Meteor science

Estimating meteor rates using Bayesian inference

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A method for estimating the true meteor rate λ from a small number of observed meteors n is derived. We employ Bayesian inference with a Poissonian likelihood function. We discuss the choice of a suitable prior and propose the adoption of Jeffreys prior, $P(\lambda) = \lambda^{-0.5}$, which yields an expectation value $E(\lambda) = n + 0.5$ for any $n \geq 0$. We update the ZHR meteor activity formula accordingly, and explain how 68%- and 95%-confidence intervals can be computed.

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1 Introduction

The formula commonly used to estimate the Zenithal Hourly Rate (ZHR) of meteors as given in the Handbook of the International Meteor Organization (IMO; Rendtel & Arlt, 2008) and used in many meteor activity graphs (e.g., Arlt & Barentsen, 2006), is given by:

$$\mathrm{E(ZHR)} = \frac{n_{\mathrm{tot}} + 1}{T} \quad \text{with} \quad \sigma = \frac{\sqrt{n_{\mathrm{tot}} + 1}}{T}, \qquad (1)$$

where n_{tot} is the total number of meteors counted in a number of observing intervals ($n_{\mathrm{tot}} = \sum_i n_i$) and T is the observing time weighted by a given correction factor for each interval ($T = \sum_i T_{\mathrm{eff},i}/C_i$).

The use of $n_{\mathrm{tot}} + 1$ rather than n_{tot} often surprises observers, because it yields a ZHR which is larger than zero even when no meteors are observed. Whilst Arlt (1999) already indicated that $n_{\mathrm{tot}}+1$ is used to account for the effects of small-number statistics, this paper will explain the formula in more detail. However, contrary to the earlier publications, we will suggest that $n_{\mathrm{tot}}+0.5$ rather than $n_{\mathrm{tot}} + 1$ is the most appropriate formula.

We will first describe the problem of small-number statistics in §2. We then describe the solution using Bayesian inference in §3, followed by a discussion on the choice of the prior assumptions in §4. In §5 we explain how confidence intervals may be computed, and in §6 we provide examples. Finally in §7 we discuss the effect of correction factors and in §8 we present the conclusions.

2 Problem description

As explained by Bias (2011) in a recent issue of this journal, the *observed* meteor rate often tends to underestimate the *true* meteor rate due to small-number statistics. This may be understood using the following example: if we would attempt to estimate the frequency of winning the lottery based on a small group of 10 players, we are most likely to find that none of these players have ever won and that the winning frequency equals zero. Of course the true probability of winning the lottery is slightly larger than zero, but the quantity cannot reliably be obtained by extrapolating from a small number of players.

An identical effect occurs when the ZHR is extrapolated from a low number of meteors. Indeed, even when zero meteors are observed in an interval of finite length, the true average rate may well have been larger than zero because we know that the interplanetary space is not empty. The observer might have been unlucky, or the interval might have been short with respect to the true rate. Such situation may occur even during major showers, e.g. when computing rates for 1-minute intervals. The ZHR formula must take this into account in order to produce reliable estimates in all situations.

3 Solution: Bayesian inference

A common technique used in statistics to estimate parameters in the presence of sparse data is called *Bayesian inference*. In brief, one constructs a parameterized model which one thinks describes the source of the data. The probability of this model to have produced the data is then computed for each possible set of parameters, taking into account any known prior constraints on the parameters. The resulting set of probabilities is called the posterior distribution, from which expectation values and confidence intervals for the free parameters may be derived.

