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Higher current algebras, homotopy Manin triples, and a rectilinear adelic complex

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ABSTRACT

The notion of a Manin triple of Lie algebras admits a generalization, to dg Lie algebras, in which various properties are required to hold only up to homotopy.

This paper introduces two classes of examples of such *homotopy Manin triples*. These examples are associated to analogs in complex dimension two of, respectively, the punctured formal 1-disc, and the complex plane with multiple punctures. The dg Lie algebras which appear include certain *higher current algebras* in the sense of Faonte, Hennion and Kapranov [18].

We work in a ringed space we call *rectilinear space*, and one of the tools we introduce is a model of the derived sections of its structure sheaf, whose construction is in the spirit of the adelic complexes for schemes due to Parshin and Beilinson.

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1. Introduction

1.1. Higher current algebras and homotopy Manin triples

Manin triples are fundamental objects in integrable systems and quantum groups. Recall that a Manin triple $(\mathfrak{a}, \langle - \mid - \rangle, \mathfrak{a}_+, \mathfrak{a}_-)$ is a Lie algebra $\mathfrak{a}$ with a symmetric nondegenerate invariant bilinear form, and two isotropic Lie subalgebras $\mathfrak{a}_\pm \subset \mathfrak{a}$ such that $\mathfrak{a} =_{\mathbb{C}} \mathfrak{a}_+ \oplus \mathfrak{a}_-$ as vector spaces [12]. One important class of examples arises when $\mathfrak{a}$ is a current algebra: namely, one may take

$$\mathfrak{a} = \mathfrak{g} \otimes \mathbb{C}((z)), \quad \mathfrak{a}_+ = \mathfrak{g} \otimes \mathbb{C}[[z]], \quad \mathfrak{a}_- = \mathfrak{g} \otimes z^{-1}\mathbb{C}[z^{-1}] \tag{1}$$

and $\langle x \otimes f(z) \mid y \otimes g(z) \rangle = \kappa\big(x|y\big)\,\mathrm{res}_z\, f(z)g(z)$, with $\mathfrak{g}$ any finite-dimensional simple Lie algebra over $\mathbb{C}$ and $\kappa\big(-|-\big)$ its standard bilinear form. (See, e.g., [9,3].)

Here the commutative algebras $\mathbb{C}[[z]]$, of formal Taylor series, and $\mathbb{C}((z))$, of formal Laurent series, can be seen as algebras of functions

$$\mathbb{C}[[z]] \cong \Gamma(\mathrm{Disc}_1, \hat{\mathcal{O}}), \qquad \mathbb{C}((z)) \cong \Gamma(\mathrm{Disc}_1^\times, \hat{\mathcal{O}}),$$

on, respectively, the formal disc Disc_1, and the punctured formal disc $\mathrm{Disc}_1^\times := \mathrm{Disc}_1 \setminus \{\mathrm{pt.}\}$, in complex dimension one.

It is natural to try to extend this to higher dimensions, but an apparent obstacle arises: in dimension $n \geq 2$, the structure sheaf $\hat{\mathcal{O}}$ admits no more sections over the punctured disc $\mathrm{Disc}_n^\times = \mathrm{Disc}_n \setminus \{\mathrm{pt.}\}$ than it does over the disc Disc_n itself: $\Gamma(\mathrm{Disc}_n^\times, \hat{\mathcal{O}}) = \Gamma(\mathrm{Disc}_n, \hat{\mathcal{O}}) \cong \mathbb{C}[[z_1, \dots, z_n]]$. So the would-be higher current algebra $\mathfrak{g} \otimes \Gamma(\mathrm{Disc}_n^\times, \hat{\mathcal{O}})$ seems to be "missing all the negative modes". However, Faonte, Hennion and Kapranov [18] – and see also [33,34,25] – make the following observation: one may replace these algebras of functions by their derived analogs, $R\Gamma(\mathrm{Disc}_2, \hat{\mathcal{O}})$ and $R\Gamma(\mathrm{Disc}_2^\times, \hat{\mathcal{O}})$. By definition $R\Gamma(\mathrm{Disc}_2^\times, \hat{\mathcal{O}})$ is a cochain complex whose cohomology computes the sheaf cohomology of $\hat{\mathcal{O}}$, as we recall in §2.4 below. And it is well known that the structure sheaf $\hat{\mathcal{O}}$ has higher cohomology when $n \geq 2$. In this paper we focus exclusively on the case of dimension $n = 2$, where one has

