2] $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2},$$

where $e_I := \sum_{i \in I} e_i \in \mathbb{R}^n$, and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

It is well-known that the image of the Grassmannian $Gr_{k,n}$ under the moment map is the (k,n)-*hypersimplex* $\Delta_{k,n}$, which is the convex hull of the points e_I where I runs over $\binom{[n]}{k}$. If one restricts the moment map to $Gr_{k,n}^{\geq 0}$ then the image is again the hypersimplex $\Delta_{k,n}$ [68, Proposition 7.10].

We will consider the restriction of the moment map to positroid cells of $Gr_{k,n}^{\geq 0}$.

Definition 2.4. Given a positroid cell S_π of $Gr_{k,n}^{\geq 0}$, we let $\Gamma_\pi^\circ = \mu(S_\pi)$, and $\Gamma_\pi = \overline{\mu(S_\pi)}$.

There are a number of natural questions to ask. What do the Γ_π look like, and how can one characterize them? On which positroid cells is the moment map injective? The images Γ_π of (closures of) positroid cells are called *positroid polytopes*; we will explore their nice properties in Section 3.

One of our main motivations is to understand *positroid dissections* of the hypersimplex.

Definition 2.5. Let $\mathcal{C} = \{\Gamma_\pi\}$ be a collection of positroid polytopes, with $\{S_\pi\}$ a collection of positroid cells of $Gr_{k,n}^{\geq 0}$. We say that $\mathcal{C}$ is a *positroid dissection* of $\Delta_{k,n}$ if we have that:

- $\dim \Gamma_\pi = n - 1$ for each $\Gamma_\pi \in \mathcal{C}$
- pairs of two distinct positroid polytopes Γ_π° and $\Gamma_{\pi'}^\circ$ in the collection are disjoint
- $\cup_\pi \Gamma_\pi = \Delta_{k,n}$, i.e. the union of the images of the cells is dense in $\Delta_{k,n}$.

2 We remark that there is another version of the moment map called the *algebraic moment map*, which we will briefly discuss later, see Definition 3.18.

We say that a positroid dissection $\mathcal{C} = \{\Gamma_\pi\}$ of $\Delta_{k,n}$ is a *positroid tiling* (or simply a *tiling*) of $\Delta_{k,n}$ if μ is injective on each S_π.

Question 2.6. Let $\mathcal{C} = \{\Gamma_\pi\}$ be a collection of positroid polytopes, with $\{S_\pi\}$ positroid cells of $Gr_{k,n}^{\geq 0}$. When is $\mathcal{C}$ a positroid dissection of $\Delta_{k,n}$? When is it a positroid tiling?

2.3 The $\widetilde{Z}$-map and the amplituhedron

Building on [1], Arkani-Hamed and Trnka [6] recently introduced a beautiful new mathematical object called the *(tree) amplituhedron*, which is the image of the positive Grassmannian under a map $\widetilde{Z}$ induced by a totally positive matrix Z.

Definition 2.7. For $a \leq b$, define $\mathrm{Mat}_{a,b}^{>0}$ as the set of real $a \times b$ matrices whose $a \times a$ minors are all positive. Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. The *amplituhedron map* $\tilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ is defined by $\tilde{Z}(C) := CZ$, where C is a $k \times n$ matrix representing an element of $Gr_{k,n}^{\geq 0}$ and CZ is a $k \times (k+m)$ matrix representing an element of $Gr_{k,k+m}$. The *amplituhedron* $\mathcal{A}_{n,k,m}(Z) \subseteq Gr_{k,k+m}$ is the image $\tilde{Z}(Gr_{k,n}^{\geq 0})$.

In special cases the amplituhedron recovers familiar objects. If Z is a square matrix, i.e. $k + m = n$, then $\mathcal{A}_{n,k,m}(Z)$ is isomorphic to the positive Grassmannian. If $k = 1$, then it follows from [64] that $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* in projective space $\mathbb{P}^m$. If $m = 1$, then $\mathcal{A}_{n,k,1}(Z)$ can be identified with the complex of bounded faces of a cyclic hyperplane arrangement [43].

We will consider the restriction of the $\widetilde{Z}$-map to positroid cells of $Gr_{k,n}^{\geq 0}$.

Definition 2.8. Given a positroid cell S_π of $Gr_{k,n}^{\geq 0}$, we let $Z_\pi^\circ = \widetilde{Z}(S_\pi)$, and $Z_\pi = \overline{\widetilde{Z}(S_\pi)}$. We refer to Z_π° and Z_π as *open Grasstopes* and *Grasstopes* respectively.

As in the case of the hypersimplex, one of our main motivations is to understand *positroid dissections* of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$.

Definition 2.9. Let $\mathcal{C} = \{Z_\pi\}$ be a collection of Grasstopes, with $\{S_\pi\}$ a collection of positroid cells of $Gr_{k,n}^{\geq 0}$. We say that $\mathcal{C}$ is a *positroid dissection* of $\mathcal{A}_{n,k,m}(Z)$ if we have that:

- $\dim Z_\pi = mk$ for each $Z_\pi \in \mathcal{C}$
- pairs of distinct open Grasstopes Z_π° and $Z_{\pi'}^\circ$ in the collection are disjoint
- $\cup_\pi Z_\pi = \mathcal{A}_{n,k,m}(Z)$.

We say that a positroid dissection $\mathcal{C} = \{Z_\pi\}$ of $\mathcal{A}_{n,k,m}(Z)$ is a *positroid tiling* (or simply a *tiling*) of $\mathcal{A}_{n,k,m}(Z)$ if $\widetilde{Z}$ is injective on each S_π.

Remark 2.10. Let $\mathcal{S}$ be an index set for cells of $Gr_{k,n}^{\geq 0}$. It is expected that if Z and Z' both lie in $\mathrm{Mat}_{k+m,n}^{>0}$, then $\{Z_\pi\}_{\pi\in\mathcal{S}}$ is a positroid tiling (respectively, dissection) of $\mathcal{A}_{n,k,m}(Z)$ if and only if $\{Z'_\pi\}_{\pi\in\mathcal{S}}$ is a positroid tiling (respectively, dissection) of $\mathcal{A}_{n,k,m}(Z')$.

The results we prove in this paper will be independent of Z.

Question 2.11. Let $\mathcal{C} = \{Z_\pi\}$ be a collection of Grasstopes, with $\{S_\pi\}$ positroid cells of $Gr_{k,n}^{\geq 0}$. When is $\mathcal{C}$ a positroid dissection of $\mathcal{A}_{n,k,m}(Z)$? When is it a positroid tiling?

In this paper we will primarily focus on the case $m = 2$ (with the exception of Section 11, where we give some generalizations of our results and conjectures to general even m). (Positroid) tilings of the amplituhedron have also been studied in [6], [27], [2], [44], [33], [28]. Very recently the paper [16] constructed (with proof) many tilings of the $m = 2$ amplituhedron. The $m = 2$ amplituhedron has also been studied in [7] (which gave an alternative description of it in terms of sign patterns; see also [44]), in [47] (which described the boundary stratification of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$), and in [46] (which discussed its relation to cluster algebras). Note that our notion of dissection above is the same as the notion of subdivision from [33, Definition 7.1]. (However, we prefer the word "dissection," as the word "subdivision" is often used to indicate that there are constraints on how the boundaries match up.)

3 Positroid polytopes and the moment map

In this section we study *positroid polytopes*, which are images of positroid cells of $Gr_{k,n}^{\geq 0}$ under the moment map $\mu : Gr_{k,n}^{\geq 0} \to \mathbb{R}^n$. We recall some of the known properties of matroid and positroid polytopes, we give a new characterization of positroid polytopes (see Theorem 3.9), and we describe when the moment map is an injection on a positroid cell, or equivalently, when the moment map restricts to a homeomorphism from the closure of a positroid cell to the corresponding positroid polytope (see Proposition 3.15 and Proposition 3.16).

3.1 Matroid polytopes

The torus $T = (\mathbb{C}^*)^n$ acts on $Gr_{k,n}$ by scaling the columns of a matrix representative A. We let TA denote the orbit of A under the action of T, and $\overline{TA}$ its closure. It follows from classical work of Atiyah [11] and Guillemin-Sternberg [34] that the image $\mu(\overline{TA})$

is a convex polytope, whose vertices are the images of the torus-fixed points, i.e. the vertices are the points e_I such that $p_I(A) \neq 0$.

This motivates the notion of *matroid polytope*. Note that any full rank $k \times n$ matrix A gives rise to a matroid $M(A) = ([n], \mathcal{B})$, where $\mathcal{B} = \{I \in \binom{[n]}{k} \mid p_I(A) \neq 0\}$.

Definition 3.1. Given a matroid $M = ([n], \mathcal{B})$, the (basis) *matroid polytope* Γ_M of M is the convex hull of the indicator vectors of the bases of M:

$$\Gamma_M := \mathrm{convex}\{e_B \mid B \in \mathcal{B}\} \subset \mathbb{R}^n.$$

The following elegant characterization of matroid polytopes is due to Gelfand, Goresky, MacPherson, and Serganova.

Theorem 3.2 ([31]). Let $\mathcal{B}$ be a collection of subsets of $[n]$ and let $\Gamma_{\mathcal{B}} := \mathrm{convex}\{e_B \mid B \in \mathcal{B}\} \subset \mathbb{R}^n$. Then $\mathcal{B}$ is the collection of bases of a matroid if and only if every edge of $\Gamma_{\mathcal{B}}$ is a parallel translate of $e_i - e_j$ for some $i, j \in [n]$.

The dimension of a matroid polytope is determined by the number of connected components of the matroid. Recall that a matroid which cannot be written as the direct sum of two nonempty matroids is called *connected*.

Proposition 3.3 ([53]). Let M be a matroid on E. For two elements $a, b \in E$, we set $a \sim b$ whenever there are bases B_1, B_2 of M such that $B_2 = (B_1 - \{a\}) \cup \{b\}$. The relation $\sim$ is an equivalence relation, and the equivalence classes are precisely the connected components of M.

Proposition 3.4 ([15]). For any matroid, the dimension of its matroid polytope is $\dim \Gamma_M = n - c$, where c is the number of connected components of M.

We note that there is an inequality description of any matroid polytope.

Proposition 3.5 ([69]). Let $M = ([n], \mathcal{B})$ be any matroid of rank k, and let $r_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ be its rank function. Then the matroid polytope Γ_M can be described as

$$\Gamma_M = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = k, \sum_{i \in A} x_i \leq r_M(A) \text{ for all } A \subset [n] \right\}.$$

3.2 Positroid polytopes

In this paper we are interested in *positroids*; these are the matroids $M(A)$ associated to $k \times n$ matrices A with maximal minors all nonnegative.

In Definition 3.1, we defined the matroid polytope Γ_M to be the convex hull of the indicator vectors of the bases of the matroid M. We can of course apply the same definition to any positroid M, obtaining the *positroid polytope* Γ_M. On the other hand, in Definition 2.4, for each positroid cell S_π, we defined $\Gamma_\pi = \overline{\mu(S_\pi)}$ to be the closure of the image of the cell under the moment map. Fortunately these two objects coincide.

Proposition 3.6. [68, Proposition 7.10] Let M be the positroid associated to the positroid cell S_π. Then $\Gamma_M = \Gamma_\pi = \mu(\overline{S_\pi}) = \overline{\mu(S_\pi)}$.

The first statement in Theorem 3.7 below was proved in [9, Corollary 5.4] (and generalized to the setting of Coxeter matroids in [68, Theorem 7.13].) The second statement follows from the proof of [68, Theorem 7.13].

Theorem 3.7. Every face of a positroid polytope is a positroid polytope. Moreover, every face $\Gamma_{\pi'}$ of a positroid polytope Γ_π has the property that $S_{\pi'} \subset \overline{S_\pi}$.

There is a simple inequality characterization of positroid polytopes.

Proposition 3.8. [9, Proposition 5.7] A matroid M of rank k on $[n]$ is a positroid if and only if its matroid polytope Γ_M can be described by the equality $x_1 + \cdots + x_n = k$ and inequalities of the form

$$\sum_{\ell \in [i,j]} x_\ell \leq r_{ij}, \text{ with } i,j \in [n].$$

Here $[i,j]$ is the cyclic interval given by $[i,j] = \{i, i+1, \ldots, j\}$ if $i < j$ and $[i,j] = \{i, i+1, \ldots, n, 1, \ldots, j\}$ if $i > j$.

We now give a new characterization of positroid polytopes. In what follows, we use Sab as shorthand for $S \cup \{a,b\}$, etc.

Theorem 3.9. Let M be a matroid of rank k on the ground set $[n]$, and consider the matroid polytope Γ_M. It is a positroid polytope (i.e. M is a positroid) if and only if all of its two-dimensional faces are positroid polytopes.

Moreover, if M fails to be a positroid polytope, then Γ_M has a two-dimensional face F with vertices $e_{Sab}, e_{Sad}, e_{Sbc}, e_{Scd}$, for some $1 \le a < b < c < d \le n$ and S of size $k-2$ disjoint from $\{a,b,c,d\}$.

Remark 3.10. A different characterization of positroids in terms of faces of their matroid polytopes was given in [58, Proposition 6.4], see also [58, Lemma 6.2 and Lemma 6.3]. There are also some related ideas in the proof of [25, Lemma 30].

By Theorem 3.7, every two-dimensional face of Γ_M is a positroid polytope. To prove the other half of Theorem 3.9, we use the following lemma.

Lemma 3.11. Let M be a matroid of rank k on $[n]$ which has two connected components, i.e. $M = M_1 \oplus M_2$ such that the ground sets of M_1 and M_2 are S and $T = [n] \setminus S$. Suppose that $\{S, T\}$ fails to be a noncrossing partition of $[n]$, in other words, there exists $a < b < c < d$ (in cyclic order) such that $a, c \in S$ and $b, d \in T$. Then Γ_M has a two-dimensional face which is not a positroid polytope; in particular, that face is a square with vertices $e_{Sab}, e_{Sad}, e_{Sbc}, e_{Scd}$, for some $1 \le a < b < c < d \le n$ and S of size $k-2$ disjoint from $\{a,b,c,d\}$.

Proof. By Proposition 3.3, we have bases Aa and Ac of M_1 and also bases Bb and Bd of M_2. We can find a linear functional on Γ_{M_1} given by a vector in $\mathbb{R}^S$ whose dot product is maximized on the convex hull of e_{Aa} and e_{Ac} (choose the vector w such that $w_h = 1$ for $h \in A$, $w_h = \frac{1}{2}$ for $h = a$ or $h = c$, and $w_h = 0$ otherwise); therefore there is an edge in Γ_{M_1} between e_{Aa} and e_{Ac}. Similarly, there is an edge in Γ_{M_2} between e_{Bb} and e_{Bd}. Therefore $\Gamma_M = \Gamma_{M_1} \times \Gamma_{M_2}$ has a two-dimensional face whose vertices are $e_{ABab}, e_{ABad}, e_{ABbc}, e_{ABcd}$. This is not a positroid polytope because $\{ab, ad, bc, cd\}$ are not the bases of a rank 2 positroid. ■

Proposition 3.12. Let M be a connected matroid. If all of the two-dimensional faces of Γ_M are positroid polytopes, then Γ_M is a positroid polytope (i.e. M is a positroid).

Proof. Suppose for the sake of contradiction that Γ_M is not a positroid polytope.

Since Γ_M is not a positroid polytope, then by Proposition 3.5 and Proposition 3.8, it has a facet F of the form $\sum_{i \in S} x_i = r_M(S)$, where S is not a cyclic interval. In other words, S and $T = [n] \setminus S$ fail to form a noncrossing partition. Each facet of Γ_M is the matroid polytope of a matroid with two connected components, so by the greedy algorithm for matroids (see e.g. [9, Proposition 2.12]), F must be the matroid polytope of $M|S \oplus M/S$. But now by Lemma 3.11, F has a two-dimensional face which is not a positroid polytope. ■

We now complete the proof of Theorem 3.9.

Proof. We start by writing M as a direct sum of connected matroids $M = M_1 \oplus \cdots \oplus M_l$. Let $S_1, \ldots, S_l$ be the ground sets of $M_1, \ldots, M_l$. By [9, Lemma 7.3], either one of the M_i's fails to be a positroid, or $\{S_1, \ldots, S_l\}$ fails to be a non-crossing partition of $[n]$. If one of the M_i's fails to be a positroid, then by Proposition 3.12, Γ_{M_i} has a two-dimensional face which fails to be a positroid. But then so does $\Gamma_M = \Gamma_{M_1} \times \cdots \times \Gamma_{M_l}$. On the other hand, if $\{S_1, \ldots, S_l\}$ fails to be a non-crossing partition of $[n]$, then by Lemma 3.11, Γ_M has a two-dimensional face which fails to be a positroid. This completes the proof. ■

Our next goal is to use Proposition 3.4 to determine when the moment map restricted to a positroid cell is a homeomorphism. To do so, we need to understand how to compute the number of connected components of a positroid. The following result comes from [9, Theorem 10.7] and its proof. We say that a permutation π of $[n]$ is *stabilized-interval-free (SIF)* if it does not stabilize any proper interval of $[n]$; that is, $\pi(I) \neq I$ for all intervals $I \subsetneq [n]$.

Proposition 3.13. Let S_π be a positroid cell of $Gr_{k,n}^{\geq 0}$ and let M_π be the corresponding positroid. Then M_π is connected if and only if π is a SIF permutation of $[n]$. More generally, the number of connected components of M_π equals the number of connected components of any reduced plabic graph associated to π.

Example 3.14. Consider the permutation $\pi = (5, 3, 4, 2, 6, 7, 1)$ (which in cycle notation is $(234)(1567)$. Then there are two minimal-by-inclusion cyclic intervals such that $\pi(I) = I$, namely $[2, 4]$ and $[5, 1]$, and hence the matroid M_π has two connected components. (Note that $[1, 7]$ is also a cyclic interval with $\pi([1, 7]) = [1, 7]$ but it is not minimal-by-inclusion.)

Proposition 3.15. Consider a positroid cell $S_\pi \subset Gr_{k,n}^{\geq 0}$ and let M_π be the corresponding positroid. Then the following statements are equivalent:

1. the moment map restricts to an injection on S_π
2. the moment map is a homeomorphism from $\overline{S_\pi}$ to Γ_π
3. $\dim S_\pi = \dim \Gamma_\pi = n - c$, where c is the number of connected components of the matroid M_π.