In our case, the data is the observed meteor rate n (in arbitrary time T). Our only free model parameter is the true meteor rate λ (i.e. ZHR). There are many values of λ which may explain a given n, each having the conditional probability $P(\lambda|n)$. This notation means the probability of finding λ given that one has seen n meteors. This is the posterior probability described earlier which may be estimated using the theorem by Bayes:

$$P(\lambda|n) = \frac{P(n|\lambda)\, P(\lambda)}{P(n)}, \qquad (2)$$

where $P(n|\lambda)$ is the generative model, i.e. the known probability to see n meteors for a given true rate λ. We may assume that meteors appear in a random way following a Poissonian law for independent events:

$$P(n|\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}. \qquad (3)$$

The function $P(n)$ serves to normalize the distribution to unity, while $P(\lambda)$ expresses what we know about how probable each of the possible true rates λ are. This function is called the prior and, ideally, should be a probability distribution on its own. The challenge is to decide what we can assume about $P(\lambda)$ beforehand?

4 Choice of the prior

The criteria for choosing a suitable prior $P(\lambda)$ is a subject of debate in the statistical community. On one hand, one may decide to construct a prior based on previous evidence, for example the meteor activity in the past decade. This is called an *informative prior*. On the other hand, one may prefer a prior which contains only vague or general information and is not biased towards past observations. This is called an *objective prior*.

Whether or not it is appropriate to include previous observations in the computation of meteor rates is a philosophical question. However, given the intrinsically variable nature of meteor showers, we suggest that an objective prior is the only practical approach. We discuss the choice of such prior in what follows.

4.1 Uniform prior

A natural choice is a uniform prior $P(\lambda)$ = constant, i.e. all values of λ are assumed to be equally likely. A uniform prior leads to a distribution which cannot be normalized, and is called an improper prior. Any limit on ZHR, and be it very high, makes the distribution normalizable though, that is, asymptotically a uniform prior is not a problem.

Let us see what the uniform prior implies for the inferred rate. The expectation value of the true meteor rate is defined as the sum of every possible value of λ times its posterior probability. (In this paper, we also use the terms "average" or "mean" as synonyms for expectation value.) For a continuous quantity such as λ, the expectation value is a normalized integral over all λ which, for a uniform prior, reads:

$$\begin{aligned}
\mathrm{E}(\lambda) &= \int_0^\infty \lambda\, P(\lambda|n)\,\mathrm{d}\lambda \\
&= \int_0^\infty \lambda\, \frac{\lambda^n}{n!}\, e^{-\lambda}\,\mathrm{d}\lambda \\
&= (n+1)\int_0^\infty \frac{\lambda^{n+1}}{(n+1)!}\, e^{-\lambda}\,\mathrm{d}\lambda \\
&= n+1. \qquad (4)
\end{aligned}$$

Similarly, the error estimate for $\mathrm{E}(\lambda)$ follows from the mathematical definition of the variance, often referred to as σ^2:

$$\begin{aligned}
\sigma^2(\lambda) &= \mathrm{E}[(\lambda-\mathrm{E}(\lambda))^2] \\
&= \int_0^\infty P(\lambda|n)\,(\lambda-(n+1))^2\,\mathrm{d}\lambda \\
&= \int_0^\infty \frac{(n+1)(n+2)\lambda^{n+2}e^{-\lambda}}{(n+2)!} - \\
&\quad -\frac{2(n+1)^2\lambda^{n+1}e^{-\lambda}}{(n+1)!} + \\
&\quad +\frac{(n+1)^2\lambda^n e^{-\lambda}}{n!}d\lambda \\
&= (n+1)(n+2)-(n+1)^2 \\
&= n+1. \qquad (5)
\end{aligned}$$

The expectation value of the true activity rate is thus given by $(n+1)$ with spread of the distribution of $\sigma = \sqrt{n+1}$. These quantities are then simply divided by the weighted time T to obtain the ZHR. This explains the formula given in the IMO Handbook.