$$H^\bullet(\mathrm{Disc}_2^\times, \hat{\mathcal{O}}) \cong \begin{cases} \mathbb{C}[[w, z]] & \bullet = 0 \\ w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] & \bullet = 1 \\ 0 & \text{otherwise.} \end{cases}$$

One sees that natural candidates for the "missing" negative modes reemerge in the first cohomology. Moreover, as stressed in [18], the derived sections form more than just a cochain complex: $R\Gamma(\mathrm{Disc}_2^\times, \hat{\mathcal{O}})$ is naturally a differential graded (dg) commutative algebra (unique up to zigzags of quasi-isomorphisms). Thus, one obtains the dg Lie algebra

$$\mathfrak{g} \otimes R\Gamma(\mathrm{Disc}_2^\times, \hat{\mathcal{O}}),$$

which it is natural to call a *higher current algebra*.

The first goal of this paper is to give an analog of the Manin triple (1) above, in complex dimension two, following this philosophy. To do so, we first need to clarify what it should mean to give a Manin triple in the differential graded setting. We give our definition in Section 6: roughly speaking, it says that a *homotopy Manin triple (in dg Lie algebras)* is a triple of dg Lie algebras and maps of dg Lie algebras between them

$$\mathfrak{a}_+ \xrightarrow{\iota_+} \mathfrak{a} \xleftarrow{\iota_-} \mathfrak{a}_-,$$

such that there is a homotopy equivalence $\mathfrak{a} \simeq \mathfrak{a}_+ \oplus \mathfrak{a}_-$ of dg vector spaces; together with an invariant pairing $\langle - \mid - \rangle$ on $\mathfrak{a}$ which is non-degenerate up to homotopy, and for which $\mathfrak{a}_\pm$ are isotropic, again up to homotopy.

Manin L_∞-triples – or strongly homotopy Manin triples of L_∞ algebras – have been defined previously in [37]. The definition we give here is compatible with that definition in a sense we discuss in Appendix C.

Then our first result, Theorem 14, gives an example of such a homotopy Manin triple. We call it a "local" homotopy Manin triple because it is associated to the formal punctured polydisc at a point in complex dimension two. Namely it has

$$\mathfrak{a} = \mathfrak{g} \otimes R\Gamma(\mathrm{PDisc}_2^\times, \hat{\mathcal{O}}), \qquad \mathfrak{a}_+ = \mathfrak{g} \otimes R\Gamma(\mathrm{PDisc}_2, \hat{\mathcal{O}}) \cong \mathfrak{g} \otimes \mathbb{C}[[w]] \otimes \mathbb{C}[[z]] \tag{2}$$

and $\mathfrak{a}_-$ a certain differential graded Lie algebra whose cohomology is a copy of

$$H^1(\mathfrak{a}) \cong \mathfrak{g} \otimes w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}]$$

sitting in cohomological degree one.

(We shall explain the meaning of the *poly*disc, $\mathrm{PDisc}_2^\times$, in a moment.)

Next, our main result is Theorem 24, which exhibits what we call a "global" homotopy Manin triple. It is an analog, in complex dimension two, of another important and familiar class of Manin triples, namely those of the form

$$\mathfrak{a} = \mathfrak{g} \otimes \bigoplus_{i=1}^{N} \mathbb{C}((x - a_i)), \quad \mathfrak{a}_+ = \mathfrak{g} \otimes \bigoplus_{i=1}^{N} \mathbb{C}[[x - a_i]]$$

and

$$\mathfrak{a}_- = \mathfrak{g} \otimes \mathbb{C}(x)^{\infty}_{a_1,\dots,a_N} \tag{3}$$

where $\mathbb{C}(x)^{\infty}_{a_1,\dots,a_N}$ denotes the commutative algebra of rational expressions in x vanishing at ∞ and singular at most at the finitely many distinct points $a_1, \dots, a_N \in \mathbb{C}$. One may think of these points

$$a_1, \dots, a_N$$

as *marked points* or *punctures* in the complex plane. This family of Manin triples is – after introducing a central extension – closely related to rational Gaudin models [16] and rational conformal blocks on the Riemann sphere; see [14] and references therein.