Proof. Suppose that (1) holds, i.e. that the moment map is an injection when restricted to a cell S_π. Then $\dim \Gamma_\pi = \dim S_\pi$. By [68, Proposition 7.12], the positroid variety X_π is a toric variety if and only if $\dim \Gamma_\pi = \dim S_\pi$, so this implies that X_π is a toric variety, and $\overline{S_\pi}$ is its nonnegative part. It is well-known that the moment map is a homeomorphism when restricted to the nonnegative part of a toric variety [29, Section 4.2], so it follows that μ is a homeomorphism on $\overline{S_\pi}$. Therefore (1) implies (2). But obviously (2) implies (1).

Now suppose that (2) holds. Since Γ_π is the moment map image of $\overline{S_\pi}$, it follows that $\dim \Gamma_\pi = \dim S_\pi$, and by Proposition 3.4, we have that $\dim \Gamma_\pi = n - c$, where c is the number of connected components of the matroid M_π. Therefore (2) implies (3).

Now suppose (3) holds. Then by [68, Proposition 7.12], X_π must be a toric variety, and so the moment map restricts to a homeomorphism from $\overline{S_\pi}$ to Γ_π. So (3) implies (2). ■

Proposition 3.16. Consider a positroid cell $S_\pi \subset Gr_{k,n}^{\geq 0}$ and let M_π be the corresponding positroid. Then the moment map is a homeomorphism from $\overline{S_\pi}$ to $\Gamma_\pi \subset \mathbb{R}^n$ if and only if any reduced plabic graph associated to π is a forest. The $(n-1)$-dimensional cells S_π on which the moment map is a homeomorphism to their image are precisely those cells whose reduced plabic graphs are trees.

Proof. This follows from Proposition 3.15 and Proposition 3.13, together with the fact that we can read off the dimension of a positroid cell from any reduced plabic graph G for it as the number of regions of G minus 1. ■

Remark 3.17. The connected $(n-1)$-dimensional positroid cells S_π of $Gr_{k,n}^{\geq 0}$ are precisely those $(n-1)$-dimensional cells where π is a single cycle of length n.

As an alternative to the moment map from Definition 2.3, we can also consider the *algebraic moment map* as in [60], defined as follows.[3]

Definition 3.18. Let A be a $k \times n$ matrix representing a point of $Gr_{k,n}$. The *algebraic moment map* $\tilde{\mu} : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\tilde{\mu}(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)| e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|}.$$

3 The reference [60] defines this map for toric varieties, but it makes sense for $Gr_{k,n}$.

Lemma 3.19. Proposition 3.15 and Proposition 3.16 hold verbatim after replacing *moment map* by *algebraic moment map*. In particular, if S_π is a positroid cell whose reduced plabic graph is a tree, then $\tilde{\mu}$ is an injection on S_π and $\Gamma_\pi = \tilde{\mu}(\overline{S_\pi})$.

Proof. We note that both the moment map and the algebraic moment map are homeomorphisms when restricted to the nonnegative part of a toric variety [60, Theorem 8.5], [29, Section 4.2]. Therefore the proofs of Proposition 3.15 and Proposition 3.16 hold when we use the algebraic moment map. ■

Proposition 3.20. We have $\tilde{\mu}(Gr_{k,n}^{\geq 0}) = \Delta_{k,n}$.

Proof. It follows immediately from the definition that $\tilde{\mu}(A)$ will always be a convex combination of the points e_I for $I \in \binom{[n]}{k}$ so $\tilde{\mu}(Gr_{k,n}^{\geq 0}) \subseteq \Delta_{k,n}$.

In the other direction, choose any positroid tiling $\{S_\pi\}$ of $\Delta_{k,n}$, e.g. as in Proposition 10.4. Then by Lemma 3.19 and the definition of positroid tiling, we have $\tilde{\mu}(\overline{S_\pi}) = \Gamma_\pi$ and $\bigcup \Gamma_\pi = \Delta_{k,n}$. It follows that $\tilde{\mu}(Gr_{k,n}^{\geq 0}) = \Delta_{k,n}$. ■

4 Dissecting the hypersimplex and the amplituhedron

In this section we provide two recursive recipes for dissecting the hypersimplex $\Delta_{k+1,n}$, and dissecting the amplituhedron $\mathcal{A}_{n,k,2}(Z)$; the recipe for dissecting the $m = 2$ amplituhedron was proposed in [44, Section 4.1] and proved in [16]. These recursive recipes are completely parallel: as we will see in Section 5, the cells of corresponding dissections are in bijection with each other via the *T-duality map* on positroid cells. Since these two recursions are analogous to the BCFW recurrence (which gives tilings of the $m = 4$ amplituhedron), we refer to them as *BCFW-style* recurrences.

4.1 BCFW dissections of the hypersimplex

Definition 4.1. Let G (resp. G') be a reduced plabic graph with $n-1$ boundary vertices, associated to a positroid cell of $Gr_{k+1,n-1}^{\geq 0}$ (resp. $Gr_{k,n-1}^{\geq 0}$), which do not have a loop at vertex $n-1$. We define $\mathrm{i}_{\mathrm{pre}}$ (resp. $\mathrm{i}_{\mathrm{inc}}$) to be the map which takes G (resp. G') and replaces the $(n-1)$st boundary vertex with a trivalent internal white (resp. black) vertex attached to boundary vertices $n-1$ and n, as in the middle (resp. rightmost) graph of Figure 1.

Abusing notation slightly, we also use $\mathrm{i}_{\mathrm{pre}}$ and $\mathrm{i}_{\mathrm{inc}}$ to denote the corresponding maps on decorated permutations, positroid cells and their images under the moment and amplituhedron maps.

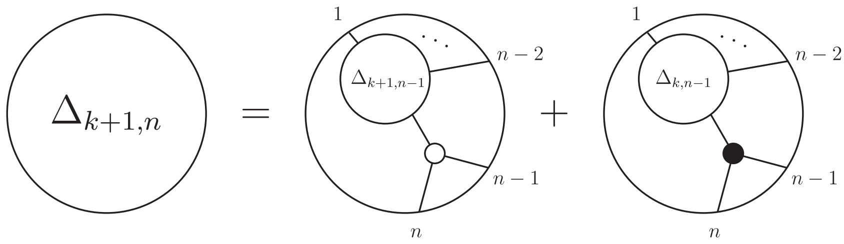


Fig. 1. A BCFW-style recursion for dissecting the hypersimplex. There is a parallel recursion obtained from this one by cyclically shifting all boundary vertices of the plabic graphs by i (modulo n).

Remark 4.2. Using Section 12, it is straightforward to verify that both $\mathfrak{i}_{\text{pre}}(G)$ and $\mathfrak{i}_{\text{inc}}(G')$ are reduced plabic graphs for cells of $Gr_{k+1,n}^{\geq 0}$. Moreover, we can in fact define $\mathfrak{i}_{\text{pre}}(G)$ (resp. $\mathfrak{i}_{\text{inc}}(G')$) on any reduced plabic graph for $Gr_{k+1,n-1}^{\geq 0}$ (resp. $Gr_{k,n-1}^{\geq 0}$) which does not have a black (resp. white) lollipop at vertex $n-1$, and will again have that $\mathfrak{i}_{\text{pre}}(G)$ and $\mathfrak{i}_{\text{inc}}(G')$ represent cells of $Gr_{k+1,n}^{\geq 0}$.

Using Definition 12.7, it is easy to determine the effect of $\mathfrak{i}_{\text{pre}}$ and $\mathfrak{i}_{\text{inc}}$ on decorated permutations. We leave the proof of the following lemma as an exercise.

Lemma 4.3. If $\pi = (a_1, a_2, \ldots, a_{n-1})$ is a decorated permutation such that $(n-1) \mapsto a_{n-1}$ is not a black fixed point, then $\mathfrak{i}_{\text{pre}}(\pi) = (a_1, a_2, \ldots, a_{n-2}, n, a_{n-1})$.

If $\pi = (a_1, a_2, \ldots, a_{n-1})$ is a decorated permutation such that $(n-1) \mapsto a_{n-1}$ is not a white fixed point, then $\mathfrak{i}_{\text{inc}}(\pi) = (a_1, a_2, \ldots, a_{j-1}, n, a_{j+1}, \ldots, a_{n-1}, n-1)$ where $j = \pi^{-1}(n-1)$.

Remark 4.4. Lemma 4.3 can be equivalently expressed in terms of Ⅎ-diagrams (see [54] or [44, Section 2]). If D is the Ⅎ-diagram associated to π as in the first paragraph of Lemma 4.3, then $\mathfrak{i}_{\text{pre}}(D)$ is obtained from D by adding a new column to the left of D, where the new column consists of a single $+$ at the bottom. If D is the Ⅎ-diagram associated to π as in the second paragraph of Lemma 4.3, then $\mathfrak{i}_{\text{inc}}(D)$ is obtained from D by adding a new row at the bottom of D, where the row consists of a single box containing a $+$.

Theorem 4.5 (BCFW recursion for the hypersimplex). Let $\mathcal{C}_{k+1,n-1}$ (respectively $\mathcal{C}_{k,n-1}$) be a collection of positroid polytopes which dissects the hypersimplex $\Delta_{k+1,n-1}$ (resp.

$\Delta_{k,n-1}$). Then

$$\mathcal{C}_{k+1,n} = \mathrm{i}_{\mathrm{pre}}(\mathcal{C}_{k+1,n-1}) \cup \mathrm{i}_{\mathrm{inc}}(\mathcal{C}_{k,n-1})$$

dissects $\Delta_{k+1,n}$.

We use the term *BCFW dissection* (respectively, *BCFW tiling*) to refer to any dissection or tiling that has the form $\mathcal{C}_{k,n}$ from Theorem 4.5.

Diagrammatically, Theorem 4.5 is depicted in Fig. 1.

Remark 4.6. Because of the cyclic symmetry of the positive Grassmannian and the hypersimplex (see e.g. Theorem 7.4) there are $n-1$ other versions of Theorem 4.5 (and Figure 1) in which all plabic graph labels get shifted by i modulo n (for $1 \le i \le n-1$).

Proof. The hypersimplex $\Delta_{k+1,n}$ is cut out by the inequalities $0 \le x_i \le 1$, as well as the equality $\sum_i x_i = k+1$. We will show that Figure 1 represents the partition of $\Delta_{k+1,n}$ into two pieces, with the middle graph representing the piece cut out by $x_{n-1} + x_n \le 1$, and the rightmost graph representing the piece cut out by $x_{n-1} + x_n \ge 1$.

Toward this end, it follows from Proposition 12.6 that if G is a reduced plabic graph representing a cell of $Gr^{\ge 0}_{k+1,n-1}$, such that the positroid M_G has bases $\mathcal{B}$, then the bases of $M_{\mathrm{i}_{\mathrm{pre}}(G)}$ are precisely $\mathcal{B} \sqcup \{(B \setminus \{n-1\}) \cup \{n\} \mid B \in \mathcal{B}, n-1 \in B\}$. In particular, each basis of $M_{\mathrm{i}_{\mathrm{pre}}(G)}$ may contain at most one element of $\{n-1, n\}$.

Meanwhile, it follows from Proposition 12.6 that if G is a reduced plabic graph representing a cell of $Gr^{\ge 0}_{k,n-1}$, such that the positroid M_G has bases $\mathcal{B}$, then the bases of $M_{\mathrm{i}_{\mathrm{inc}}(G)}$ are precisely $\{B \cup \{n\} \mid B \in \mathcal{B}\} \sqcup \{B \cup \{n-1\} \mid B \in \mathcal{B}, n-1 \notin B\}$. In particular, each basis of $M_{\mathrm{i}_{\mathrm{inc}}(G)}$ must contain at least one element of $\{n-1, n\}$.

It is now a straightforward exercise (using e.g. [9, Proposition 5.6]) to determine that if $\mathcal{C}_{k+1,n-1}$ is a collection of cells in $Gr^{\ge 0}_{k+1,n-1}$ whose images dissect $\Delta_{k+1,n-1}$ then the images of $\mathrm{i}_{\mathrm{pre}}(\mathcal{C}_{k+1,n-1})$ dissect the subset of $\Delta_{k+1,n}$ cut out by the inequality $x_{n-1} + x_n \le 1$. Similarly for $\mathrm{i}_{\mathrm{inc}}(\mathcal{C}_{k,n-1})$ and the subset of $\Delta_{k+1,n}$ cut out by $x_{n-1} + x_n \ge 1$. ■

Example 4.7. Let $n = 5$ and $k = 2$. We will use Theorem 4.5 to obtain a dissection of $\Delta_{k+1,n} = \Delta_{3,5}$. We start with a dissection of $\Delta_{3,4}$ coming from the plabic graph shown below (corresponding to the decorated permutation $(4, 1, 2, 3)$), and a dissection of $\Delta_{2,4}$ (corresponding to the permutations $(2, 4, 1, 3)$ and $(3, 1, 4, 2)$). Applying the theorem leads

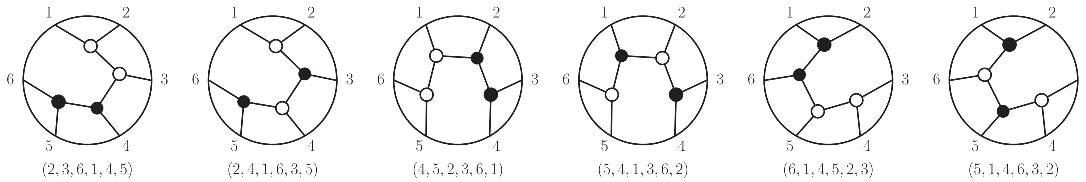
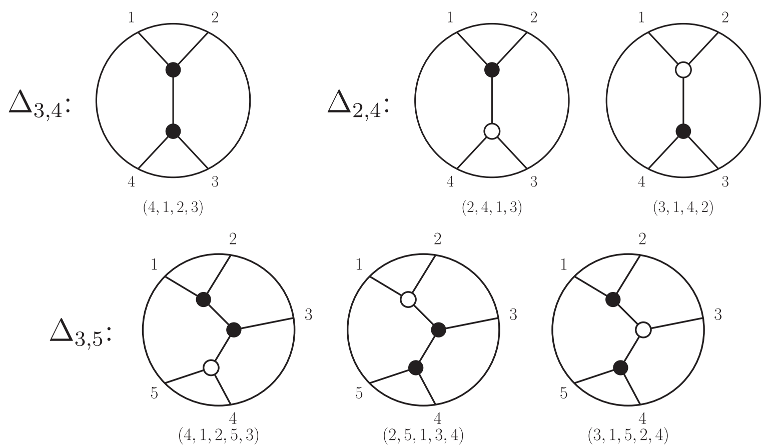


Fig. 2. An example of a dissection of $\Delta_{3,6}$ that cannot be obtained from the BCFW-style recursion in Theorem 4.5.

to the three plabic graphs in the bottom line, which correspond to the permutations $(4, 1, 2, 5, 3), (2, 5, 1, 3, 4), (3, 1, 5, 2, 4)$.



Remark 4.8. It is worth pointing out that our BCFW-style recursion does not provide all possible dissections of the hypersimplex. This comes from the fact that in each step of the recursion we divide the hypersimplex into two pieces, while there are some dissections coming from 3-splits (a k-split is a coarsest subdivision with k maximal faces and a common face of codimension $k-1$). The simplest example of a dissection which cannot be obtained from the recursion can be found already for $\Delta_{3,6}$ and is depicted in Figure 2.

4.2 BCFW dissections of the $m = 2$ amplituhedron

We now introduce some maps on plabic graphs, and recall a result of Bao and He [16].

Definition 4.9. Let G be a reduced plabic graph with $n-1$ boundary vertices, associated to a positroid cell of $Gr_{k,n-1}^{\geq 0}$. We define ι_{pre} to be the map which takes G

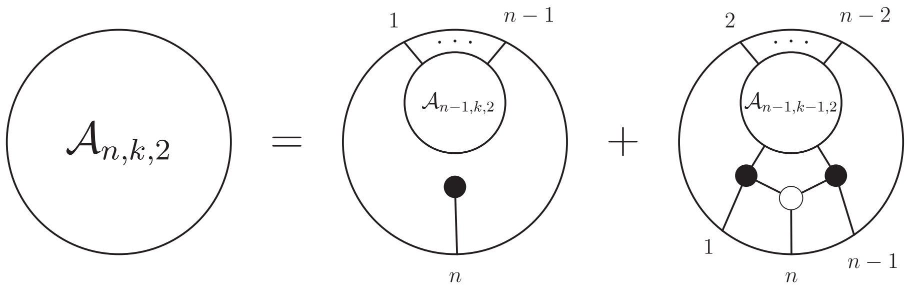


Fig. 3. A BCFW-style recursion for dissecting the amplituhedron. There is a parallel recursion obtained from this one by cyclically shifting all boundary vertices of the plabic graphs by i (modulo n).

and adds a black lollipop at a new boundary vertex n, as shown in the middle graph of Figure 3. Similarly, we define ι_{inc} to be the map on a plabic graph G' for $Gr^{\geq 0}_{k-1,n-1}$ which modifies G', changing the graph locally around vertices $1, n, n-1$, as shown at the right of Figure 3.

Remark 4.10. The resulting graph $\iota_{\text{pre}}(G)$ is a reduced plabic graph for a cell of $Gr^{\geq 0}_{k,n}$. It is not hard to show that, if G' does not have white fixed points at vertices 1 or $n-1$, then $\iota_{\text{inc}}(G')$ is a reduced plabic graph for a cell of $Gr^{\geq 0}_{k,n}$.

Abusing notation slightly, we also use ι_{pre} and ι_{inc} to denote the corresponding maps on positroid cells and positroid polytopes, decorated permutations, etc. Using Definition 12.7, one can also determine the effect of ι_{pre} and ι_{inc} on decorated permutations (and Ⅎ-diagrams). We leave the proof of the following lemma as an exercise.

Lemma 4.11. Let $\pi = (a_1, a_2, \ldots, a_{n-1})$ be a decorated permutation on $n-1$ letters. Then $\iota_{\text{pre}}(\pi) = (a_1, a_2, \ldots, a_{n-2}, a_{n-1}, n)$, where n is a black fixed point.