4.2 Exponential prior

A uniform prior may not be ideal if we want to express the fact that very large rates are very unlikely. Almost all rates ever obtained are below say 1000 meteors per hour. A suitable prior will decrease with rate, and it is mathematically convenient to use power functions like

$$P(\lambda) = 1/\lambda^{1-\alpha}, \qquad (6)$$

where $0 \le \alpha \le 1$. Such a power-law has the advantage that we can use Γ-functions for the derivation of $E(\lambda)$. The resulting expectation value now involves the prior and is

$$E(\lambda) = \frac{\int_0^\infty \lambda\, P(\lambda)P(n|\lambda)d\lambda}{\int_0^\infty P(\lambda)P(n|\lambda)d\lambda}, \qquad (7)$$

where the lower integral is to normalize the posterior distribution. Now, let's make use of the Γ-functions for which

$$n! = n\Gamma(n) = \Gamma(n+1) = \int_0^\infty \lambda^n e^{-\lambda}d\lambda \qquad (8)$$

holds. We do not need to know how Γ is computed, we only need its properties to derive the expectation value:

$$\begin{aligned}
E(\lambda) &= \frac{\int_0^\infty \frac{1}{\lambda^{1-\alpha}}\frac{\lambda^n\,\lambda}{\Gamma(n+1)}e^{-\lambda}d\lambda}{\int_0^\infty \frac{1}{\lambda^{1-\alpha}}\frac{\lambda^n}{\Gamma(n+1)}e^{-\lambda}d\lambda} \\
&= \frac{\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}}{\frac{\Gamma(n+\alpha)}{\Gamma(n+1)}} \\
&= \frac{(n+\alpha)\Gamma(n+\alpha)}{\Gamma(n+\alpha)} \\
&= n+\alpha. \qquad (9)
\end{aligned}$$

Similarly, the variance is:

$$\begin{aligned}
\sigma^2(\lambda) &= E\left[\Big(\lambda-E(\lambda)\Big)^2\right] = \\
&= \frac{\int_0^\infty P(\lambda)P(n|\lambda)\Big(\lambda-(n+\alpha)\Big)^2 d\lambda}{\int_0^\infty P(\lambda)P(n|\lambda)d\lambda} \\
&= \frac{(n+1+\alpha)(n+\alpha)\Gamma(n+\alpha)}{\Gamma(n+\alpha)} - \\
&\quad -\frac{2(n+\alpha)^2\Gamma(n+\alpha)+(n+\alpha)^2\Gamma(n+\alpha)}{\Gamma(n+\alpha)}
\end{aligned}$$

$$= (n+1+\alpha)(n+\alpha) - 2(n+\alpha)^2 + (n+\alpha)^2$$
$$= n+\alpha, \qquad (10)$$

The expectation value of the true activity rate is thus given by $(n+\alpha)$ with a spread $\sigma = \sqrt{n+\alpha}$, for a given prior $1/\lambda^{1-\alpha}$. Again, these priors are normalizable for $0 \le \alpha \le 1$ by enforcing a limitation of the valid range.

One question remains: which is the most appropriate value to adopt for α? On one hand, the uniform prior ($\alpha = 1$) yields the expectation value $n+1$, which is likely to overestimate the meteor rate due to assumption that any arbitrarily large rate is equally likely as the zero rate. On the other hand, the prior $1/\lambda$ ($\alpha = 0$) yields a "traditional" extrapolation $E(\lambda) = n$, which is likely to underestimate the rate as explained previously. It appears appropriate to adopt a prior somewhere in-between $0 < \alpha < 1$.

4.3 Jeffreys prior; $\alpha = 0.5$

The problem of choosing a suitable prior for a Poissonian process exists in other fields (e.g. radioactive decay counts, neutrino detections). An axiomatic solution has previously been proposed by Harold Jeffreys. He required that a prior should be "invariant under reparameterization", i.e. a prior should not depend on the variable investigated (Jeffreys 1946, 1961; Kass & Wasserman 1996). One could, for example, be interested in the rate λ as well as in the mean time between events $\mu = 1/\lambda$. The priors for both expectation values should be compatible. For general relations between different quantities, the requirement of compatibility is written mathematically as

$$P(\lambda)d\lambda = P(\mu)d\mu. \qquad (11)$$

For a Poissonian distribution, such a general compatibility is achieved for a prior $\alpha = 0.5$. In this case, the estimate is not specific to either computing the rate or the mean time lapse or something else.