The homotopy Manin triple $(\mathfrak{a}, \mathfrak{a}_+, \mathfrak{a}_-)$ we introduce in Theorem 24 is also defined by a finite collection of distinct marked points,

$$(w_1, z_1), \dots, (w_N, z_N),$$

which now live in $\mathbb{C}^2$. For $\mathfrak{a}$ and $\mathfrak{a}_+$ it has direct sums of copies of the dg Lie algebras mentioned in (2) above, one for each marked point:

$$\mathfrak{a} \cong \bigoplus_{i=1}^{N} \mathfrak{g} \otimes R\Gamma(\mathrm{PDisc}_2^{\times}(w_i, z_i), \hat{\mathcal{O}}), \qquad \mathfrak{a}_+ \cong \bigoplus_{i=1}^{N} \mathfrak{g} \otimes R\Gamma(\mathrm{PDisc}_2(w_i, z_i), \hat{\mathcal{O}}),$$

and it has

$$\mathfrak{a}_- = \mathfrak{g}_{\mathrm{Global}},$$

a certain dg Lie algebra which we shall describe in detail in Section 8 below.

We should stress that the existence of this homotopy Manin triple of Theorem 24 should not be seen as particularly surprising, conceptually. It is essentially implicit already in [18]: see Proposition 1.1.4 there and Proposition 8 below. Our main goal in the present paper is rather to give explicit *models*, in dg Lie algebras, for the various derived algebras above and the maps between them, and to give explicit descriptions of the various homotopies. Our hope is that these models will prove to be useful for doing concrete calculations.

Now we indicate how we construct these models and introduce the other main theme of the present paper, which is what we call the *rectilinear* setting.

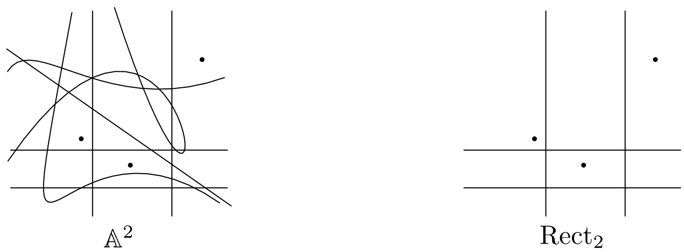
1.2. The rectilinear setting

At this point we are about to do something which, to an algebraic geometer, will probably appear mildly barbarous.

Rather than starting with the usual affine space $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[w, z]$, in this paper we are going to start instead with the product *in topological spaces* of two copies of the affine line $\mathbb{A}^1$. We shall call the resulting space,

$$\mathrm{Rect}_2 := \mathbb{A}^1 \times^{\mathbf{Top}} \mathbb{A}^1,$$

rectilinear space. It has the same set of closed points as $\mathbb{A}^2$, namely $\{(a, b) \in \mathbb{C}^2\}$, but very many fewer generalized points: it has points corresponding to the rectilinear lines $w = a$ and $z = a$, $a \in \mathbb{C}$, but it lacks points corresponding to all the other algebraic curves:



We get a ringed space $(\mathrm{Rect}_2, \mathcal{O})$ where $\mathcal{O}$ is the sheaf of commutative algebras whose spaces of local sections are spanned by products $f(w)g(z)$ of rational functions.