Let $\pi = (a_1, a_2, \ldots, a_{n-1})$ be a decorated permutation; assume that neither positions 1 nor $n-1$ are white fixed points. Let $h = \pi^{-1}(n-1)$. Then $\iota_{\text{inc}}(\pi)$ is the permutation such that $1 \mapsto n-1$, $h \mapsto n$, $n \mapsto a_1$, and $j \mapsto a_j$ for all $j \neq 1, h, n$.

The construction below is closely related to the recursion from [44, Definition 4.4], which is a sort of $m = 2$ version of the BCFW recurrence.

Theorem 4.12 (BCFW recursions for the $m = 2$ amplituhedron). [16, Theorem A] Let $\mathcal{C}_{n-1,k,2}$ (respectively $\mathcal{C}_{n-1,k-1,2}$) be a collection of Grasstopes which dissects the $m = 2$

amplituhedron $\mathcal{A}_{n-1,k,2}(Z')$ (resp. $\mathcal{A}_{n-1,k-1,2}(Z'')$). Then

$$\mathcal{C}_{n,k,2} = \iota_{\text{pre}}(\mathcal{C}_{n-1,k,2}) \cup \iota_{\text{inc}}(\mathcal{C}_{n-1,k-1,2})$$

dissects $\mathcal{A}_{n,k,2}(Z)$.

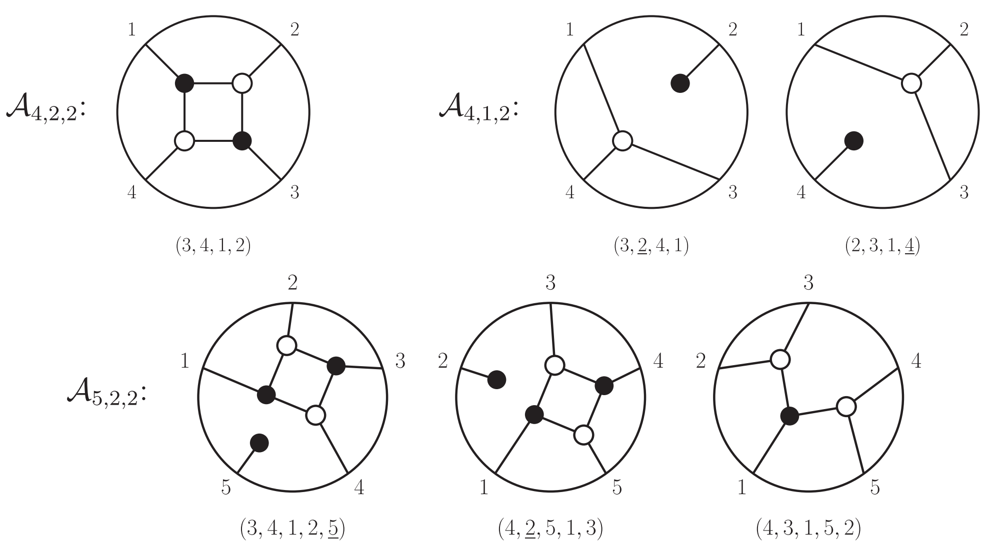
We use the term *BCFW dissection* (respectively, *BCFW tiling*) to refer to any dissection or tiling that has the form $\mathcal{C}_{k,n}$ from Theorem 4.12.

Diagrammatically, Theorem 4.12 reads as follows:

Remark 4.13. Because of the cyclic symmetry of the positive Grassmannian and the amplituhedron (see e.g. Theorem 7.5) there are $n-1$ other versions of Theorem 4.5 (and Figure 1) in which all plabic graph labels get shifted by i modulo n (for $1 \leq i \leq n-1$).

Note that [16] worked in the setting of *positroid tilings* – i.e. they were only considering collections of cells that map injectively from the positive Grassmannian to the amplituhedron – but Theorem 4.12 holds in the more general setting of dissections.

Example 4.14. Let $n = 5$ and $k = 2$. We will use Theorem 4.12 to obtain a dissection of $\mathcal{A}_{n,k,2}(Z) = \mathcal{A}_{5,2,2}$. We start with a dissection of $\mathcal{A}_{4,2,2}$ coming from the plabic graph shown below (corresponding to the decorated permutation $(3,4,1,2)$), and a dissection of $\mathcal{A}_{4,1,2}$ (corresponding to the permutations $(3,\underline{2},4,1)$ and $(2,3,1,\underline{4})$). Applying the theorem leads to the three plabic graphs in the bottom line, which correspond to the permutations $(3,4,1,2,\underline{5}), (4,\underline{2},5,1,3), (4,3,1,5,2)$.



5 The T-duality map

In this section we define the *T-duality map* (previously defined in [44, Definition 4.5]), from certain positroid cells of $Gr^{\geq 0}_{k+1,n}$ to positroid cells of $Gr^{\geq 0}_{k,n}$, and we prove many remarkable properties of it. We will subsequently explain, in Theorem 6.5, how the T-duality map gives a correspondence between tilings (and more generally dissections) of the hypersimplex $\Delta_{k+1,n}$ and the amplituhedron $\mathcal{A}_{n,k,2}(Z)$.

To get a preview of the phenomenon we will illustrate, compare the decorated permutations labelling the plabic graphs in Example 4.7 and Example 4.14; can you spot the correspondence? (This correspondence will be explained in Theorem 6.5.)

5.1 T-duality as a map on permutations

Definition 5.1. We define the *T-duality* map from loopless decorated permutations on $[n]$ to coloopless decorated permutations on $[n]$ as follows. Given a loopless decorated permutation $\pi = (a_1, a_2, \dots, a_n)$ (written in list notation) on $[n]$, we define the decorated permutation $\hat{\pi}$ by $\hat{\pi}(i) = \pi(i-1)$, so that $\hat{\pi} = (a_n, a_1, a_2, \dots, a_{n-1})$, where any fixed points in $\hat{\pi}$ are declared to be loops. Equivalently, $\hat{\pi}$ is obtained from π by composing π with the permutation $\pi_0 = (n, 1, 2, \dots, n-1)$ in the symmetric group, $\hat{\pi} = \pi_0 \circ \pi$.

Recall that an anti-excedance of a decorated permutation is a position i such that $\pi(i) < i$, or $\pi(i) = i$ and i is a coloop. Our first result shows that T-duality is a bijection between loopless cells of $Gr^{\geq 0}_{k+1,n}$ and coloopless cells of $Gr^{\geq 0}_{k,n}$.

Lemma 5.2. The T-duality map $\pi \mapsto \hat{\pi}$ is a bijection between the loopless permutations on $[n]$ with $k+1$ anti-excedances, and the coloopless permutations on $[n]$ with k anti-excedances. Equivalently, the T-duality map is a bijection between loopless positroid cells of $Gr^{\geq 0}_{k+1,n}$ and coloopless positroid cells of $Gr^{\geq 0}_{k,n}$.

Proof. The second statement follows from the first by Section 12, so it suffices to prove the first statement. Let $\pi = (a_1, \dots, a_n)$ be a loopless permutation on $[n]$ with $k+1$ anti-excedances; then $\hat{\pi} = (a_n, a_1, \dots, a_{n-1})$. Consider any i such that $1 \leq i \leq n-1$. Suppose i is a position of a anti-excedance, i.e. either $a_i < i$ or $a_i = \bar{i}$. Then the letter a_i appears in the $(i+1)$st position in $\hat{\pi}$, and since $a_i < i+1$, we again have an anti-excedance. On the other hand, if i is *not* a position of an anti-excedance, i.e. $a_i > i$ (recall that π is loopless), then in the $(i+1)$st position of $\hat{\pi}$ we have $a_i \geq i+1$. By Definition 5.1 if we have a fixed point in position $i+1$ (i.e. $a_i = i+1$) this is a loop, and so position $i+1$ of $\hat{\pi}$ will not be a anti-excedance. Therefore if $I \subset [n-1]$ is the positions of the anti-excedances located in the first $n-1$ positions of π, then $I+1$ is the positions of the anti-excedances located in positions $\{2,3,\ldots,n\}$ in $\hat{\pi}$.

Now consider position n of π. Because π is loopless, n will be the position of a anti-excedance in π. And because $\hat{\pi}$ is defined to be coloopless, 1 will never be the position of a anti-excedance in $\hat{\pi}$. Therefore the number of anti-excedances of $\hat{\pi}$ will be precisely one less than the number of anti-excedances of π.

It is easy to reverse this map so it is a bijection. ■

Remark 5.3. Since by Lemma 5.2 the map $\pi \mapsto \hat{\pi}$ is a bijection, we can also talk about the inverse map from coloopless permutations on $[n]$ with k anti-excedances to loopless permutations on $[n]$ with $k+1$ anti-excedances. We denote this inverse map by $\pi \mapsto \check{\pi}$.

Remark 5.4. Our map $\pi \mapsto \hat{\pi}$ is in fact a special case of the map ρ_A introduced by Benedetti-Chavez-Tamayo in [14, Definition 23] (in the case where $A = \emptyset$).

5.2 T-duality as a map on cells

While we have defined the T-duality map as a map $\pi \mapsto \hat{\pi}$ on the permutations labelling positroid cells, it can be shown that it is induced from a map on the corresponding cells. We will follow here the derivation in [1] and define a Q-map which maps elements of the positroid cell S_π of $Gr^{\geq 0}_{k+1,n}$ to the positroid cell $S_{\hat{\pi}}$ of $Gr^{\geq 0}_{k,n}$. Note that in much of this section we allow m to be any positive even integer.

Definition 5.5. Let $\lambda \in Gr_{\frac{m}{2},n}$. We say that λ is *generic* if $p_I(\lambda) \neq 0$ for all $I \in \binom{[n]}{\frac{m}{2}}$.

For $m=2$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$ is generic in $\mathbb{R}^n$ if $\lambda_i \neq 0$ for all $i = 1, \ldots, n$.

Lemma 5.6. Given $C = (c_1, c_2, \ldots, c_n)$ representing an element of $Gr_{k+\frac{m}{2},n}$ where c_i are columns of C, then C contains a generic $\frac{m}{2}$-plane if and only if $\operatorname{rank}\left(\{c_i\}_{i\in I}\right) = \frac{m}{2}$ for all $I \in \binom{[n]}{\frac{m}{2}}$.

Proof. If a generic $\frac{m}{2}$-plane $\lambda \in M(\frac{m}{2}, n)$ is contained in C, then there is a matrix $h \in M(\frac{m}{2}, k+\frac{m}{2})$ such that $\lambda = h \cdot C$. Then $p_I(\lambda) = \sum_{J \in \binom{[k+\frac{m}{2}]}{\frac{m}{2}}} p_J(h) C^I_J$, with $I \in \binom{[n]}{\frac{m}{2}}$. If $\operatorname{rank}\left(\{c_i\}_{i\in I}\right) = \frac{m}{2}$ then there exist $J_I \in \binom{[k+\frac{m}{2}]}{\frac{m}{2}}$ such that $C^I_{J_I} \neq 0$, therefore it is enough to choose h such that $p_{J_I}(h) \neq 0$ in order to guarantee $\lambda = h \cdot C$ is generic. Vice-versa if we assume $\operatorname{rank}\left(\{c_i\}_{i\in I}\right) < \frac{m}{2}$ then $C^I_J = 0$ for all $J \in \binom{[k+\frac{m}{2}]}{\frac{m}{2}}$ and this would imply $p_I(\lambda) = 0$. ■

If we specialize to the $m = 2$ case, we have the following:

Lemma 5.7. Let S_π be a positroid cell in $Gr^{\geq 0}_{k+1,n}$. Then S_π is loopless if and only if every vector space $V \in S_\pi$ contains a generic vector.

Lemma 5.8. Let S_π be a positroid cell. If every vector space $V \in S_\pi$ contains a generic $\frac{m}{2}$-plane then $\pi(i) \geq i + \frac{m}{2}$ (as an affine permutation, see Definition 12.3) for all i.

Proof. Let $C = (c_1, c_2, \ldots, c_n)$ be a matrix representing V, listed as a sequence of column vectors. Let us assume that there exists a such that $\pi(a) \leq a + \frac{m}{2} - 1$. Then $c_a \in \operatorname{span}\{c_{a+1}, \ldots, c_{a+\frac{m}{2}-1}\}$ and, in particular, $r[a; a + \frac{m}{2} - 1] < \frac{m}{2}$. The proof follows immediately from Lemma 5.6. ■

Definition 5.9. For a positroid cell $S_\pi \subset Gr^{\geq 0}_{k+\frac{m}{2},n}$ and $\lambda \in Gr_{\frac{m}{2},n}$ a generic vector of an element $V \in S_\pi$, we define

$$S_\pi^{(\lambda)} := \{W \in S_\pi : \lambda \subset W\}.$$

Let $C_\pi^{(\lambda)}$ be matrix representatives for elements in $S_\pi^{(\lambda)}$. It is always possible to find an invertible row transformation which bring $C_\pi^{(\lambda)}$ into the form

$$C_\pi^{(\lambda)} = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{\frac{m}{2},1} & \lambda_{\frac{m}{2},2} & \cdots & \lambda_{\frac{m}{2},n} \\ c_{\frac{m}{2}+1,1} & c_{\frac{m}{2}+1,2} & \cdots & c_{\frac{m}{2}+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\frac{m}{2}+k,1} & c_{\frac{m}{2}+k,2} & \cdots & c_{\frac{m}{2}+k,n} \end{pmatrix} \tag{5.1}$$

Let us define a linear transformation $Q^{(\lambda)} : \mathbb{R}^n \mapsto \mathbb{R}^n$ represented by the $n \times n$ matrix $Q^{(\lambda)}$ with elements[4]

$$Q^{(\lambda)}_{ab} = \sum_{i=0}^{\frac{m}{2}} (-1)^i \, \delta_{a,b-\frac{m}{2}+i} \, p_{b-\frac{m}{2},\ldots,b-\frac{m}{2}+i-1,\frac{m}{2}+i+1,\ldots,b}(\lambda), \quad a, b, \in [n]. \tag{5.2}$$

Here we used the notation where $\delta_{ab} = 1$ when $a = b$ and $\delta_{ab} = 0$ otherwise.

4 Notice that our definition differs from the one found in [1] for $m = 4$. They are however related to each other by a cyclic shift and rescaling each column of $Q^{(\lambda)}$.

It is easy to show that $\lambda Q^{(\lambda)} = 0$ and that $Q^{(\lambda)}$ has rank $n - \frac{m}{2}$. Let us define $\hat{C}_\pi^{(\lambda)} = C_\pi^{(\lambda)} \cdot Q^{(\lambda)}$, then

$$\hat{C}_\pi^{(\lambda)} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ \hat{c}_{\frac{m}{2}+1,1} & \hat{c}_{\frac{m}{2}+1,2} & \dots & \hat{c}_{\frac{m}{2}+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{c}_{\frac{m}{2}+k,1} & \hat{c}_{\frac{m}{2}+k,2} & \dots & \hat{c}_{\frac{m}{2}+k,n} \end{pmatrix}. \tag{5.3}$$

It is easy to check that $\mathrm{span}\{\hat{c}_a, \hat{c}_{a+1}, \dots, \hat{c}_b\} \subset \mathrm{span}\{c_{a-\frac{m}{2}}, c_{a-\frac{m}{2}+1}, \dots, c_b\}$ and moreover that for consecutive maximal minors we have: $p_{a-\frac{m}{2},\dots a,\dots a+k-1}(C)$ is proportional to $p_{a,\dots,a+k-1}(\hat{C})$. Then, the matrix $Q^{(\lambda)}$ projects elements of $S_\pi^{(\lambda)}$ into $S_{\hat{\pi}}$, with

$$\hat{\pi}(i) = \pi(i - \frac{m}{2}). \tag{5.4}$$

The proof of this fact closely follows the one found in [1, page 75].

For $m = 2$ we get the explicit form of $Q^{(\lambda)}$ is:

$$Q_{ab}^{(\lambda)} = \delta_{a,b-1}\lambda_b - \delta_{a,b}\lambda_{b-1}\,, \quad a, b \in [n]. \tag{5.5}$$

Moreover, we have the following relation between consecutive minors

$$p_{a,a+1,\dots,a+k-1}(\hat{C}) = (-1)^k \lambda_a \dots \lambda_{a+k-2}\, p_{a-1,a,\dots,a+k-1}(C). \tag{5.6}$$

Remark 5.10. In order for the T-duality map to be a well-defined (on affine permutations), we require that both $i \le \pi(i) \le n + i$ and $i \le \hat{\pi}(i) \le n + i$ are satisfied. Given that $\hat{\pi}(i) = \pi(i - \frac{m}{2})$, this implies extra conditions on allowed permutations, i.e. $\pi(i) \ge i + \frac{m}{2}$ and $\hat{\pi}(i) \le i + n - \frac{m}{2}$. We observe that the operation in (5.4) is then well-defined for the cells S_π^λ, by Lemma 5.8. Finally, for $m = 2$ these conditions correspond to lack of loops (resp. coloops) for π (reps. $\hat{\pi}$).

Proposition 5.11 (How T-duality affects dimensions of cells). Let S_π be a loopless cell of $Gr_{k+1,n}^{\ge 0}$. Then $S_{\hat{\pi}}$ is a coloopless cell of $Gr_{k,n}^{\ge 0}$, and $\dim(S_{\hat{\pi}}) - 2k = \dim(S_\pi) - (n-1)$. In particular, if $\dim S_\pi = n - 1$, then $\dim S_{\hat{\pi}} = 2k$.