Because there are no further statistical principles to decide which prior is to be preferred, we suggest the use of Jeffreys prior

$$P(\lambda) = 1/\lambda^{0.5}, \qquad (12)$$

for future estimates of the rate. The expectation value for the meteor rate is then

$$E(\lambda) = (n+0.5) \pm \sqrt{n+0.5}. \qquad (13)$$

An illustration of the posterior distribution $P(\lambda|n)$ under the assumption of Jeffreys' prior is shown in Figure 1 for different values of n.

Note that in most cases, when n is large ($n \gg 0.5$), the differences between the various priors are negligable.

5 Confidence intervals

The standard deviation $\sigma = \sqrt{n+0.5}$ characterizes the spread in the posterior distribution around the expectation value. In the case of a Gaussian distribution, the standard deviation corresponds to a confidence interval (i.e. the 68%-confidence interval of a Gaussian distribution is located between -1σ and $+1\sigma$ from the mean).

However, the posterior $P(\lambda|n)$ does not follow a Gaussian shape and is more similar to a Poissonian distribution (though a Gaussian shape is approached for large n). The posterior is asymmetric with a tail towards high meteor rates, which makes it somewhat misleading to characterize the uncertainty with a single number. A better way to characterize the uncertainty is to compute the (asymmetric) error margins of the 68%-confidence interval. This may be done as follows.

Given the posterior distribution

$$P(\lambda|n) = \frac{\lambda^{n-1+\alpha} e^{-\lambda}}{\Gamma(n+\alpha)} \qquad (14)$$

The corresponding cumulative distribution function is

$$P'(\lambda \le \lambda_{\text{marg}}|n) = \int_0^{\lambda_{\text{marg}}} \frac{t^{n-1+\alpha} e^{-t}}{\Gamma(n+\alpha)} dt = 1 - \frac{\Gamma(\alpha+n, \lambda_{\text{marg}})}{\Gamma(\alpha+n)}, \qquad (15)$$

with the incomplete Γ function which for integers $n > 0$ can be computed as

$$\Gamma(n, \lambda_{\text{marg}}) = (n-1)!\, e^{-\lambda_{\text{marg}}} \sum_{k=0}^{n-1} \frac{\lambda_{\text{marg}}^k}{k!} \qquad (16)$$

By integrating the cumulative distribution function numerically for different probabilities ($P' = 2.5\%$, 16%, 84%, and 97.5%), we obtain useful quantiles which correspond to the central 68%- and 95%-confidence intervals.

These quantiles are shown in Table 1, given as a multiplier of the true rate. For example, after computing the ZHR, one may look up the corresponding relative margins for a given n_{tot} in Table 1 and obtain the 68%-interval by computing ZHR $\cdot\, \delta_{68,\text{low}}$ (negative margin) and ZHR $\cdot\, \delta_{68,\text{high}}$ (positive margin).

It is interesting to note that the 68% interval approaches symmetry from $n_{\text{tot}} \gtrsim 3$, while the 95% interval is more sensitive to the wings and remains asymmetric even beyond $n_{\text{tot}} \gg 1000$.

For n_{tot} larger than about 30, the 68%-interval approaches a Gaussian shape and the margins can be approximated using $\sigma = \sqrt{n_{\text{tot}} + 0.5}/T$ $(= \text{ZHR}/\sqrt{n_{\text{tot}} + 0.5})$.

Finally, we remind the reader that the margins in Table 1 only represent the uncertainty which is due to Poissonian statistics. The true uncertainty of a ZHR estimate is likely to be somewhat larger due to observing errors (e.g., uncertainty in the limiting magnitude determination). Fortunately, the impact of such observing errors is likely to be small when data from a sufficient number of independent observers is averaged.