Similarly, locally, in place of the formal 2-disc $\mathrm{Disc}_2 = \mathrm{Spec}\,\mathbb{C}[[w,z]]$, we shall work with the product in topological spaces of two copies of the formal 1-disc,

$$\mathrm{PDisc}_2 := \mathrm{Disc}_1 \times^{\mathbf{Top}} \mathrm{Disc}_1,$$

which we shall call the formal rectilinear polydisc. Like Disc_2, it has a single closed point, $(0,0)$, and generalized points corresponding to the lines $w=0$ and $z=0$, but it lacks points corresponding to all the other germs of algebraic curves:

Disc_2

PDisc_2

This choice breaks symmetry, and results in a space with fewer good properties: $\mathbb{A}^2$ is an affine scheme; Rect_2 is merely a ringed space. To motivate it, let us give some background on the problem in mathematical physics we are seeking to develop tools to address.

1.3. Motivation

We are ultimately interested in solving the spectral problem for integrable quantum field theories in $1+1$ spacetime dimensions. The specific approach we have in mind originates in the paper [15], in which Feigin and Frenkel make the important observation that certain integrable quantum field theories can be seen as quantum Gaudin models of affine type. See also [17]. And indeed, Vicedo [54] showed that large classes of classical integrable field theories can be realized as classical Gaudin models of affine type. See also [10,39,40,1].

By "solving the spectral problem for an integrable quantum theory", we mean roughly speaking defining a hierarchy of mutually commuting conserved quantities ("higher Hamiltonians") and then characterizing their joint spectra on suitable classes of representations.

Quantum Gaudin models of *finite* type are among the best-understood quantum integrable systems. In particular, there is a complete description of their higher Hamiltonians, which generate what is known variously as a Bethe or Gaudin algebra [21,45,46,53], and of the spectra of these higher Hamiltonians, which is given in terms of certain local systems called opers [4,20,47]. This description can be interpreted as the geometric incarnation of the Langlands correspondence [22].

At least from one perspective, tools from chiral conformal field theory – namely, vertex algebras and rational conformal blocks – are key to establishing these results. These rational conformal blocks are defined *on the spectral plane*, i.e. the copy of the complex plane $\mathbb{C}$ with marked points which defines any rational Gaudin model; a Gaudin model of finite type is not in any sense a field theory itself.

Quantum Gaudin models of affine type are less systematically understood (though see [15,17,42,43,57,35,23]). One problem is that it is difficult to extend tools which work in finite type to affine types; see e.g. [58] for an attempt to do so for the Wakimoto construction/Feigin-Frenkel homomorphism. In any case, any attempt to generalise from a finite type algebra $\mathfrak{g}$ to an affine type algebra $\widehat{\mathfrak{g}}$ is almost bound to be missing half of the story if both are merely regarded as Kac-Moody algebras, because it ignores the geometrical interpretation of $\widehat{\mathfrak{g}}$ as a centrally extended current algebra, associated to a punctured disc; which is to say, morally speaking, that it ignores the fact that the Gaudin model is describing a field theory. Indeed, vertex algebras, and the additional control they give over the representation theory of $\widehat{\mathfrak{g}}$, should again enter the picture, but these vertex algebras should be associated to a copy of $\mathbb{C}$ which is quite distinct from the spectral plane. In the case of chiral conformal field theories at least, this new copy of $\mathbb{C}$ is morally the *worldsheet* of the field theory.

Thus, one expects to be in a situation in which there are two copies of the complex plane in play,

$$\mathbb{C}_{\text{spectral plane}} \times \mathbb{C}_{\text{worldsheet}},$$

whose interpretations are conceptually quite distinct. Let w, z be the Cartesian coordinates. It is natural to think that one will ultimately want to attach representation-theoretic data to:

- rectilinear lines, e.g. fibres $(w=a)$ over points in the spectral plane,
- points (a,b), and probably also
- rectilinear flags, e.g.

$$(a,b) \subset (w=a) \subset \mathbb{C}^2. \tag{4}$$

It is much less clear that one needs other curves such as $(w=z)$ or $(w=z^2)$ in this context; at the very least, the rectilinear lines and flags certainly enjoy a preferred status. To say much the same thing another way: one expects to encounter functions belonging to $\mathbb{C}(w) \otimes \mathbb{C}(z)$, such as $\frac{1}{w-a}\frac{1}{z-b}$, but the function $\frac{1}{w-z}$, for example, looks very strange in this context since the physical meanings of the coordinates w and z are so different.