Proof. Let us translate Definition 5.1 into the language of affine permutations. Then T-duality maps a $(k+1, n)$-bounded affine permutation π_a into a (k, n)-bounded affine permutation $\hat{\pi}_a = \pi_a \circ t$, with $t : \mathbb{Z} \to \mathbb{Z}$ the map $i \mapsto i - 1$. By [54, Proposition 17.10] and Section 12, the codimension of the positroid cell S_{ν_a} equals the length $\ell(\nu_a)$ of the associated affine permutation ν_a. Clearly the map t preserves the set of inversions, and hence the length, of affine bounded permutations, i.e. $\ell(\pi_a) = \ell(\hat{\pi}_a)$. Therefore the codimensions of $S_{\pi_a} \subseteq Gr^{\geq 0}_{k+1,n}$ and $S_{\hat{\pi}_a} \subseteq Gr^{\geq 0}_{k,n}$ are equal:

$$(k+1)(n-k-1) - \dim(S_{\pi_a}) = k(n-k) - \dim(S_{\hat{\pi}_a}), \tag{5.7}$$

from which the claim of the proposition follows immediately. ■

Remark 5.12. Alternatively, one may prove the above result by mimicking an argument of a similar statement given in [1, pages 75-76].

6 T-duality relates tiles, tilings, and dissections

In this section we will compare the positroid tiles and tilings (and more generally, dissections) of the hypersimplex $\Delta_{k+1,n}$ with those of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$. Again, we will see that T-duality connects them! Our main result of this section is Theorem 6.5, which says that T-duality provides a bijection between the BCFW tilings/dissections of the hypersimplex $\Delta_{k+1,n}$, and the BCFW tilings/dissections of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$.

The $2k$-dimensional cells of $Gr^{\geq 0}_{k,n}$ which have full-dimensional image in $\mathcal{A}_{n,k,2}(Z)$ were studied in [46] and called *generalized triangles*. In this paper we will refer to the above objects as *positroid tiles* defined as follows.

Definition 6.1 (Positroid tiles of $\mathcal{A}_{n,k,2}$). Let S_π be a $2k$-dimensional cell of $Gr^{\geq 0}_{k,n}$ such that $\dim Z_\pi = \dim S_\pi$, and the restriction of the amplituhedron map $\widetilde{Z}$ to S_π is an injection. Then we call Z_π a *positroid tile* of $\mathcal{A}_{n,k,2}(Z)$.

A conjectural description of positroid tiles was given in [46]:

Definition 6.2. We say that a collection of convex polygons (which have $p_1, \ldots, p_r$ vertices) inscribed in a given n-gon is a collection of k *non-intersecting triangles in an n-gon* if each pair of such polygons intersects in at most a vertex and if the total number of triangles needed to triangulate all polygons in the collection is k, i.e. $(p_1 - 2) + \ldots + (p_r - 2) = k$.

It was conjectured and experimentally checked in [46] that positroid tiles in $\mathcal{A}_{n,k,2}(Z)$ are in bijection with collections of 'k non-intersecting triangles in a n-gon'. Moreover, one can read off the cell S_π of $Gr^{\geq 0}_{k,n}$ corresponding to a positroid tile of $\mathcal{A}_{n,k,2}(Z)$ using the combinatorics of the collection of k non-intersecting triangles in an n-gon, see [46, Section 2.4]. The basic idea is to associate a row vector to each of the non-intersecting triangles, with generic entries at the positions of the triangle vertices (and zeros everywhere else). This way one constructs a $k \times n$ matrix whose matroid is the matroid for S_π.

Borrowing the terminology of Definition 6.1, we make the following definition.

Definition 6.3 (Positroid tiles of $\Delta_{k+1,n}$). Let S_π be an $(n-1)$-dimensional cell of $Gr^{\geq 0}_{k+1,n}$ such that the moment map μ is an injection on S_π. Then we say the image $\Gamma_\pi := \overline{\mu(S_\pi)}$ in $\Delta_{k+1,n}$ is a *positroid tile* in $\Delta_{k+1,n}$.

We have already studied the positroid tiles in $\Delta_{k+1,n}$ in Proposition 3.16: they come from $(n-1)$-dimensional positroid cells whose matroid is connected, or equivalently, they come from the positroid cells whose reduced plabic graphs are trees. And since these are positroid cells in $Gr^{\geq 0}_{k+1,n}$, each such plabic graph, when drawn as a trivalent graph, is a tree with n leaves with precisely k internal black vertices. By simply taking the planar dual of these tree, we get the following:

Proposition 6.4. There is a bijective map between positroid tiles in $\Delta_{k+1,n}$ and collections of k non-intersecting triangles in an n-gon.

Proof. Consider a collection of non-intersecting polygons inside an n-gon $\mathcal{P} = (P_1, \ldots, P_r)$ and its complement $\overline{\mathcal{P}} = (\overline{P}_1, \ldots \overline{P}_{\bar{r}})$. Let us choose a triangulation of all polygons into triangles $\mathcal{P} \to \mathcal{T} = (T_1, \ldots, T_k)$ and $\overline{\mathcal{P}} \to \overline{\mathcal{T}} = (\overline{T}_1, \ldots, \overline{T}_{n-k-2})$. Associate a black vertex to the middle of each triangle T and a white vertex with to middle of each triangle $\overline{T}$. Finally, connect each pair of vertices corresponding to triangles sharing an edge and draw an edge through each boundary of the n-gon. This way we get a tree graph with exactly k black and $n-k-2$ white vertices. Hence it is a plabic graph for the cell $S_\pi \subset G^{\geq 0}_{k+1,n}$ corresponding to a plabic tile of $\Delta_{k+1,n}$. ■

In the following theorem we show that T-duality relates BCFW tilings and dissections of the hypersimplex and amplituhedron.

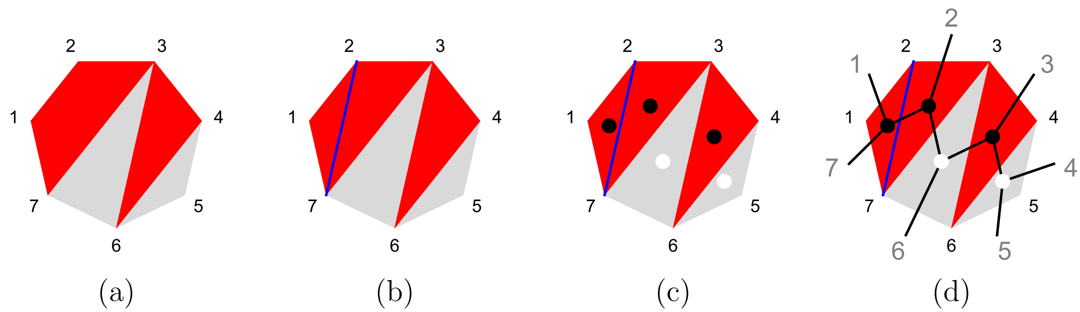


Fig. 4. The map in Proposition 6.4 for $\pi = \{4, 7, 1, 6, \underline{5}, 3, 2\} \in Gr^{\geq 0}_{3,7}$: (a) positroid tile label, (b) A triangulation of collections $\mathcal{P}$ and $\bar{\mathcal{P}}$, (c) Assigning vertices, (d) Plabic graph of $\breve{\pi} = \{7, 1, 6, 5, 3, 2, 4\} \in Gr^{\geq 0}_{4,7}$

Theorem 6.5 (BCFW tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ are T-dual). The T-duality map provides a bijection between the BCFW tilings of the hypersimplex $\Delta_{k+1,n}$ and the BCFW tilings of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$. That is, the collection $\{\Gamma_\pi\}$ of positroid polytopes constructed in Theorem 4.12 is a positroid tiling of $\Delta_{k+1,n}$ if and only if the T-dual collection $\{Z_{\hat{\pi}}\}$ of Grasstopes is a positroid tiling of $\mathcal{A}_{n,k,2}(Z)$. The same statement holds if we replace the word "tiling" with "dissection".

Proof. We prove this by induction on $k + n$, using Theorem 4.5 and Theorem 4.12. It suffices to show:

- if $\{\Gamma_\pi\}_{\pi\in\mathcal{C}}$ dissects $\Delta_{k+1,n-1}$ and $\{Z'_{\hat{\pi}}\}_{\pi\in\hat{\mathcal{C}}}$ dissects $\mathcal{A}_{n-1,k,2}(Z')$ then for any $\pi \in \mathcal{C}$, $\widehat{\mathrm{i}_{\mathrm{pre}}(\pi)} = \iota_{\mathrm{pre}}(\hat{\pi})$.
- if $\{\Gamma_\pi\}_{\pi\in\mathcal{C}}$ dissects $\Delta_{k,n-1}$ and $\{Z''_{\hat{\pi}}\}_{\pi\in\hat{\mathcal{C}}}$ dissects $\mathcal{A}_{n-1,k-1,2}(Z'')$ then for any $\pi \in \mathcal{C}$, $\widehat{\mathrm{i}_{\mathrm{inc}}(\pi)} = \iota_{\mathrm{inc}}(\hat{\pi})$.

Let $\pi = (a_1, \ldots, a_{n-1})$ be a decorated permutation. We first verify the first statement. Then $\mathrm{i}_{\mathrm{pre}}(\pi) = (a_1, a_2, \ldots, a_{n-2}, n, a_{n-1})$, so $\widehat{\mathrm{i}_{\mathrm{pre}}(\pi)} = (a_{n-1}, a_1, a_2, \ldots, a_{n-2}, n)$, where n is a black fixed point. Meanwhile, $\hat{\pi} = (a_{n-1}, a_1, a_2, \ldots, a_{n-2})$, so $\iota_{\mathrm{pre}}(\hat{\pi}) = (a_{n-1}, a_1, a_2, \ldots, a_{n-2}, n)$, where n is a black fixed point.

We now verify the second statement. Let $j = \pi^{-1}(n-1)$. Then we have that $\mathrm{i}_{\mathrm{inc}}(\pi) = (a_1, a_2, \ldots, a_{j-1}, n, a_{j+1}, \ldots, a_{n-1}, n-1)$, and $\widehat{\mathrm{i}_{\mathrm{inc}}(\pi)} = (n-1, a_1, a_2, \ldots, a_{j-1}, n, a_{j+1}, \ldots, a_{n-1})$. Meanwhile $\hat{\pi} = (a_{n-1}, a_1, a_2, \ldots, a_{n-2})$. Then it is straightforward to verify that $\iota_{\mathrm{inc}}(\hat{\pi})$ is exactly the permutation $\widehat{\mathrm{i}_{\mathrm{inc}}(\pi)} = (n-1, a_1, a_2, \ldots, a_{j-1}, n, a_{j+1}, \ldots, a_{n-1})$, as desired. ■

We now see that T-duality relates positroid tiles of the hypersimplex and the amplituhedron.

Proposition 6.6. Suppose the positroid polytope Γ_π is a positroid tile of the hypersimplex $\Delta_{k+1,n}$. Then the T-dual Grasstope $Z_{\hat{\pi}}$ is a positroid tile of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ for all $Z \in \mathrm{Mat}_{n,k+2}^{>0}$.

Proof. By Proposition 3.16, the fact that μ is injective implies that a (any) reduced plabic graph G representing S_π must be a (planar) tree. But then by Theorem 4.5 (see Figure 1), G has a black or white vertex which is incident to two adjacent boundary vertices i and $i+1$ (modulo n), and hence appears in some tiling of the hypersimplex (and specifically on the right-hand side of Figure 1).

Applying Theorem 6.5, we see that $\hat{\pi}$ appears in some tiling of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$. It follows that $\widetilde{Z}$ is injective on $S_{\hat{\pi}}$. ■

By Proposition 6.6 and Proposition 6.4, collections of k non-intersecting triangles in an n-gon label both positroid tiles of $\Delta_{k+1,n}$ and, via T-duality, positroid tiles of $\mathcal{A}_{n,k,2}(Z)$. We conjecture that this labelling is compatible with the way [46] associates collections of k non-intersecting triangles in an n-gon with positroid tiles of $\mathcal{A}_{n,k,2}(Z)$.

Using Proposition 6.6, Proposition 3.13 and Proposition 3.15, we obtain the following.

Corollary 6.7. The $\widetilde{Z}$-map is an injection on all $2k$-dimensional cells of the form $S_{\hat{\pi}} \subset Gr_{k,n}^{\geq 0}$, where π is a SIF permutation and $\dim S_\pi = n-1$.

We know from Proposition 3.15 that the moment map is an injection on the cell S_π of $Gr_{k,n}^{\geq 0}$ precisely when $\dim S_\pi = n-c$, where c is the number of connected components of the positroid of π. We have experimentally checked the following statement for these cells.

Conjecture 6.8. Let S_π be a loopless $(n-c)$-dimensional cell of $Gr_{k+1,n}^{\geq 0}$ with c connected components (for c a positive integer). Then $S_{\hat{\pi}}$ is a coloopless $(2k+1-c)$-dimensional cell of $Gr_{k,n}^{\geq 0}$ on which $\widetilde{Z}$ is injective.

Note that the statement that $S_{\hat{\pi}}$ is coloopless of dimension $(2k+1-c)$ follows from Lemma 5.2 and Proposition 5.11. Moreover the $c=1$ case of the conjecture is Proposition 6.6.

While Theorem 6.5 shows that T-duality relates the large class of BCFW tilings/dissections of $\Delta_{k+1,n}$ to the corresponding large class of BCFW tilings/dissections of $\mathcal{A}_{n,k,2}(Z)$, not all tilings/dissections arise from a BCFW-style recursion. Nevertheless, we conjecture the following.

Conjecture 6.9 (Tilings and dissections of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ are T-dual). A collection of positroid polytopes $\{\Gamma_\pi\}$ is a tiling (respectively, dissection) of $\Delta_{k+1,n}$ if and only if for all $Z \in \mathrm{Mat}^{>0}_{n,k+2}$ the collection of T-dual Grasstopes $\{Z_{\hat{\pi}}\}$ is a tiling (respectively, dissection) of $\mathcal{A}_{n,k,2}(Z)$.

This conjecture is supported by Theorem 6.5, Proposition 10.4 and results of Section 7 (which relates parity duality and T-duality), and will be explored in a subsequent work[5]. We have also checked the conjecture using Mathematica, see Section 10.

7 T-duality, cyclic symmetry and parity duality

In this section we discuss the relation of T-duality to *parity duality*, which relates dissections of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ with dissections of $\mathcal{A}_{n,n-m-k,m}(Z')$. The definition of parity duality was originally inspired by the physical operation of parity conjugation in quantum field theory – more specifically, in the context of scattering amplitudes in $\mathcal{N} = 4$ Super-Yang-Mills, where amplitudes can be computed from the geometry of $\mathcal{A}_{n,k,4}(Z)$ [6]. Furthermore, the conjectural formula of Karp, Williams, and Zhang [44] for the number of cells in each tiling of the amplituhedron is invariant under the operation of swapping the parameters k and $n - m - k$ and hence is consistent with parity duality: this motivated further works, see [28, Section 2.4] and [33]. In particular, [33] gave an explicit bijection between dissections of $\mathcal{A}_{n,k,m}(Z)$ and dissections of $\mathcal{A}_{n,n-m-k,m}(Z')$, see Theorem 7.7.

In Theorem 7.3, we will explain how parity duality for $m = 2$ amplituhedra is naturally induced by a composition of the usual duality for Grassmannians ($Gr_{k,n} \simeq Gr_{n-k,n}$) and the T-duality map (between loopless cells of $Gr^{\geq 0}_{k+1,n}$ and coloopless cells of $Gr^{\geq 0}_{k,n}$). The usual Grassmannian duality gives rise to a bijection between dissections of the hypersimplex $\Delta_{k+1,n}$ and dissections of the hypersimplex $\Delta_{n-k-1,n}$. By composing this Grassmannian duality with the T-duality map (on both sides), we obtain the parity duality between dissections of $\mathcal{A}_{n,k,2}(Z)$ and $\mathcal{A}_{n,n-k-2,2}(Z')$!

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5 Since our paper appeared on arXiv, Conjecture 6.9 has been proved for tilings in [55].

Recall that our convention on dissections is that the images of all positroid cells are of full dimension $n-1$. Therefore all positroids involved in a dissection must be connected, and the corresponding decorated permutations will be fixed-point-free.

Theorem 7.1 (Grassmannian duality for dissections of the hypersimplex). Let $\{\Gamma_\pi\}$ be a collection of positroid polytopes which dissects the hypersimplex $\Delta_{k+1,n}$. Then the collection of positroid polytopes $\{\Gamma_{\pi^{-1}}\}$ dissects the hypersimplex $\Delta_{n-k-1,n}$.

Proof. If G is a plabic graph representing the positroid cell S_π, and if we swap the colors of the black and white vertices of G, we obtain a graph G' representing the positroid $S_{\pi^{-1}}$. It is not hard to see from [9] that G' and π^{-1} represent the *dual positroid* to G and π. But now the matroid polytopes Γ_π and $\Gamma_{\pi^{-1}}$ are isomorphic via the map dual : $\mathbb{R}^n \to \mathbb{R}^n$ sending $(x_1,\dots,x_n) \mapsto (1-x_1,\dots,1-x_n)$. This maps relates the two dissections in the statement of the theorem. ■

By composing the inverse map on decorated permutations $\pi \mapsto \pi^{-1}$ (which represents the Grassmannian duality of Theorem 7.1) with T-duality, we obtain the following map.

Definition 7.2. We define $\widetilde{U_{k,n}}$ to be the map between coloopless permutations on $[n]$ with k anti-excedances and coloopless permutations on $[n]$ with $n-k-2$ anti-excedances such that $\widetilde{U_{k,n}}\hat{\pi} = \widehat{\pi^{-1}}$. Equivalently, we have $(\widetilde{U_{k,n}}\pi)(i) = \pi^{-1}(i-1)-1$, where values of the permutation are considered modulo n, and any fixed points which are created are designated to be loops.

Theorem 7.3 (Parity duality from T-duality and Grassmannian duality). Let $\{Z_\pi\}$ be a collection of Grasstopes which dissects the amplituhedron $\mathcal{A}_{n,k,2}(Z)$. Then the collection of Grasstopes $\{Z_{\widetilde{U_{k,n}}\pi}\}$ dissects the amplituhedron $\mathcal{A}_{n,n-k-2,2}(Z')$.

We will prove Theorem 7.3 by using the cyclic symmetry of the positive Grassmannian and the amplituhedron, and showing (see Lemma 7.8) that up to a cyclic shift, our map $\widetilde{U_{k,n}}$ agrees with the parity duality map of [33].