6 Examples

Now let us consider the formula using a few examples. If one meteor was seen in five minutes, the rate equals

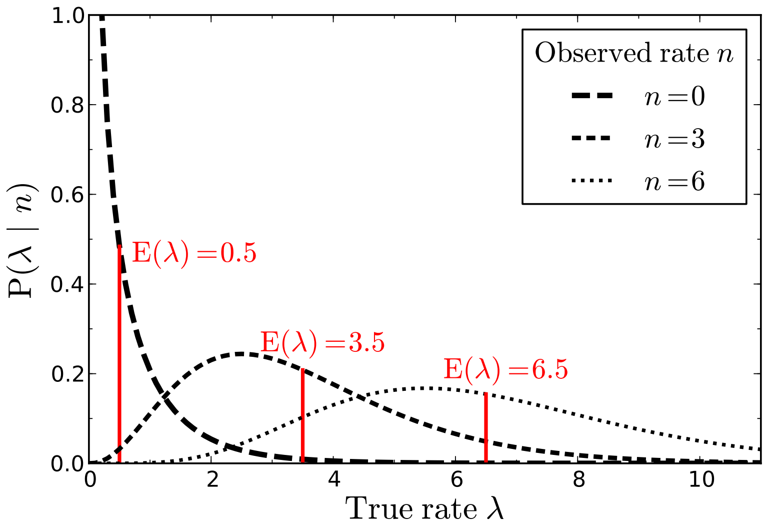

Figure 1: Posterior distribution $P(\lambda|n)$ of the true meteor rate under Jeffreys' prior ($\alpha = 0.5$), plotted for different observed rates ($n = 0, 3, 6$). Vertical lines indicate the position of the expectation values $\mathrm{E}(\lambda) = n + 0.5$.

Table 1: Margins of the 68%- and 95%-confidence intervals, given as multipliers to the ZHR. After computing ZHR $= (n_{\mathrm{tot}} + 0.5)/T$, the ±-values can be obtained by computing ZHR $\cdot\, \delta_{68,\mathrm{low}}$ (negative margin) and ZHR $\cdot\, \delta_{68,\mathrm{high}}$ (positive margin) for the appropriate n_{tot}.

n_{tot}	$\delta_{95,\mathrm{low}}$	$\delta_{68,\mathrm{low}}$	$\delta_{68,\mathrm{high}}$	$\delta_{95,\mathrm{high}}$
0	-1.00	**-0.96**	**+0.99**	+4.02
1	-0.93	**-0.72**	**+0.73**	+2.12
2	-0.83	**-0.59**	**+0.59**	+1.57
3	-0.76	**±0.51**		+1.29
4	-0.70	**±0.45**		+1.11
5	-0.65	**±0.41**		+0.99
6	-0.61	**±0.38**		+0.90
7	-0.58	**±0.36**		+0.83
8	-0.56	**±0.34**		+0.78
9	-0.53	**±0.32**		+0.73
10	-0.51	**±0.30**		+0.69
11	-0.49	**±0.29**		+0.66
12	-0.48	**±0.28**		+0.63
13	-0.46	**±0.27**		+0.60
14	-0.45	**±0.26**		+0.58
15	-0.43	**±0.25**		+0.56
16	-0.42	**±0.24**		+0.54
17	-0.41	**±0.24**		+0.52
18	-0.40	**±0.23**		+0.50
19	-0.39	**±0.22**		+0.49
20	-0.39	**±0.22**		+0.48
22	-0.37	**±0.21**		+0.45
24	-0.36	**±0.20**		+0.43
26	-0.34	**±0.19**		+0.42
28	-0.33	**±0.19**		+0.40
30	-0.32	**±0.18**		+0.38

$E(\mathrm{ZHR}) = \frac{n+0.5}{T} = 18.0^{+13.1}_{-13.0}$. When zero meteors are seen during five minutes, the rate equals ZHR $= 6.0^{+5.9}_{-5.8}$. The error margins are actually so close to symmetric that we can always give a single value for the error bars, i.e. ZHR $= 18 \pm 13$ and ZHR $= 6 \pm 6$ respectively. Although the rounded margins suggest so, the lower margin is not zero!