It is these intuitions which lead us to consider the rectilinear space $(\mathrm{Rect}_2, \mathcal{O})$ we sketched above.

With this digression on motivations complete, let us return to describing the contents of the present paper.

1.4. The rectilinear adelic complex

In view of the discussion above, one would like to have an explicit model for the derived spaces of sections $R\Gamma(-, \mathcal{O})$ of the structure sheaf $\mathcal{O}$ on rectilinear space Rect_2, and one would like this model to have the property of being well-adapted to eventually attaching data to rectilinear lines and flags, as well as closed points.

That leads us to construct a model of $R\Gamma(-, \mathcal{O})$ in the spirit of the adelic complexes for schemes due to Parshin [52] and Beilinson [5]. Their construction involves associating a commutative algebra to each flag of subschemes. The algebra associated to a given flag is defined by an elegant and rather intricate procedure of repeated localizations and completions.

In our case, what considering merely the topological product $\mathrm{Rect}_2 = \mathbb{A}^1 \times^{\mathbf{Top}} \mathbb{A}^1$ of affine lines buys us is that (a) we get a much smaller set of flags, consisting only of the rectilinear flags and (b) the algebras attached to flags are simpler to describe. The resulting *rectilinear adelic complex* is given in Section 4, where the main result is Theorem 6.

Subsequently, when we move to considering the homotopy Manin triples of Theorem 24, we need only a *finite* collection of flags, built from the finite set of those rectilinear lines which intersect our chosen collection of marked points $(w_i, z_i)_{i=1}^N$.

In each case, the rectilinear flags form a semisimplicial set, and this gives rise to a semi*co*simplicial commutative algebra. Then, by applying the Thom-Whitney functor whose definition we recall in Section 5, we get a dg commutative algebra.

1.5. Triangular decompositions of enveloping algebras

Our final collection of results concerns the universal enveloping algebras. Recall that, in the usual case of Lie algebras, a Manin triple $(\mathfrak{a}, \langle - | - \rangle, \mathfrak{a}_+, \mathfrak{a}_-)$ encodes in particular a decomposition of $\mathfrak{a}$ as the direct sum in vector spaces of two Lie subalgebras, $\mathfrak{a} \cong_{\mathbb{C}} \mathfrak{a}_- \oplus \mathfrak{a}_+$. That in turn gives rise to an isomorphism

$$U(\mathfrak{a}) \cong U(\mathfrak{a}_-) \otimes U(\mathfrak{a}_+)$$

between the enveloping algebras; it is an isomorphism of vector spaces and, moreover, of $(U(\mathfrak{a}_-), U(\mathfrak{a}_+))$-bimodules [11].

In our present setting, of homotopy Manin triples of dg Lie algebras, we get something similar, at least in the special case that the homotopy equivalence of dg vector spaces $\mathfrak{a} \simeq \mathfrak{a}_- \oplus \mathfrak{a}_+$ is actually a strong deformation retract (we recall the definitions in §3.9 and Section 9 below)

$$\mathfrak{a}_- \oplus \mathfrak{a}_+ \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathfrak{a} \circlearrowleft$$

That turns out to be true of our local homotopy Manin triple from Section 7. The resulting triangular decomposition is given in Section 9, Corollary 29. On the other hand it is not true of the global homotopy Manin triple of Section 8. The remainder of Section 9 is dedicated to introducing a certain modification of the global homotopy Manin triple for which this extra condition does hold; see Theorem 33 and its Corollary 34.

(The need for strong deformation retracts here is related to the fact that we insist on staying with dg Lie algebras and their dg associative enveloping algebras, rather than going to the less concrete but perhaps conceptually more natural setting of L_∞ algebras and their envelopes. This choice is motivated by nothing deeper than the authors' impression that PBW-type statements become rather subtle for L_∞ algebras; see [38] and references therein including [2,44].)

1.6. Outline

This paper is structured as follows.