The totally nonnegative Grassmannian exhibits a beautiful *cyclic symmetry* [54]. Let us represent an element of $Gr_{k,n}^{\geq 0}$ by a $k\times n$ matrix, encoded by the sequence of n columns $\langle v_1,\dots,v_n\rangle$. We define the *(left) cyclic shift map* σ to be the map which sends $\langle v_1,\dots,v_n\rangle$ to the point $\langle v_2,\dots,v_n,(-1)^{k-1}v_1\rangle$, which one can easily verify lies in $Gr_{k,n}^{\geq 0}$. Since the cyclic shift maps positroid cells to positroid cells, for π a decorated permutation, we define $\sigma\pi$ to be the decorated permutation such that $S_{\sigma\pi} = \sigma(S_\pi)$. It is easy to see that $\sigma\pi(i) = \pi(i+1) - 1$. (Note that under the cyclic shift, a fixed point of π at position $i+1$ gets sent to a fixed point of $\sigma\pi$ at position i; we color fixed points accordingly.) Meanwhile the inverse operation, the *right cyclic shift* σ^{-1} satisfies $(\sigma^{-1}\pi)(i) = \pi(i-1) + 1$. We use σ^t (respectively, σ^{-t}) to denote the repeated application of σ (resp. σ^{-1}) t times, so that $(\sigma^t\pi)(i) := \pi(i+t) - t$ and $(\sigma^{-t}\pi)(i) := \pi(i-t) + t$.

The next result follows easily from the definitions.

Theorem 7.4 (Cyclic symmetry for dissections of the hypersimplex). Let $\{\Gamma_\pi\}$ be a collection of positroid polytopes which dissects the hypersimplex $\Delta_{k+1,n}$. Then the collection of positroid polytopes $\{\Gamma_{\sigma\pi}\}$ dissects $\Delta_{k+1,n}$.

Proof. Let $\sigma_{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $(x_1, \dots, x_n) \mapsto (x_2, \dots, x_n, x_1)$. Clearly $\sigma_{\mathbb{R}}$ is an isomorphism mapping the hypersimplex $\Delta_{k+1,n}$ back to itself. Moreover, applying the cyclic shift σ to a positroid has the effect of simply shifting all its bases, so the matroid polytope of $\sigma\pi$ satisfies $\Gamma_{\sigma\pi} = \sigma_{\mathbb{R}}(\Gamma_\pi)$. The result now follows. ∎

The above cyclic symmetry for dissections of the hypersimplex also has an analogue for the amplituhedron.

Theorem 7.5 (Cyclic symmetry for dissections of the amplituhedron). [16, Corollary 3.2] Let $\{\Gamma_\pi\}$ be a collection of Grasstopes which dissects the amplituhedron $\mathcal{A}_{n,k,m}(Z)$, with m even. Then the collection of Grasstopes $\{Z_{\sigma\pi}\}$ also dissects $\mathcal{A}_{n,k,m}(Z)$.

In order to make contact with [33], we introduce a map $U_{k,n}$ on (coloopless) decorated permutations as follows.

Definition 7.6. We define $U_{k,n}$ to be the map from coloopless permutations on $[n]$ with k anti-excedances to coloopless permutations on $[n]$ with $n-k-2$ anti-excedances such that $(U_{k,n}\pi)(i) = \pi^{-1}(i+k) + (n-k-2)$, where values of the permutation are considered modulo n, and any fixed points which are created are designated to be loops.

It is not hard to see that this map is equivalent to the parity duality from [33] for $m = 2$. In particular we have the following theorem:

Theorem 7.7. [33, Theorem 7.2] Let $\{Z_\pi\}$ be a collection of Grasstopes which dissects the amplituhedron $\mathcal{A}_{n,k,2}(Z)$. Then the collection of Grasstopes $\{Z_{U_{k,n}\pi}\}$ dissects the amplituhedron $\mathcal{A}_{n,n-k-2,2}(Z')$.

Lemma 7.8. For fixed n and k, the maps $\widetilde{U_{k,n}}$ and $U_{k,n}$ are related by the cyclic shift map

$$\widetilde{U_{k,n}} = \sigma^{-(k+1)} \circ U_{k,n}. \tag{7.1}$$

Proof. Since $(U_{k,n}\pi)(i) = \pi^{-1}(i+k) + (n-k-2)$, we have that $(\sigma^{-(k+1)} \circ U_{k,n}\pi)(i) = \pi^{-1}((i+k)-(k+1)) + (n-k-2) + (k+1) = \pi^{-1}(i-1) + n - 1$, which is exactly $\widetilde{U_{k,n}}$ (mod n). ■

We now prove Theorem 7.3.

Proof. This result follows immediately from Theorem 7.5, Theorem 7.7, and Lemma 7.8. ■

Remark 7.9. From Theorem 7.4 and Theorem 7.5 it is clear that if we redefine the T-duality map in Definition 5.1 by composing it with any cyclic shift σ^a (for a an integer), the main properties of the map will be preserved. In particular, any statement about dissections of the hypersimplex versus the corresponding ones of the amplituhedron will continue to hold, along with the parity duality.

Remark 7.10. Parity duality has a nice graphical interpretation when we represent positroid tiles of $\mathcal{A}_{n,k,2}(Z)$ as collection of k non-intersecting triangles in an n-gon. The Grassmannian duality of $Gr^{\geq 0}_{k+1,n}$ amounts to swapping black and white vertices in the plabic graphs, and when we compose it with the T-duality map, by Proposition 6.4, results in taking the complementary polygons inside the n-gon. We end up with a collection of $n-k-2$ non-intersecting triangles in the n-gon.

8 Good and bad dissections of the hypersimplex and the amplituhedron

Among all possible positroid dissections, there are some with particularly nice features, which we will call "good", as well as others with rather unpleasant properties. We show below examples of both a good and a bad dissection.

Example 8.1. Let us study the following tiling of $\mathcal{A}_{6,2,2}(Z)$:

$$\mathcal{C}_1 = \left\{Z_{\pi^{(1)}}, Z_{\pi^{(2)}}, Z_{\pi^{(3)}}, Z_{\pi^{(4)}}, Z_{\pi^{(5)}}, Z_{\pi^{(6)}}\right\}$$

with

$$\pi^{(1)} = (\underline{1}, \underline{2}, 5, 6, 3, 4), \quad \pi^{(2)} = (\underline{1}, 3, 6, 5, 2, 4), \quad \pi^{(3)} = (\underline{1}, 4, 6, 2, \underline{5}, 3)\,,$$
$$\pi^{(4)} = (2, 6, \underline{3}, 5, 1, 4), \quad \pi^{(5)} = (2, 6, 4, 1, \underline{5}, 3), \quad \pi^{(6)} = (3, 6, 1, 4, \underline{5}, 2)\,.$$

All elements of $\mathcal{C}_1$ are 4-dimensional positroid tiles. The tiling $\mathcal{C}_1$ is a refinement of the following dissection

$$\mathcal{C}_2 = \left\{Z_{\pi^{(1)}}, Z_{\pi^{(7)}}, Z_{\pi^{(8)}}, Z_{\pi^{(6)}}\right\}$$

with

$$\pi^{(7)} = (1, 4, 6, 5, 2, 3)\,, \qquad \pi^{(8)} = (2, 6, 4, 5, 1, 3)\,.$$

The dissection $\mathcal{C}_2$ has the property that if a pair of tiles intersect along a 3-dimensional surface then this surface is an image of another positroid cell in $Gr^{\geq 0}_{2,6}$:

$$Z_{\pi^{(1)}} \cap Z_{\pi^{(7)}} = Z_{(\underline{1},\underline{2},6,5,3,4)}$$
$$Z_{\pi^{(7)}} \cap Z_{\pi^{(8)}} = Z_{(\underline{1},6,4,5,2,3)}$$
$$Z_{\pi^{(8)}} \cap Z_{\pi^{(6)}} = Z_{(2,6,1,\underline{4},\underline{5},3)}$$

and all remaining pairs of tiles intersect along lower dimensional surfaces. We consider the dissection $\mathcal{C}_2$ "good" because all its elements have compatible codimension one boundaries. However, the dissection $\mathcal{C}_1$ does not have this property. Let us observe that

$$Z_{\pi^{(2)}} \cup Z_{\pi^{(3)}} = Z_{\pi^{(7)}}$$
$$Z_{\pi^{(4)}} \cup Z_{\pi^{(5)}} = Z_{\pi^{(8)}}$$

We expect that, after we subdivide $Z_{\pi^{(7)}}$ and $Z_{\pi^{(8)}}$, the boundary $Z_{(\underline{1},6,4,5,2,3)}$ which they share will also get subdivided. This however happens in two different ways and we do not get compatible codimension one faces for the dissection $\mathcal{C}_1$. It is a similar picture to the one we get when we consider polyhedral subdivisions of a double square pyramid: it is possible to subdivide it into two pieces along its equator, and then further subdivide each pyramid into two simplices. However, in order to get a polyhedral triangulation of the double square pyramid, we need to do it in a compatible way, along the same diagonal of the equatorial square.

Therefore, we prefer to work with dissections where the boundaries of the strata interact nicely. Toward this end, we introduce the following notion of *good dissection*.

Definition 8.2. Let $\mathcal{C} = \{\Gamma_{\pi^{(1)}}, \ldots, \Gamma_{\pi^{(\ell)}}\}$ be a dissection of $\Delta_{k+1,n}$. We say that $\mathcal{C}$ is a *good dissection* of $\Delta_{k+1,n}$ if the following condition is satisfied: for $i \neq j$, if $\Gamma_{\pi^{(i)}} \cap \Gamma_{\pi^{(j)}}$ has codimension one, then $\Gamma_{\pi^{(i)}} \cap \Gamma_{\pi^{(j)}}$ equals Γ_π, where Γ_π is a facet of both $\Gamma_{\pi^{(i)}}$ and $\Gamma_{\pi^{(j)}}$.

Note that the above condition is equivalent to requiring that $\mathcal{C}$ is a *polyhedral subdivision* of $\Delta_{k+1,n}$. To make the analogous notion for amplituhedron, we need to define facets.

Definition 8.3. Let $Z_\pi \subset \mathcal{A}_{n,k,m}(Z)$ be a Grasstope. We say that $Z_{\pi'}$ is a *facet* of Z_π if it is maximal by inclusion among the Grasstopes satisfying the following properties: the cell $S_{\pi'}$ is contained in $\overline{S_\pi}$; $Z_{\pi'}$ is contained in the boundary of Z_π; $Z_{\pi'}$ has codimension 1 in Z_π.

Definition 8.4. Let $\mathcal{C} = \{Z_{\pi^{(1)}}, \ldots, Z_{\pi^{(\ell)}}\}$ be a collection of Grasstopes of $\mathcal{A}_{n,k,2}(Z)$. We say that $\mathcal{C}$ is a *good dissection* of $\mathcal{A}$ if the following condition is satisfied: for $i \neq j$, if $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ has codimension one, then $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ equals Z_π, where Z_π is a facet of both $Z_{\pi^{(i)}}$ and $Z_{\pi^{(j)}}$.

In the following, we will conjecture that good dissections of the hypersimplex are in one-to-one correspondence with good dissections of the amplituhedron. Toward this goal, we start by providing a characterization of good intersections of positroid polytopes.

Proposition 8.5. Let $\Gamma_{\pi^{(1)}}$ and $\Gamma_{\pi^{(2)}}$ be two $(n-1)$-dimensional positroid polytopes whose intersection $\Gamma_{\pi^{(1)}} \cap \Gamma_{\pi^{(2)}}$ is a polytope of dimension $n-2$. Then $\Gamma_{\pi^{(1)}} \cap \Gamma_{\pi^{(2)}}$ is a positroid polytope of the form $\Gamma_{\pi^{(3)}}$, where $\pi^{(3)}$ is a loopless permutation.

Proof. By Theorem 3.7, $\Gamma_{\pi^{(1)}} \cap \Gamma_{\pi^{(2)}}$ is a positroid polytope and hence has the form $\Gamma_{\pi^{(3)}}$, for some decorated permutation $\pi^{(3)}$. (Using Proposition 3.4, the fact that $\dim(\Gamma_{\pi^{(3)}}) = n-2$ implies that the positroid associated to $\pi^{(3)}$ has precisely two connected components.)

Now we claim that the positroid associated to $\pi^{(3)}$ is loopless. In general there is an easy geometric way of recognizing when a matroid M is loopless from the polytope Γ_M: M is loopless if and only if Γ_M is not contained in any of the n facets of the hypersimplex of the type $x_i = 0$ for $1 \leq i \leq n$. Since $\Gamma_{\pi^{(3)}}$ arises as the codimension 1 intersection of two full-dimensional matroid polytopes contained in $\Delta_{k+1,n}$ it necessarily meets the interior of the hypersimplex and hence the matroid must be loopless. ∎

Remark 8.6. Recall that the T-duality map is well-defined on positroid cells whose matroid is connected, and more generally, loopless. Proposition 8.5 implies that if we consider two cells $S_{\pi^{(1)}}$ and $S_{\pi^{(2)}}$ of $Gr^{\geq 0}_{k+1,n}$ whose matroid is connected and whose moment map images (necessarily top-dimensional) intersect in a common facet, then that facet is the moment map image of a loopless cell $S_{\pi^{(3)}}$. Therefore we can apply the T-duality map to all three cells $S_{\pi^{(1)}}$, $S_{\pi^{(2)}}$, and $S_{\pi^{(3)}}$.

Conjecture 8.7. Let $S_{\pi^{(1)}}$ and $S_{\pi^{(2)}}$ be two positroid cells in $Gr^{\geq 0}_{k,n}$ corresponding to coloopless permutations $\pi^{(1)}$ and $\pi^{(2)}$. Let $\dim Z^{\circ}_{\pi^{(1)}} = \dim Z^{\circ}_{\pi^{(2)}} = 2k$ with $Z_{\pi^{(1)}} \cap Z_{\pi^{(2)}} = Z_{\pi^{(3)}}$, where $S_{\pi^{(3)}} \subset G^{\geq 0}_{k,n}$ is such that $\dim Z^{\circ}_{\pi^{(3)}} = 2k - 1$. Then $\pi^{(3)}$ is a coloopless permutation.

Remark 8.8. Conjecture 8.7 guarantees that if we consider two positroid cells with top-dimensional images in the amplituhedron $\mathcal{A}_{n,k,2}(Z)$, which have a facet in common, then the positroid cell corresponding to this facet is coloopless and therefore we can apply the T-duality map to it.

Finally we arrive at a conjecture connecting good dissections of hypersimplex and amplituhedron, which we confirmed experimentally.

Conjecture 8.9. The collection of positroid polytopes $\{\Gamma_\pi\}$ is a good tiling (respectively, good dissection) of $\Delta_{k+1,n}$ if and only if, for all $Z \in \mathrm{Mat}^{>0}_{n,k+2}$, the collection of T-dual Grasstopes $\{Z_{\hat{\pi}}\}$ is a good tiling (respectively, good dissection) of $\mathcal{A}_{n,k,2}(Z)$.

9 The positive tropical Grassmannian and positroid subdivisions

The goal of this section is to use the positive tropical Grassmannian to understand the regular positroid subdivisions of the hypersimplex. In Section 10, we will apply the

T-duality map to these regular positroid subdivisions of the hypersimplex, to obtain subdivisions of the amplituhedron which have very nice properties.

The *tropical Grassmannian* – or rather, an outer approximation of it called the *Dressian* – controls the regular matroidal subdivisions of the hypersimplex [39], [61, Proposition 2.2]. There is a positive subset of the tropical Grassmannian, called the *positive tropical Grassmannian*, which was introduced by Speyer and the third author in [65]. The positive tropical Grassmannian equals the positive Dressian, and as we will show in Proposition 9.12, it controls the regular *positroid* subdivisions of the hypersimplex.

Remark 9.1. We've learned since circulating the first draft of this paper that some of our results in this section regarding positroid subdivisions of the hypersimplex and the positive tropical Grassmannian, though not previously in the literature, were known or anticipated by various other experts including David Speyer, Nima Arkani-Hamed, Thomas Lam, Marcus Spradlin, Nick Early, Felipe Rincon, Jorge Olarte. There is some related work in [25] and the upcoming [5].

9.1 The tropical Grassmannian, the Dressian, and their positive analogues

Definition 9.2. Given $e = (e_1, \dots, e_N) \in \mathbb{Z}^N_{\geq 0}$, we let $\mathbf{x}^e$ denote $x_1^{e_1} \dots x_N^{e_N}$. Let $E \subset \mathbb{Z}^N_{\geq 0}$. For $f = \sum_{e \in E} f_e \mathbf{x}^e$ a nonzero polynomial, we denote by $\mathrm{Trop}(f) \subset \mathbb{R}^N$ the set of all points $(X_1, \dots, X_N)$ such that, if we form the collection of numbers $\sum_{i=1}^N e_i X_i$ for e ranging over E, then the minimum of this collection is not unique. We say that $\mathrm{Trop}(f)$ is the *tropical hypersurface associated to* f.

In our examples, we always consider polynomials f with real coefficients. We also have a positive version of Definition 9.2.

Definition 9.3. Let $E = E^+ \sqcup E^- \subset \mathbb{Z}^N_{\geq 0}$, and let f be a nonzero polynomial with real coefficients which we write as $f = \sum_{e \in E^+} f_e \mathbf{x}^e - \sum_{e \in E^-} f_e \mathbf{x}^e$, where all of the coefficients f_e are nonnegative real numbers. We denote by $\mathrm{Trop}^+(f) \subset \mathbb{R}^N$ the set of all points $(X_1, \dots, X_N)$ such that, if we form the collection of numbers $\sum_{i=1}^N e_i X_i$ for e ranging over E, then the minimum of this collection is not unique and furthermore is achieved for some $e \in E^+$ and some $e \in E^-$. We say that $\mathrm{Trop}^+(f)$ is the *positive part of* $\mathrm{Trop}(f)$.

The Grassmannian $Gr_{k,n}$ is a projective variety which can be embedded in projective space $\mathbb{P}^{\binom{[n]}{k}-1}$, and is cut out by the *Plücker ideal*, that is, the ideal of relations satisfied by the Plücker coordinates of a generic $k \times n$ matrix. These relations include the *three-term Plücker relations* defined below.