If zero meteors were seen in four hours, the rate equals to ZHR $= 0.125^{+0.124}_{-0.120}$ Indeed, rates can only be constrained to values close to zero when no meteors are observed for a very long period. A more "normal" case for a minor shower would be say a total of 12 meteors in a total of 4 hours, delivering ZHR $= 3.1 \pm 0.9$. A larger number of meteors of say 34 meteors in 11 hours gives simply ZHR $= 3.1 \pm 0.5$ with the above error for large meteor numbers.

7 The influence of correction factors

In the previous sections we have ignored the specific correction factors which are used to obtain a standardized ZHR (e.g. to account for limiting magnitude and radiant elevation). Given two rates; one without correction, $\hat{\lambda}$ and one with correction, λ, we have assumed there is a factor f which does *not* depend on the rate:

$$f\,\lambda = \hat{\lambda} \tag{17}$$

Indeed, for a set of N observing periods being combined in one expectation value for λ, all having different correction factors f_i, we obtain

$$\begin{aligned} E(\lambda) &= \frac{\int_0^\infty \frac{1}{\lambda^{1-\alpha}}\lambda \frac{(f_1\lambda)^{n_1}(f_2\lambda)^{n_2}\cdots(f_N\lambda)^{n_N}}{n_1!n_2!\cdots n_N!} e^{-\sum f_i\lambda} d\lambda}{\int_0^\infty \frac{1}{\lambda^{1-\alpha}} \frac{(f_1\lambda)^{n_1}(f_2\lambda)^{n_2}\cdots(f_N\lambda)^{n_N}}{n_1!n_2!\cdots n_N!} e^{-\sum f_i\lambda} d\lambda} \\ &= \frac{\Gamma(n_{\mathrm{tot}}+1+\alpha)}{(\sum f_i)^{n_{\mathrm{tot}}+1+\alpha}} \frac{(\sum f_i)^{n_{\mathrm{tot}}+\alpha}}{\Gamma(n_{\mathrm{tot}}+\alpha)} \\ &= \frac{n_{\mathrm{tot}}+\alpha}{\sum f_i}. \end{aligned} \tag{18}$$

In other words, our method may be applied regardless of the values of the correction factors.

8 Discussion and conclusion

In conclusion, we recommend to compute ZHR values using the term $n + 0.5$:

$$\mathrm{E(ZHR)} = \frac{(n_{\mathrm{tot}} + 0.5)\ r^{6.5-\mathrm{lm}}}{T_{\mathrm{eff}} \sin h_{\mathrm{R}}}, \tag{19}$$

with error margins:

$$\Delta\mathrm{ZHR} = \frac{\mathrm{ZHR}}{\sqrt{n_{\mathrm{tot}} + 0.5}}. \tag{20}$$

Note that for $n_{\mathrm{tot}} \leq 30$, the error margins listed in Table 1 should be used instead of Eqn. 20 to obtain a 68%-confidence interval.

This method of computing the ZHR adds a small bias taking into account the asymmetry of possible rates. In particular, it is essential to adopt the method whenever rates are based on less than ~ 10 meteors. Such situations commonly occur when a major shower is analysed using very short (e.g. 1-minute) intervals. When computing a rate based on a number of observing periods (indexed with i), *never ever* compute the rate from $n_i + 0.5$ for individual periods and average them afterwards. A more accurate estimate is based on the sum of meteors from these observing periods, and hence on $n_{\mathrm{tot}} + 0.5$. Finally, it is, of course, much better to have enough observations and large enough n_{tot} that the subtleties of choosing a good prior are no longer important, i.e. $n_{\mathrm{tot}} \gg 0.5$.

Comprehensive information about Bayesian inference in general and the choice of priors can be found in e.g. Kass &Wasserman (1996) and Bolstad (2007; Chapter 10 for priors).

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