After defining rectilinear space $(\mathrm{Rect}_2, \mathcal{O})$ and the formal rectilinear polydisc $(\mathrm{PDisc}_2^\times, \hat{\mathcal{O}})$ in Section 2, we introduce in Section 3 the semisimplicial set of rectilinear flags and associated semicosimplicial algebras. Then in Section 4 we introduce the rectilinear adelic complex.

The Thom-Whitney, or Thom-Sullivan, construction is recalled in Section 5.

Then in Section 6 we give our notion of what it means to have a homotopy Manin triple, before giving our two classes of examples in Section 7 and Section 8. These sections contain the main results of the paper, Theorem 14 and Theorem 24. Finally, results about the enveloping dg associative algebras are collected in Section 9.

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2. Rectilinear space and the formal rectilinear polydisc

We work over $\mathbb{C}$. The tensor product $\otimes$ means $\otimes_{\mathbb{C}}$ throughout.

2.1. The affine line $\mathbb{A}^1$ and the formal disc Disc_1

Recall that one thinks of the polynomial algebra $\mathbb{C}[z]$ as the algebra of functions on the affine line $\mathbb{A}^1 := \mathrm{Spec}\,\mathbb{C}[z]$, its prime spectrum. The prime ideals of $\mathbb{C}[z]$ are 0 and $(z-a)\mathbb{C}[z]$ for every $a \in \mathbb{C}$, so that as a set

$$\mathbb{A}^1 = \{\eta\} \sqcup \mathbb{C},$$

where η is the generic point. The nonempty open sets in the Zariski topology are the complements $\mathbb{A}^1 \setminus \{c_1, \dots, c_k\}$ of finite collections of closed points $c_i \in \mathbb{C}$, and the structure sheaf

$$\mathcal{O} : \mathbf{Open}(\mathbb{A}^1)^{\mathrm{op}} \to \mathbf{CAlg}; \quad \mathcal{O}(\mathbb{A}^1 \setminus \{c_1, \dots, c_k\}) = \mathbb{C}(z)_{c_1,\dots,c_k}$$

assigns to such an open set the algebra $\mathbb{C}(z)_{c_1,\dots,c_k}$ of rational expressions in z singular at most at the missing points. The stalk at $a \in \mathbb{C}$ is $\mathcal{O}_a := \varinjlim_{U \ni a} \mathcal{O}(U) = S_a^{-1}\mathbb{C}[z]$, the localization of $\mathbb{C}[z]$ away from the ideal generated by $(z-a)$ or in other words the algebra of rational expressions in z with no singularity at $z=a$. At the generic point η the stalk is $\mathcal{O}_\eta = \mathbb{C}(z)$, the field of all rational expressions in z.

The completion of $\mathbb{C}[z]$ with respect to the maximal ideal $(z-a)\mathbb{C}[z]$ is the algebra $\mathbb{C}[[z-a]]$ of formal power series. One thinks of it as the algebra of functions on the <u>formal disc</u> at a, $\mathrm{Disc}_1(a) := \mathrm{Spec}\,\mathbb{C}[[z-a]]$. The prime ideals of $\mathbb{C}[[z-a]]$ are 0 and the unique maximal ideal $(z-a)\mathbb{C}[[z-a]]$, so that as a set the formal disc at a has exactly two points,

$$\mathrm{Disc}_1(a) = \{\eta, a\},$$

namely the generic point which we again call η, and the closed point a. (When we wish to refer to an abstract copy of the formal disc, we shall sometimes write pt. for the closed point $\mathrm{Disc}_1 = \{\eta, \mathrm{pt.}\}$.) The only nonempty open sets of $\mathrm{Disc}_1(a)$ are $\mathrm{Disc}_1(a)$ itself and the <u>punctured formal disc</u> at a, $\mathrm{Disc}_1^\times(a) := \mathrm{Disc}_1 \setminus \{a\} = \{\eta\}$. The structure sheaf, which shall denote by $\hat{\mathcal{O}}$, is given by