Definition 9.4. Let $1 < a < b < c < d \leq n$ and choose a subset $S \in \binom{[n]}{k-2}$ which is disjoint from $\{a, b, c, d\}$. Then $p_{Sac}p_{Sbd} = p_{Sab}p_{Scd} + p_{Sad}p_{Sbc}$ is a *three-term Plücker relations* for the Grassmannian $Gr_{k,n}$. Here Sac denotes $S \cup \{a, c\}$, etc.

Definition 9.5. Given S, a, b, c, d as in Definition 9.4, we say that the *tropical three-term Plücker relation holds* if

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$ or
- $P_{Sab} + P_{Scd} = P_{Sad} + P_{Sbc} \leq P_{Sac} + P_{Sbd}$.

And we say that the *positive tropical three-term Plücker relation holds* if either of the first two conditions above holds.

Definition 9.6. The *tropical Grassmannian* Trop $Gr_{k,n} \subset \mathbb{R}^{\binom{[n]}{k}}$ is the intersection of the tropical hypersurfaces Trop(f), where f ranges over all elements of the Plücker ideal. The *Dressian* $\mathrm{Dr}_{k,n} \subset \mathbb{R}^{\binom{[n]}{k}}$ is the intersection of the tropical hypersurfaces Trop(f), where f ranges over all three-term Plücker relations.

Similarly, the *positive tropical Grassmannian* $\mathrm{Trop}^+ Gr_{k,n} \subset \mathbb{R}^{\binom{[n]}{k}}$ is the intersection of the positive tropical hypersurfaces $\mathrm{Trop}^+(f)$, where f ranges over all elements of the Plücker ideal. The *positive Dressian* $\mathrm{Dr}^+_{k,n} \subset \mathbb{R}^{\binom{[n]}{k}}$ is the intersection of the positive tropical hypersurfaces $\mathrm{Trop}^+(f)$, where f ranges over all three-term Plücker relations.

Note that the Dressian $Dr_{k,n}$ (respectively, the positive Dressian $Dr^+_{k,n}$) is the subset of $\mathbb{R}^{\binom{[n]}{k}}$ where the tropical (respectively, positive tropical) three-term Plücker relations hold.

In general, the Dressian $\mathrm{Dr}_{k,n}$ is much larger than the tropical Grassmannian Trop $Gr_{k,n}$ – for example, the dimension of the Dressian $\mathrm{Dr}_{3,n}$ grows quadratically is n, while the dimension of the tropical Grassmannian Trop $Gr_{3,n}$ is linear in n [35]. However, the situation for their positive parts is different.

Theorem 9.7. [66]. The positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{k,n}$ equals the positive Dressian $\mathrm{Dr}^+_{k,n}$.

Definition 9.8. We say that a point $\{P_I\}_{I\in\binom{[n]}{k}} \in \mathbb{R}^{\binom{[n]}{k}}$ is a *(finite) tropical Plücker vector* if it lies in the Dressian $\mathrm{Dr}_{k,n}$, i.e. for every three-term Plücker relation, it lies in the associated tropical hypersurface. And we say that $\{P_I\}_{I\in\binom{[n]}{k}}$ is a *positive tropical Plücker vector*, if it lies in the positive Dressian $\mathrm{Dr}^+_{k,n}$ (equivalently, the positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{k,n}$), i.e. for every three-term Plücker relation, it lies in the positive part of the associated tropical hypersurface.

Example 9.9. For $Gr_{2,4}$, there is only one Plücker relation, $p_{13}p_{24} = p_{12}p_{34} + p_{14}p_{23}$. The Dressian $\mathrm{Dr}_{2,4} \subset \mathbb{R}^{\binom{[4]}{2}}$ is defined to be the set of points $(P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}) \in \mathbb{R}^6$ such that

- $P_{13} + P_{24} = P_{12} + P_{34} \le P_{14} + P_{23}$ or
- $P_{13} + P_{24} = P_{14} + P_{23} \le P_{12} + P_{34}$ or
- $P_{12} + P_{34} = P_{14} + P_{23} \le P_{13} + P_{24}$.

And $\mathrm{Dr}^+_{2,4} = \mathrm{Trop}^+ Gr_{2,4} \subset \mathbb{R}^{\binom{[4]}{2}}$ is defined to be the set of points $(P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}) \in \mathbb{R}^6$ such that

- $P_{13} + P_{24} = P_{12} + P_{34} \le P_{14} + P_{23}$ or
- $P_{13} + P_{24} = P_{14} + P_{23} \le P_{12} + P_{34}$

9.2 The positive tropical Grassmannian and positroid subdivisions

Recall that $\Delta_{k,n}$ denotes the (k,n)-hypersimplex, defined as the convex hull of the points e_I where I runs over $\binom{[n]}{k}$. Consider a real-valued *height function* $\{I\} \mapsto P_I$ on the vertices of $\Delta_{k,n}$. We define a polyhedral subdivision $\mathcal{D}_P$ of $\Delta_{k,n}$ as follows: consider the points $(e_I, P_I) \in \Delta_{k,n} \times \mathbb{R}$ and take their convex hull. Take the lower faces (those whose outwards normal vector have last component negative) and project them back down to $\Delta_{k,n}$; this gives us the subdivision $\mathcal{D}_P$. We will omit the subscript P when it is clear from context. A subdivision obtained in this manner is called *regular*.

Remark 9.10. A lower face F of the regular subdivision defined above is determined by some vector $\lambda = (\lambda_1, \ldots, \lambda_n, -1)$ whose dot product with the vertices of the face F is maximized. So if F is the matroid polytope of a matroid M with bases $\mathcal{B}$, this is equivalent to saying that $\lambda_{i_1} + \cdots + \lambda_{i_k} - P_I = \lambda_{j_1} + \cdots + \lambda_{j_k} - P_J > \lambda_{h_1} + \cdots + \lambda_{h_k} - P_H$ for any two bases $I, J \in \mathcal{B}$ and $H \notin \mathcal{B}$.

Given a subpolytope Γ of $\Delta_{k,n}$, we say that Γ is *matroidal* if the vertices of Γ, considered as elements of $\binom{[n]}{k}$, are the bases of a matroid M, i.e. $\Gamma = \Gamma_M$.

The following result is originally due to Kapranov [39]; it was also proved in [61, Proposition 2.2].

Theorem 9.11. The following are equivalent.

- The collection $\{P_I\}_{I\in\binom{[n]}{k}}$ is a tropical Plücker vector.
- The one-skeleta of $\mathcal{D}_P$ and $\Delta_{k,n}$ are the same.
- Every face of $\mathcal{D}_P$ is matroidal.

Given a subpolytope Γ of $\Delta_{k,n}$, we say that Γ is *positroid* if the vertices of Γ, considered as elements of $\binom{[n]}{k}$, are the bases of a positroid M, i.e. $\Gamma = \Gamma_M$. We now give a positroid version of Proposition 9.11.

Theorem 9.12. The following are equivalent.

- The collection $\{P_I\}_{I\in\binom{[n]}{k}}$ is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is positroid.

Proof. Suppose that the collection $\{P_I\}_{I\in\binom{[n]}{k}}$ are positive tropical Plücker coordinates. Then in particular they are tropical Plücker coordinates, and so by Proposition 9.11, every face of $\mathcal{D}_P$ is matroidal.

Suppose that one of those faces Γ_M fails to be positroid. Then by Theorem 3.9, Γ_M (and hence $\mathcal{D}_P$) has a two-dimensional face with vertices $e_{Sab}, e_{Sad}, e_{Sbc}, e_{Scd}$, for some $1 \leq a < b < c < d \leq n$ and S of size $k-2$ disjoint from $\{a, b, c, d\}$. By Remark 9.10, this means that there is a vector $\lambda = (\lambda_1, \ldots, \lambda_n, -1)$ whose dot product is maximized at the face F. In particular, if we compare the value of the dot product at vertices of F versus e_{Sac} and e_{Sbd}, we get $\lambda_a + \lambda_b - P_{Sab} = \lambda_c + \lambda_d - P_{Scd} = \lambda_a + \lambda_d - P_{Sad} = \lambda_b + \lambda_c - P_{Sbc}$ is greater than either $\lambda_a + \lambda_c - P_{Sac}$ or $\lambda_b + \lambda_d - P_{Sbd}$. But then

$$\lambda_a+\lambda_b-P_{Sab}+\lambda_c+\lambda_d-P_{Scd}=\lambda_a+\lambda_d-P_{Sad}+\lambda_b+\lambda_c-P_{Sbc}>\lambda_a+\lambda_c - P_{Sac} + \lambda_b + \lambda_d - P_{Sbd},$$

which implies that

$$P_{Sab} + P_{Scd} = P_{Sad} + P_{Sbc} < P_{Sac} + P_{Sbd},$$

which contradicts the fact that $\{P_I\}$ is a collection of positive tropical Plücker coordinates.

Suppose that every face of $\mathcal{D}_P$ is positroid. Then every face is in particular matroidal, and so by Proposition 9.11, the collection $\{P_I\}_{I\in\binom{[n]}{k}}$ are tropical Plücker coordinates. Suppose that they fail to be positive tropical Plücker coordinates. Then there is some $S \in \binom{[n]}{k-2}$ and $a < b < c < d$ disjoint from S such that $P_{Sab} + P_{Scd} = P_{Sad} + P_{Sbc} < P_{Sac} + P_{Sbd}$. We will obtain a contradiction by showing that $\mathcal{D}_P$ has a two-dimensional (non-positroid) face with vertices $e_{Sab}, e_{Sad}, e_{Sbc}, e_{Scd}$, for some $1 \le a < b < c < d \le n$ and S of size $k-2$ disjoint from $\{a, b, c, d\}$.

To show that these vertices form a face, choose some large number N which is greater than the absolute value of any of the tropical Plücker coordinates, i.e. $N > \max\{|P_I|\}_{I\in\binom{[n]}{k}}$. We define a vector $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ by setting

$$\lambda_i = \begin{cases} \frac{1}{2}(P_{Sab} + P_{Sac} + P_{Sad}) & \text{for i=a} \\ \frac{1}{2}(P_{Sab} + P_{Sbc} + P_{Sbd}) & \text{for i=b} \\ \frac{1}{2}(P_{Sac} + P_{Sbc} + P_{Scd}) & \text{for i=c} \\ \frac{1}{2}(P_{Sad} + P_{Sbd} + P_{Scd}) & \text{for i=d} \\ \frac{3}{2}N & \text{for } i \in S \\ -\frac{3}{2}N & \text{for } i \notin S \cup \{a, b, c, d\}. \end{cases}$$

We now compute the lower face of $\mathcal{D}_P$ determined by vector $\lambda := (\lambda_1, \dots, \lambda_n, -1)$, using Remark 9.10. Clearly any point (e_I, P_I) of $\Delta_{k,n} \times \mathbb{R}$ maximizing the dot product with λ must have $e_I \in \{e_{Sab}, e_{Sac}, e_{Sad}, e_{Sbc}, e_{Sbd}, e_{Scd}\}$. The relation $P_{Sab} + P_{Scd} = P_{Sad} + P_{Sbc} < P_{Sac} + P_{Sbd}$ implies that the lower face of $\mathcal{D}_P$ determined by λ has vertices $e_{Sab}, e_{Sad}, e_{Sbc}, e_{Scd}$. ■

It follows from Proposition 9.12 that the regular subdivisions of $\Delta_{k+1,n}$ consisting of positroid polytopes are precisely those of the form $\mathcal{D}_P$, where $P = \{P_I\}$ is a positive tropical Plücker vector. This motivates the following definition.

Definition 9.13. We say that a positroid dissection of $\Delta_{k+1,n}$ is a *regular positroid subdivision* if it has the form $\mathcal{D}_P$, where $P = \{P_I\} \in \mathbb{R}^{\binom{[n]}{k}}$ is a positive tropical Plücker vector.

Remark 9.14. Every regular subdivision of a polytope is a polytopal subdivision, and so in particular it is a good dissection (see Definition 8.2).

9.3 Fan structures on the Dressian and positive Dressian

As described in [35], there are two natural fan structures on the (positive) Dressian: the *Plücker fan*, and the *secondary fan.*

We say that two elements of the Dressian, i.e. two tropical Plücker vectors $\{P_I\}_{I\in\binom{[n]}{k}}$ and $\{P'_I\}_{I\in\binom{[n]}{k}} \in \mathbb{R}^{\binom{[n]}{k}}$, lie in the same cone of the *Plucker fan* if for each S, a, b, c, d as in Definition 9.5, the same inequality holds for both $\{P_{Sac}, P_{Sbd}, P_{Sab}, P_{Scd}, P_{Sad}, P_{Sbc}\}$ and $\{P'_{Sac}, P'_{Sbd}, P'_{Sab}, P'_{Scd}, P'_{Sad}, P'_{Sbc}\}$. In particular, the maximal cones in the Plücker fan structure are the cones where the inequalities from Definition 9.5 are all strict.

On the other hand, using Proposition 9.11 and Proposition 9.12, we say that two elements of the Dressian, i.e. two tropical Plücker vectors $\{P_I\}_{I\in\binom{[n]}{k}}$ and $\{P'_I\}_{I\in\binom{[n]}{k}} \in \mathbb{R}^{\binom{[n]}{k}}$, lie in the same cone of the *secondary fan* if the matroidal subdivisions $\mathcal{D}_P$ and $\mathcal{D}_{P'}$ coincide. In particular, the maximal cones in the secondary fan structure are the cones corresponding to the unrefinable positroid subdivisions.

In [35] it was shown that for the Dressian $Dr_{3,n}$, the Plücker fan structure and the secondary fan structure coincide. And in [52, Theorem 14] it was shown that the fan structures coincide for general Dressians $Dr_{k,n}$. We can now just refer to the fan structure on $Dr^+_{k,n} = \mathrm{Trop}^+ Gr_{k,n}$ without specifying either "Plücker fan" or "secondary fan."

We have the following result.

Corollary 9.15. A collection $\mathcal{C} = \{S_\pi\}$ of positroid cells of $Gr^{\geq 0}_{k,n}$ gives a regular positroid tiling of $\Delta_{k,n}$ (see Definition 2.5) if and only if this tiling has the form $\mathcal{D}_P$, for $P = \{P_I\}_{I\in\binom{[n]}{k}}$ a positive tropical Plücker vector from a maximal cone of $\mathrm{Trop}^+ Gr_{k,n}$.

Proof. Suppose that a collection $\{S_\pi\}$ of positroid cells of $Gr^{\geq 0}_{k,n}$ is a regular positroid tiling; in other words, the images of the cells $\{S_\pi\}$ under the moment map are the top-dimensional positroid polytopes in the subdivision $\mathcal{D}_P$ of $\Delta_{k,n}$, and the moment map is an injection on each S_π. Therefore by Proposition 3.15 and Proposition 3.16, $\dim S_\pi = n-1$, each positroid M_π is connected, and the reduced plabic graph associated to π is a (planar) tree.

We claim that the collection $\{S_\pi\}$ gives an unrefineable possible positroid subdivision of the hypersimplex. That is, there is no nontrival way to subdivide one of the positroid polytopes Γ_π into two full dimensional positroid polytopes. If we *can* subdivide Γ_π as above, and there is another full-dimensional positroid polytope $\Gamma_{\pi'}$

strictly contained in Γ_π, then the bases of $M_{\pi'}$ are a subset of the bases of Γ_π, and hence the cell $S_{\pi'}$ lies in the closure of S_π. But then a reduced plabic graph G' for $S_{\pi'}$ can be obtained by deleting some edges from a reduced plabic graph G for S_π; this means that G' has fewer faces than G and hence has the corresponding cell has smaller dimension, which is a contradiction, so the claim is true.

But now the fact that $\{S_\pi\}$ gives an unrefineable positroid subdivision means that it came from a maximal cone of $\mathrm{Trop}^+ Gr_{k,n}$.

Conversely, consider a regular positroid subdivision $\mathcal{D}_P$ coming from a maximal cone of $\mathrm{Trop}^+ Gr_{k,n}$. Then the subdivision $\mathcal{D}_P$ (which we identify with its top-dimensional pieces $\{S_\pi\}$) is an unrefineable positroid subdivision. In other words, none of the positroid polytopes Γ_π can be subdivided into two full-dimensional positroid polytopes, which in turn means that the reduced plabic graph corresponding to π must be a tree. This implies that the moment map is an injection on each S_π and hence $\{S_\pi\}$ gives a regular positroid tiling of $\Delta_{k,n}$. ■

Corollary 9.16. The number of regular positroid tilings of the hypersimplex $\Delta_{k,n}$ equals the number of maximal cones in the positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{k,n}$.

The fact that the Plücker fan structure and the secondary fan structure on $\mathrm{Trop}^+ Gr_{k,n}$ coincide also implies that the f-vector of $\mathrm{Trop}^+ Gr_{k,n}$ reflects the number of positroid subdivisions of $\Delta_{k,n}$ (with maximal cones corresponding to unrefineable subdivisions and rays corresponding to coarsest subdivisions).

10 Subdivisions of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ from $\mathrm{Trop}^+ Gr_{k+1,n}$

In Section 8, we discussed the fact that arbitrary dissections of the hypersimplex and the amplituhedron can have rather unpleasant properties, with their maximal cells intersecting badly at their boundaries. We introduced the notion of *good dissections* for the hypersimplex and amplituhedron in Definition 8.2 and Definition 8.4. Our goal in this section is to introduce a large class of good dissections for the amplituhedron which come from $\mathrm{Trop}^+ Gr_{k+1,n}$.

10.1 Regular positroid subdivisions of $\mathcal{A}_{n,k,2}(Z)$

Recall from Definition 9.13 that the *regular positroid subdivisions* of $\Delta_{k+1,n}$ are precisely the dissections $\mathcal{D}_P$ induced from height functions $P = \{P_I\} \in \mathbb{R}^{\binom{[n]}{k}}$ on the hypersimplex which are positive tropical Plücker vectors.

While we do not know how to define a notion of height function for the amplituhedron, we know from Section 5, Section 6, and Section 7 that T-duality maps dissections of $\Delta_{k+1,n}$ to the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ and preserves various nice properties along the way. We therefore apply the T-duality map from Definition 5.1 to regular positroid subdivisions of $\Delta_{k+1,n}$, to define a class of subdivisions of the $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}(Z)$ which we optimistically refer to as *regular (positroid) subdivisions*.

Definition 10.1. We say that a positroid dissection of $\mathcal{A}_{n,k,2}(Z)$ is a *regular positroid subdivision* if it has the form $\{Z_{\hat{\pi}}\}$, where $\{\Gamma_\pi\}$ is a regular positroid subdivision of $\Delta_{k+1,n}$.

As every regular positroid subdivision of $\Delta_{k+1,n}$ is a polyhedral subdivision (and hence is good), Proposition 8.9 implies the following.

Conjecture 10.2. Every regular positroid subdivision of $\mathcal{A}_{n,k,2}(Z)$ is a good dissection.

In Section 10.5 we provide some computational evidence for Conjecture 10.2. For example, for $\mathcal{A}_{6,2,2}(Z)$ and $\mathcal{A}_{7,2,2}(Z)$, every regular positroid subdivision is good, and moreover, all good dissections are regular positroid subdivisions. (This appears to also be the case for $\mathcal{A}_{8,2,2}(Z)$; but we were only able to compute the number of *tilings* in this case.) One might hope to strengthen Conjecture 10.2 and conjecture that the regular positroid subdivisions are precisely the good dissections. However, the notion of regularity is rather subtle (as usual in polyhedral geometry), and starting from $\mathcal{A}_{9,2,2}(Z)$, there are some good dissections which are not regular.

10.2 A large class of regular positroid tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$

Definition 10.3. Let T be any planar trivalent tree with n leaves (which will necessarily have $n-2$ internal vertices), embedded in a disk with the leaves labelled from 1 to n in clockwise order. Let $\mathcal{T}_{n,k}$ be the set of $\binom{n-2}{k}$ plabic graphs obtained from T by coloring precisely k of the internal vertices black, as in Figure 5.

Proposition 10.4. The cells of $Gr^{\geq 0}_{k+1,n}$ corresponding to the plabic graphs in $\mathcal{T}_{n,k}$ give a regular tiling of $\Delta_{k+1,n}$. Therefore the images of these cells under the T-duality map give a regular tiling of $\mathcal{A}_{n,k,2}(Z)$.

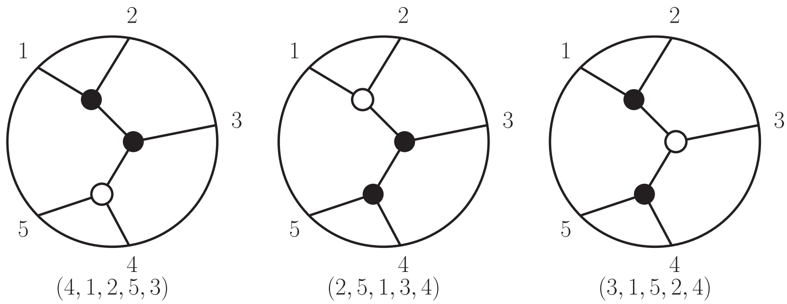

Fig. 5. The collection $\mathcal{T}_{5,2}$ of plabic graphs giving a regular subdivision of $\Delta_{3,5}$

Proof. We can use Theorem 4.5 (see Figure 1) to inductively prove that the cells corresponding to $\mathcal{T}_{n,k}$ give a tiling of $\Delta_{k+1,n}$. The fact that the cells corresponding to the plabic graphs in $\mathcal{T}_{n,k}$ give a *regular* tiling of $\Delta_{k+1,n}$ follows from [61, Theorem 8.4]. Now using Theorem 6.5, it follows that the images of these cells under the T-duality map give a tiling of $\mathcal{A}_{n,k,2}(Z)$. The fact that this tiling is regular now follows from Definition 10.1. ■

Remark 10.5. The above construction gives us C_{n-2} regular tilings of $\mathcal{A}_{n,k,2}(Z)$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number.

10.3 The fan structure for regular positroid subdivisions

We now discuss the fan structure for regular positroid subdivisions of the hypersimplex and amplituhedron.

Definition 10.6. Given two subdivisions $\{\Gamma_\pi\}$ and $\{\Gamma_{\pi'}\}$ of $\Delta_{k+1,n}$, we say that $\{\Gamma_\pi\}$ *refines* $\{\Gamma_{\pi'}\}$ and write $\{\Gamma_\pi\} \preceq \{\Gamma_{\pi'}\}$ if every Γ_π is contained in some $\Gamma_{\pi'}$.

Similarly, given two subdivisions $\{Z_\pi\}$ and $\{Z_{\pi'}\}$ of $\mathcal{A}_{n,k,2}(Z)$, we say that $\{Z_\pi\}$ *refines* $\{Z_{\pi'}\}$ and write $\{Z_\pi\} \preceq \{Z_{\pi'}\}$ if every Z_π is contained in some $Z_{\pi'}$.

Recall from Section 9.3 that we have a fan structure on $\mathrm{Trop}^+ Gr_{k+1,n}$ (the *secondary fan*, which coincides with the *Plücker fan*) which describes the regular positroid subdivisions of $\Delta_{k+1,n}$, ordered by refinement. We expect that this fan structure on $\mathrm{Trop}^+ Gr_{k+1,n}$ also describes the regular positroid subdivisions of $\mathcal{A}_{n,k,2}(Z)$.

Conjecture 10.7. The regular positroid subdivisions of $\mathcal{A}_{n,k,2}(Z)$ are parametrized by the cones of $\mathrm{Trop}^+ Gr_{k+1,n}$, with the natural partial order on the cones reflecting the refinement order on positroid subdivisions.

Conjecture 10.7 is consistent with the following conjecture.

Conjecture 10.8. Consider two regular positroid subdivisions $\{\Gamma_\pi\}$ and $\{\Gamma_{\pi'}\}$ of $\Delta_{k+1,n}$, and two corresponding positroid subdivisions $\{Z_{\hat{\pi}}\}$ and $\{Z_{\hat{\pi'}}\}$ of $\mathcal{A}_{n,k,2}(Z)$. Then we have that $\{\Gamma_\pi\} \preceq \{\Gamma_{\pi'}\}$ if and only if $\{Z_{\hat{\pi}}\} \preceq \{Z_{\hat{\pi'}}\}$

In particular, the regular positroid tilings of $\mathcal{A}_{n,k,2}(Z)$ should come precisely from the maximal cones of $\mathrm{Trop}^+ Gr_{k+1,n}$. More specifically, if $\{P_I\}$ lies in a maximal cone of $\mathrm{Trop}^+ Gr_{k+1,n}$, and $\{S_\pi\}$ is the regular positroid tiling corresponding to $\mathcal{D}_P$, then $\{S_{\hat{\pi}}\}$ should be a regular positroid tiling of $\mathcal{A}_{n,k,2}(Z)$. (Moreover, all regular positroid tilings of $\mathcal{A}_{n,k,2}(Z)$ should arise in this way.)

10.4 The f-vector of $\mathrm{Trop}^+ Gr_{k+1,n}$

In light of Conjecture 10.7, it is useful to compute the f-vector of the positive tropical Grassmannian. This is the vector $(f_0, f_1, \dots, f_d)$ whose components compute the number of cones of fixed dimension.

As shown in [65], the positive tropical Grassmannian has an n-dimensional lineality space coming from the torus action. However, one may mod out by this torus action and study the resulting fan. The method used in [65] was to show that $\mathrm{Trop}^+ Gr_{k,n}$ (a polyhedral subcomplex of $\mathbb{R}^{\binom{[n]}{k}}$) is combinatorially equivalent to an $(n-k-1)(k-1)$-dimensional fan $F_{k,n}$, obtained by using an "X-cluster" or "web" parametrization of the positive Grassmannian, and modding out by the torus action. As explained in [65, Section 6], $F_{k,n}$ is the dual fan to the Minkowski sum of the $\binom{n}{k}$ Newton polytopes obtained by writing down each Plücker coordinate in the X-cluster parametrization.

Using this technique, [65] computed the f-vector of $\mathrm{Trop}^+ Gr_{2,n}$ (which is the f-vector of the associahedron, with maximal cones corresponding to tilings of a polygon) $\mathrm{Trop}^+ Gr_{3,6}$, and $\mathrm{Trop}^+ Gr_{3,7}$. The above f-vector computations were recently extended in [4] using the notion of "stringy canonical forms" and in [12, 21] using planar arrays and matrices of Feynman diagrams. See also [22, 24, 38] for recent, physics-inspired developments in this direction. We list all known results about maximal cones in the positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{k+1,n}$ and their relation to tilings of hypersimplex $\Delta_{k+1,n}$ in Table 1.

Table 1 New results about the tilings of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ in relation to known results about the number of maximal cones of the positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{k+1,n}$.

(k,n)	Tilings	Good tilings	$\mathrm{Trop}^+ Gr_{k+1,n}$	Non-regular good tilings
$(1,n)$	C_{n-2}	C_{n-2}	C_{n-2}	0
$(2,5)$	5	5	5	0
$(2,6)$	120	48	48	0
$(2,7)$	3073	693	693	0
$(2,8)$	6 443 460	13 612	13 612	0
$(2,9)$	?	346 806	346 710	96
$(3,6)$	14	14	14	0
$(3,7)$	3073	693	693	0
$(3,8)$	?	91 496	90 608	888
$(3,9)$	?	33 182 763	30 659 424	2 523 339

Apart from the f-vector of $\mathrm{Trop}^+ Gr_{2,n}$, the known f-vectors of positive tropical Grassmannians $\mathrm{Trop}^+ Gr_{k,n}$ (with $k \leq \frac{n}{2}$) are the following:

$$\mathrm{Trop}^+ Gr_{3,6} :(1, 48, 98, 66, 16, 1)$$

$$\mathrm{Trop}^+ Gr_{3,7} :(1, 693, 2163, 2583, 1463, 392, 42, 1)$$

$$\mathrm{Trop}^+ Gr_{3,8} :(1, 13612, 57768, 100852, 93104, 48544, 14088, 2072, 120, 1)$$

$$\mathrm{Trop}^+ Gr_{4,8} :(1, 90608, 444930, 922314, 1047200, 706042, 285948, 66740, 7984, 360, 1)$$

For $\mathrm{Trop}^+ Gr_{4,9}$ it is also known that the second component of the f-vector is 30659424 [21].

Remark 10.9. The coordinate ring of the Grassmannian has the structure of a *cluster algebra* [59]. In particular, $Gr_{2,n}$, $Gr_{3,6}$, $Gr_{3,7}$, $Gr_{3,8}$ have cluster structures of finite types A_n, D_4, E_6, and E_8, respectively. As discussed in [65], there is an intriguing connection between $\mathrm{Trop}^+ Gr_{k,n}$ and the cluster structure. In particular, $F_{2,n}$ is the fan to the type A_n associahedron, while $F_{3,6}$ and $F_{3,7}$ are coarsenings of the fans associated to the D_4 and E_6 associahedra. Via our correspondence between $\mathrm{Trop}^+ Gr_{k+1,n}$ and the amplituhedron $\mathcal{A}_{n,k,2}(Z)$, the Grassmannian cluster structure on $Gr_{k+1,n}$ should be reflected in good subdivisions of $\mathcal{A}_{n,k,2}(Z)$. In particular the type A_n cluster structure should control $\mathcal{A}_{n,1,2}(Z)$ (this is apparent, since $\mathcal{A}_{n,1,2}(Z)$ is a projective polygon), while the type D_4, E_6, and E_8 cluster structures should be closely related to $\mathcal{A}_{6,2,2}(Z)$, $\mathcal{A}_{7,2,2}(Z)$, and $\mathcal{A}_{8,2,2}(Z)$.

10.5 Experimental Data.

Checks for this section[6] for small values of n and k have been performed using Wolfram Mathematica. In particular, we used the packages `positroid' [19] and `amplituhedronBoundaries' [48]. This allowed us to find the complete poset of good dissections of $\mathcal{A}_{6,2,2}$ and $\mathcal{A}_{7,2,2}$, whose f-vectors read:

$$\mathcal{A}_{6,2,2} : (1, 48, 98, 66, 16, 1)$$
$$\mathcal{A}_{7,2,2} : (1, 693, 2163, 2583, 1463, 392, 42, 1)\,.$$

These are exactly the f-vectors of the positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{3,6}$ and $\mathrm{Trop}^+ Gr_{3,7}$, respectively. For higher values of n and k, we have been able to find all (good) tilings, and our findings[7] are summarized in Table 1.

In particular, we observe that for $\mathcal{A}_{8,2,2}(Z)$ the number of good tilings agrees with the number of maximal cones in $\mathrm{Trop}^+ Gr_{3,8}$. Starting from $n = 9$, the number of good tilings is larger than the number of maximal cones in positive tropical Grassmannian. It is indeed the first example where one can find good tilings which are *not regular*. In particular, out of 346806 good tilings, 96 are not regular. Similarly, for $k = 3$ and $n = 8$, 888 good tilings of $\mathcal{A}_{8,3,2}(Z)$ are not regular. We note that these correspond exactly to degenerate matrices found in [21].

11 T-duality and the momentum amplituhedron for general (even) m

Throughout the paper we have explored the remarkable connection between the hypersimplex and the $m = 2$ amplituhedron. This was established via the T-duality map which allowed to relate positroid tiles, tilings, and dissections of both objects. It is then a natural question to wonder whether the story generalizes for any (even) m.

For $m = 4$, we know that the amplituhedron $\mathcal{A}_{n,k,4}(Z)$ encodes the geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM, expressed in momentum twistor space. Physicists have already observed a beautiful connection between this and the formulation of

6 A more detailed discussion of these checks can be found in the arXiv version of this paper (v3).
7 We also included there the results for $Gr_{3,9}^{\geq 0}$ which, by using our conjectures, can be derived from [21].

scattering amplitudes of the same theory in momentum space[8] . At the core of this connection lies the Amplitude-Wilson Loop Duality [8], which was shown to arise from a more fundamental duality in String Theory called 'T-duality' [17]. For both formulations a Grassmannian representation has been found [3, 18]: scattering amplitudes (at tree level) are computed by performing a contour integral around specific cycles inside the positive Grassmannian (what in physics is referred to as a 'BCFW contour'). If we are in momentum space, then one has to integrate over cycles corresponding to collections of $(2n-4)$-dimensional positroid cells of $Gr^{\geq 0}_{k+2,n}$. Whereas, if we are in momentum twistor space, the integral is over collections of $4k$-dimensional positroid cells of $Gr^{\geq 0}_{k,n}$. The two integrals compute the same scattering amplitude, and it was indeed shown that formulas are related by a change of variables. In particular, this implied the existence of a map between certain $(2n-4)$-dimensional positroid cells of $Gr^{\geq 0}_{k+2,n}$ and certain $4k$-dimensional positroid cells of $Gr^{\geq 0}_{k,n}$ (called 'BCFW'), which was defined in [1, Formula (8.25)]. It is easy to see that this map is exactly our T-duality map for the case $m=4$ in (5.4), up to a cyclic shift:

$$\sigma\hat{\pi}(i) = \pi(i - \frac{m}{2} + 1) - 1 = \pi(i-1) - 1. \tag{11.1}$$

Collections of $4k$-dimensional 'BCFW' positroid cells of $Gr^{\geq 0}_{k,n}$ defined from physics were conjectured to tile $\mathcal{A}_{n,k,4}(Z)$. The proof of this conjecture can be found in [44]. On the other hand, the corresponding collections of $(2n-4)$-dimensional 'BCFW' positroid cells of $Gr^{\geq 0}_{k+2,n}$ were conjectured to tile an object $\mathcal{M}_{n,k,4}(\Lambda, \widetilde{\Lambda})$ called 'momentum amplituhedron', introduced recently by two of the authors in [23][9] .

The story aligns with the philosophy of the rest of this paper. In particular, one aims to seek for an object and a map which relates its tiles, tilings (and, more generally, dissections) to the ones of $\mathcal{A}_{n,k,m}(Z)$, for general (even) m. There is a natural candidate for such a map: we have already seen that the T-duality map defined in (5.4) does indeed the job in the case of $m=2$ and $m=4$. Moreover, some of the statements which has been proven throughout the paper for $m=2$, as Proposition 5.11 and Theorem 7.3, can be generalized for general (even) m.

8 More precisely it is 'spinor helicity' space, or, equivalently (related by half-Fourier transform), in twistor space. See [1, Section 8].

9 In the paper, the momentum amplituhedron was denoted as $\mathcal{M}_{n,k}$, without the subscript '4'.

Proposition 11.1. Let S_π be a cell of $Gr^{\geq 0}_{k+\frac{m}{2},n}$ such that, as affine permutation, $\pi(i) \geq i+\frac{m}{2}$. Then $S_{\hat{\pi}}$ is a cell of $Gr^{\geq 0}_{k,n}$ such that $\hat{\pi}(i) \leq i+n-\frac{m}{2}$. Moreover, $\dim(S_{\hat{\pi}}) - mk = \dim(S_\pi) - \frac{m}{2}(n-\frac{m}{2})$. In particular, if $\dim S_\pi = \frac{m}{2}(n-\frac{m}{2})$, then $\dim S_{\hat{\pi}} = mk$.

Proof. This is a straightforward generalization of the proof of Proposition 5.11. It is enough to observe that, in the language of affine permutations, T-duality maps a $(k+m/2,n)$-bounded affine permutation π_a into a (k,n)-bounded affine permutation $\hat{\pi}_a = \pi_a \circ t^{m/2}$, with $t^{m/2} : \mathbb{Z} \to \mathbb{Z}$ the map $i \mapsto i-m/2$. Clearly, $t^{m/2}$ preserve the length of affine permutations. Hence the codimensions of $S_{\pi_a} \subseteq Gr^{\geq 0}_{k+\frac{m}{2},n}$ and $S_{\hat{\pi}_a} \subseteq Gr^{\geq 0}_{k,n}$ are equal. ■

It is also natural to think of parity duality between $\mathcal{A}_{n,k,m}(Z)$ and $\mathcal{A}_{n,n-k-m,m}(Z')$ as a composition of the Grassmannian duality and T-duality (plus cyclic shifts). Imitating Definition 7.2, let us define $\widetilde{U}_{k,n,m}(\hat{\pi}) := \widehat{\pi^{-1}}$. Then we have the following theorem:

Theorem 11.2 (Parity duality from T-duality and Grassmannian duality). Let $\{Z_\pi\}$ be a collection of Grasstopes which dissects the amplituhedron $\mathcal{A}_{n,k,m}(Z)$. Then the collection of Grasstopes $\{Z_{\widetilde{U}_{k,n,m}\pi}\}$ dissects the amplituhedron $\mathcal{A}_{n,n-k-m,m}(Z')$.

Proof. The parity duality $U_{k,n,m}$ in [33] was defined for any (even) m as: $U_{k,n,m}(\pi) := (\pi-k)^{-1} + (n-k-m)$. Then it easy to show that $U_{k,n,m} = \sigma^{k+\frac{m}{2}} \circ \widetilde{U}_{k,n,m}$. Using Theorem 7.5, the prove follows immediately. ■

Since we found a natural candidate map, we now introduce a candidate object, which would speculatively relate to $\mathcal{A}_{n,k,m}(Z)$ via the T-duality map. This is a generalization of the momentum amplituhedron $\mathcal{M}_{n,k,4}(\Lambda,\widetilde{\Lambda})$ and it is defined below.

Definition 11.3. For k,n such that $k \leq n$, define the *twisted positive part* of $Gr_{k,n}$ as:

$$Gr^{+,\tau}_{k,n} := \{X \in Gr_{k,n} : (-1)^{\mathrm{inv}(I,[n]\setminus I)} \Delta_{[n]\setminus I}(X) \geq 0\} \tag{11.2}$$

where $\mathrm{inv}(A,B) := \#\{a \in A, b \in B | a > b\}$ denotes the inversion number.

The lemma below can be found in [40, Lemma 1.11], which sketched a proof and attributed it to Hochster and Hilbert.

Lemma 11.4. Suppose $\Delta_I(V)$ are the Plücker coordinates of a point $V \in Gr_{k,n}$. Then the kernel $V^\perp \in Gr_{n-k,n}$ of V is represented by the point with Plücker coordinates $\Delta_J(V^\perp) = (-1)^{inv(J,[n]\setminus J)}\Delta_{[n]\setminus J}(V)$ for $J \in \binom{[n]}{n-k}$.

Definition 11.5. For a, b such that $a \leq b$, define $\mathrm{Mat}^{>0}_{a,b}$ the set of real $a \times b$ matrices whose $a \times a$ minors are all positive and its *twisted positive part* as

$$\mathrm{Mat}^{>0,\tau}_{a,b} := \{A \in \mathrm{Mat}_{a,b} : (-1)^{\mathrm{inv}(I,[b]\setminus I)}\Delta_{[b]\setminus I}(A) > 0\} \tag{11.3}$$

Definition 11.6 (The momentum amplituhedron). Let $\tilde{\Lambda} \in \mathrm{Mat}^{>0}_{n,k'+\frac{m}{2}}$, $\Lambda \in \mathrm{Mat}^{>0,\tau}_{n,n-k'+\frac{m}{2}}$, $k' + m/2 \leq n$. The *momentum amplituhedron map* $\Phi_{\tilde{\Lambda},\Lambda} : Gr^{\geq 0}_{k',n} \to Gr_{k',k'+\frac{m}{2}} \times Gr_{n-k',n-k'+\frac{m}{2}}$ is defined by $\Phi_{\tilde{\Lambda},\Lambda}(C) := (C\tilde{\Lambda}, C^\perp\Lambda)$, where C and $C^\perp$ are matrices representing an element of $Gr^{\geq 0}_{k',n}$ and its orthogonal in $Gr^{\geq 0,\tau}_{k',n}$ respectively, and $C\tilde{\Lambda}$ and $C^\perp\Lambda$ matrices representing an element of $Gr_{k',k'+\frac{m}{2}}$ and $Gr_{n-k',n-k'+\frac{m}{2}}$ respectively. The *momentum amplituhedron* $\mathcal{M}_{n,k',m}(\Lambda, \tilde{\Lambda}) \subseteq Gr_{k',k'+\frac{m}{2}} \times Gr_{n-k',n-k'+\frac{m}{2}}$ is the image $\Phi_{\tilde{\Lambda},\Lambda}(Gr^{\geq 0}_{k',n})$.

Proposition 11.7 (Momentum conservation). Let $(\tilde{Y}, Y)$ represent a point in $Gr_{k',k'+\frac{m}{2}} \times Gr_{n-k',n-k'+\frac{m}{2}}$ and let $\tilde{Y}^\perp$ and $Y^\perp$ be matrices representing the orthogonal complements of Y and $\tilde{Y}$, respectively. If $(\tilde{Y}, Y)$ is in the momentum amplituhedron $\mathcal{M}_{n,k',m}(\Lambda, \widetilde{\Lambda})$, then

$$(Y^\perp\Lambda^T) \cdot (\tilde{Y}^\perp\tilde{\Lambda}^T)^T = 0 \tag{11.4}$$

Proof. From the identity

$$0 = Y^\perp Y^T = Y^\perp\Lambda^T(C^\perp)^T \tag{11.5}$$

we deduce that the row-span of $Y^\perp\Lambda^T$ is included in the row-span of the orthogonal of $C^\perp$, i.e. C. Analogously, from

$$0 = \tilde{Y}^\perp\tilde{Y}^T = \tilde{Y}^\perp\tilde{\Lambda}^T C \tag{11.6}$$

we deduce that the row-span of $\tilde{Y}^\perp\tilde{\Lambda}^T$ is included in the row-span of the $C^\perp$. Therefore $Y^\perp\Lambda^T$ and $\tilde{Y}^\perp\tilde{\Lambda}^T$ belong to orthogonal subspaces and satisfy

$$(Y^\perp\Lambda^T) \cdot (\tilde{Y}^\perp\tilde{\Lambda}^T)^T = 0. \tag{11.7}$$

■

Remark 11.8. In reference to Definition 11.6, we observe that:

$$\dim\left(Gr_{k',k'+m/2} \times Gr_{n-k',n-k'+m/2}\right) = \frac{m}{2}k' + \frac{m}{2}\left(n-k'\right) = \frac{m}{2}n. \tag{11.8}$$

Moreover, Proposition 11.7 implies that the momentum amplituhedron $\mathcal{M}_{n,k,m}$ is included in a codimension $(\frac{m}{2})^2$ sub-variety of $Gr_{k',k'+m/2} \times Gr_{n-k',n-k'+m/2}$. Therefore, the dimension of $\mathcal{M}_{n,k',m}$ is at most (and conjectured to be exactly):

$$\frac{m}{2}n - \left(\frac{m}{2}\right)^2 = \frac{m}{2}\left(n - \frac{m}{2}\right). \tag{11.9}$$

We observe that, for $m = 2$, this dimension is exactly $n - 1$, which is the dimension of the hypersimplex $\Delta_{k+1,n}$; whereas, for $m = 4$, the dimension is $2n - 4$, which is the one of BCFW cells in momentum space.

Remark 11.9. For $m = 2$, Definition 11.6 reads:

$$\Phi_{\tilde{\Lambda},\Lambda} : Gr^{\geq 0}_{k',n} \to Gr_{k',k+1} \times Gr_{n-k',n-k'+1} \cong \mathbb{P}^{k'} \times \mathbb{P}^{n-k'}. \tag{11.10}$$

Moreover, the conditions in Proposition 11.7 are equivalent to:

$$\lambda \cdot \tilde{\lambda} = 0 \tag{11.11}$$

where we used the dot product in $\mathbb{R}^n$ of the vectors $\lambda := \Lambda(Y^{\perp})^T$ and $\tilde{\lambda} := \tilde{\Lambda}(\tilde{Y}^{\perp})^T$.

Note that the $m = 2$ momentum amplituhedron is not equal to the hypersimplex, as pointed out in [50].

Remark 11.10. For $m = 4$, Definition 11.6 coincides with the one in [23]. This is the positive geometry relevant for scattering amplitudes for $\mathcal{N} = 4$ SYM in spinor helicity space.

Many properties of $\mathcal{M}_{n,k,4}(\Lambda, \widetilde{\Lambda})$ have still to be explored and proven. Let Φ_π denote the image under the amplituhedron map $\Phi_{\Lambda,\tilde{\Lambda}}(\bar{S}_\pi)$ of (the closure of) a positroid cell S_π in $Gr_{k',n}$. Analogously to the amplituhedron, we call Φ_π a *positroid tile* of $\mathcal{M}_{n,k,4}(\Lambda, \widetilde{\Lambda})$ if it is full-dimensional and if the momentum amplituhedron map is injective on S_π. We also define *positroid tilings* of $\mathcal{M}_{n,k,4}(\Lambda, \widetilde{\Lambda})$ collections $\{\Phi_\pi\}$ of positroid tiles whose interior is disjoint and cover $\mathcal{M}_{n,k,4}(\Lambda, \widetilde{\Lambda})$. Then the conjecture in [23] can be stated as:

Conjecture 11.11. [23] There exists an open subset $\mathcal{P} \subset \mathrm{Mat}^{>0,\tau}_{n,k'+2} \times \mathrm{Mat}^{>0}_{n,n-k'+2}$ such that for all $(\Lambda, \widetilde{\Lambda}) \in \mathcal{P}$ a collection of positroid tiles $\{\Phi_\pi\}$ is a positroid tiling (respectively, dissection) of $\mathcal{M}_{n,k+2,4}(\Lambda, \widetilde{\Lambda})$ if and only if for all $Z \in \mathrm{Mat}^{>0}_{n,k+4}$ the collection of T-dual Grasstopes $\{Z_{\hat{\pi}}\}$ is a tiling (respectively, dissection) of $\mathcal{A}_{n,k,4}(Z)$.

Remark 11.12. [23] provided experimental evidence that a subset $\mathcal{P}$ with the properties above can be obtained by imposing positivity of *planar Mandelstam variables*. In particular, choosing the rows of $\Lambda^\perp$ and $\widetilde{\Lambda}$ on the moment curve as $(\Lambda^\perp)_{i,a} = i^a$, $\widetilde{\Lambda}_{i,\dot{a}} = i^{\dot{a}}$, with $i \in [n], a \in [k'-2], \dot{a} \in [k'+2]$ would give a point in $\mathcal{P}$.

Finally, we speculate that:

Conjecture 11.13. Let m be a multiple of 4 and $k' = k + m/2$. There exists an open subset $\mathcal{P} \subset \mathrm{Mat}^{>0,\tau}_{n,k'+\frac{m}{2}} \times \mathrm{Mat}^{>0}_{n,n-k'+\frac{m}{2}}$ such that for all $(\Lambda, \widetilde{\Lambda}) \in \mathcal{P}$ a collection $\{\Phi_\pi\}$ of positroid tiles is a tiling (respectively, dissection) of $\mathcal{M}_{n,k',m}(\Lambda, \widetilde{\Lambda})$ if and only if the collection of T-dual Grasstopes $\{Z_{\hat{\pi}}\}$ is a tiling (respectively, dissection) of $\mathcal{A}_{n,k,m}(Z)$.

12 Appendix. Combinatorics of cells of the positive Grassmannian.

In [54], Postnikov classified the cells of the positive Grassmannian, showing that the *positroid cells* could be indexed by *decorated permutations* and also equivalence classes of *reduced plabic graphs*. We review these objects here. This will give us a canonical way to label each positroid by a decorated permutation or an equivalence class of plabic graphs. We refer to reader to [54] or [44, Section 2] for more details.

Definition 12.1. A *decorated permutation* on $[n]$ is a bijection $\pi : [n] \to [n]$ whose fixed points are each colored either black (loop) or white (coloop). We denote a black fixed point i by $\pi(i) = \underline{i}$, and a white fixed point i by $\pi(i) = \overline{i}$. An *anti-excedance* of the decorated permutation π is an element $i \in [n]$ such that either $\pi^{-1}(i) > i$ or $\pi(i) = \overline{i}$. We say that a decorated permutation on $[n]$ is of *type* (k, n) if it has k anti-excedances.

For example, $\pi = (3, \underline{2}, 5, 1, 6, 8, \overline{7}, 4)$ has a loop in position 2, and a coloop in position 7. It has three anti-excedances, in positions $4, 7, 8$.

Definition 12.2. Given a $k \times n$ matrix $C = (c_1, \ldots, c_n)$ written as a list of its columns, we associate a decorated permutation $\pi := \pi_C$ as follows. We set $\pi(i) := j$ to be the label of the first column j such that $c_i \in \mathrm{span}\{c_{i+1}, c_{i+2}, \ldots, c_j\}$. If c_i is the all-zero vector, we call i a loop or black fixed point and if c_i is not in the span of the other column vectors, we call i a coloop or white fixed point. We let

$$S_\pi = \{C \in Gr_{k,n}^{\geq 0} \mid \pi_C = \pi\}.$$

Postnikov showed that S_π is a cell, and that the positive Grassmannian $Gr_{k,n}^{\geq 0}$ is the union of cells S_π where π ranges over decorated permutations of type (k, n) [54, Section 16].

Decorated permutations can be equivalently thought of as affine permutations [42].

Definition 12.3. An *affine permutation* on $[n]$ is a bijection $\pi : \mathbb{Z} \to \mathbb{Z}$ such that for all $i \in \mathbb{Z}$, $\pi(i + n) = \pi(i) + n$ and $i \leq \pi(i) \leq i + n$. If $\sum_{i=1}^{n}(\pi(i) - i) = kn$ we say π is (k, n)-*bounded*.

There is a bijection between decorated permutations of type (k, n) and (k, n)-bounded affine permutations. Given a decorated permutation π_d we can define an affine permutation π_a by the following procedure: if $\pi_d(i) > i$, then define $\pi_a(i) := \pi_d(i)$; if $\pi_d(i) < i$, then define $\pi_a(i) := \pi_d(i)+n$; if $\pi_d(i)$ is a loop then define $\pi_a(i) := i$; if $\pi_d(i)$ is a coloop then define $\pi_a(i) := i+n$. For example, under this map, the decorated permutation $\pi_d = (3, \underline{2}, 5, 1, 6, 8, \overline{7}, 4)$ in the previous example gives rise to $\pi_a = (3, 2, 5, 9, 6, 8, 15, 12)$.

Let a pair (i, j) be an *inversion* of π_a if $i, j \in \mathbb{Z}, i < j$, and $\pi_a(i) > \pi_a(j)$. Two inversions (i, j) and (i', j') are *equivalent* if $i' - i = j' - j \in n\mathbb{Z}$. Then the *length* $\ell(\pi_a)$ of π_a is defined to be the number of equivalence classes of inversions. We note that $\ell(\pi_a)$ equals the number of *alignments* of the associated decorated permutation π_d (see [54, Section 5]).

Positroid cells can also be represented by *plabic graphs*.

Definition 12.4. A *plabic graph*[10] is an undirected planar graph G drawn inside a disk (considered modulo homotopy) with n *boundary vertices* on the boundary of the disk, labelled $1, \ldots, n$ in clockwise order, as well as some *internal vertices*. Each boundary vertex is incident to a single edge, and each internal vertex is colored either black or white. If a boundary vertex is incident to a leaf (a vertex of degree 1), we refer to that leaf as a *lollipop*. We will assume that G has no internal leaves except for lollipops.

10 "Plabic" stands for *planar bi-colored*.

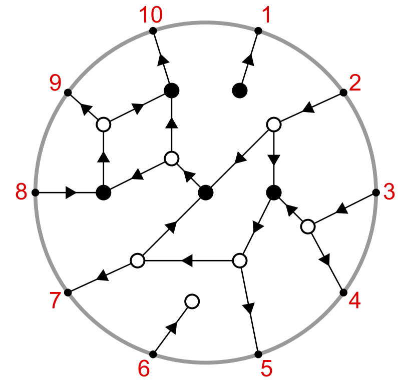

Fig. 6. A plabic graph with a perfect orientation.

Definition 12.5. A *perfect orientation* $\mathcal{O}$ of a plabic graph G is a choice of orientation of each of its edges such that each black internal vertex u is incident to exactly one edge directed away from u; and each white internal vertex v is incident to exactly one edge directed toward v. A plabic graph is called *perfectly orientable* if it admits a perfect orientation. Let $G_{\mathcal{O}}$ denote the directed graph associated with a perfect orientation $\mathcal{O}$ of G. The *source set* $I_{\mathcal{O}} \subset [n]$ of a perfect orientation $\mathcal{O}$ is the set of i which are sources of the directed graph $G_{\mathcal{O}}$. Similarly, if $j \in \bar{I}_{\mathcal{O}} := [n] - I_{\mathcal{O}}$, then j is a sink of $\mathcal{O}$.

Figure 6 shows a plabic graph with a perfect orientation. In that example, $I_{\mathcal{O}} = \{2, 3, 6, 8\}$.

All perfect orientations of a fixed plabic graph G have source sets of the same size k, where $k - (n-k) = \sum \text{color}(v) \cdot (\deg(v) - 2)$. Here the sum is over all internal vertices v, $\text{color}(v) = 1$ for a black vertex v, and $\text{color}(v) = -1$ for a white vertex; see [54]. In this case we say that G is of *type* (k, n).

Now let us connect plabic graphs to the positroids and positroid cells from Definition 2.2.

Theorem 12.6 ([54, Section 11]). Let G be a plabic graph of type (k, n). Then we have a positroid M_G on $[n]$ defined by

$$M_G = \{I_{\mathcal{O}} \mid \mathcal{O} \text{ is a perfect orientation of } G\},$$

where $I_{\mathcal{O}}$ is the set of sources of $\mathcal{O}$. Moreover, every positroid cell has the form S_{M_G} for some plabic graph G.

One can also read off the positroid from G using *flows* [67] or perfect matchings.

If a plabic graph G is *reduced* (see [54, Section 12]) or [30, Chapter 7]), we have that $S_{M_G} = S_{\pi_G}$, where π_G is the decorated permutation defined as follows.

Definition 12.7. Let G be a reduced plabic graph with boundary vertices $1, \ldots, n$. For each boundary vertex $i \in [n]$, we follow a path along the edges of G starting at i, turning (maximally) right at every internal black vertex, and (maximally) left at every internal white vertex. This path ends at some boundary vertex $\pi(i)$. By [54, Section 13], the fact that G is reduced implies that each fixed point of π is attached to a lollipop; we color each fixed point by the color of its lollipop. This defines a decorated permutation, called the *decorated trip permutation* $\pi_G = \pi$ of G.

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