$$\hat{\mathcal{O}}(\mathrm{Disc}_1(a)) = \mathbb{C}[[z-a]], \qquad \hat{\mathcal{O}}(\mathrm{Disc}_1^\times(a)) = \mathbb{C}((z-a)) \tag{5}$$

where $\mathbb{C}((z-a))$ is the algebra of formal Laurent series. Its stalks are $\hat{\mathcal{O}}_a = \mathbb{C}[[z-a]]$ and $\hat{\mathcal{O}}_\eta = \mathbb{C}((z-a))$. Note that there are embeddings of algebras $\mathcal{O}_a \hookrightarrow \hat{\mathcal{O}}_a$ and $\mathcal{O}_\eta \hookrightarrow \hat{\mathcal{O}}_\eta$ given by expanding in formal (Laurent) series in the local coordinate $z-a$.

Both the punctured affine line and the punctured formal disc are again affine schemes: $\mathbb{A}^1 \setminus \{a\} = \mathrm{Spec}\,\mathbb{C}[(z-a)^{\pm 1}]$ and $\mathrm{Disc}_1^\times(a) = \mathrm{Spec}\,\mathbb{C}((z-a))$.

2.2. Rectilinear space Rect_2

Let us denote by

$$\mathrm{Rect}_2 := \mathbb{A}^1 \times^{\mathbf{Top}} \mathbb{A}^1$$

the product in topological spaces of two copies of the affine line. Thus, Rect_2 is the set-theoretic product $\mathbb{A}^1 \times \mathbb{A}^1$, endowed with the product of the Zariski topologies. Let $w, z : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be the Cartesian coordinates. As a set, Rect_2 consists of

- All the points $(a, b) \in \mathbb{C}^2$
- All the lines $(w=a)$ for $a \in \mathbb{C}$
- All the lines $(z=b)$ for $b \in \mathbb{C}$
- The generic point E.

Here, we are adopting the suggestive notations $(w=a)$ and $(z=b)$ for (a,η) and (η, b) respectively, and we write E for the generic point (η,η). The closure of the line $(w=a)$ consists of the line and all its points: $\overline{\{(w=a)\}} = \{(w=a)\} \sqcup \{(a,b) : b \in \mathbb{C}\}$; the closure of the generic point E is all of Rect_2: $\overline{\{\mathrm{E}\}} = \mathrm{Rect}_2$. Complements of closures of lines form a base of the open sets. Thus, every nonempty open subset is the complement of finitely many closed points and the closures of finitely many lines:

$$U = \mathrm{Rect}_2 \setminus \bigcup_{i=1}^m \overline{\{(w=a_i)\}} \setminus \bigcup_{j=1}^n \overline{\{(z=b_j)\}} \setminus \bigcup_{k=1}^p \{(c_k, d_k)\}. \tag{6a}$$

Let $\mathcal{O} : \mathbf{Open}^{\mathrm{op}}_{\mathrm{Rect}_2} \to \mathbf{CAlg}$ be given by

$$\mathcal{O}(U) := \mathbb{C}(w)_{a_1,\dots,a_m} \otimes \mathbb{C}(z)_{b_1,\dots,b_n}. \tag{6b}$$

This is a sheaf in commutative algebras. Its restriction maps are all injections.

In this way, we get a ringed space $(\mathrm{Rect}_2, \mathcal{O})$ which we call *rectilinear space*. Its algebra of global sections is the polynomial algebra $\mathcal{O}(\mathrm{Rect}_2) = \mathbb{C}[w] \otimes \mathbb{C}[z] = \mathbb{C}[w,z]$. The stalk at any closed point $(a,b) \in \mathbb{C}^2 \hookrightarrow \mathrm{Rect}_2$ is

$$\mathcal{O}_{(a,b)} = S_a^{-1}\mathbb{C}[w] \otimes S_b^{-1}\mathbb{C}[z],$$

which is a local ring: its unique maximal ideal is $(w-a)\mathcal{O}_{(a,b)} + (z-b)\mathcal{O}_{(a,b)}$. However, the remaining stalks are not local rings: for example, the stalk of $\mathcal{O}$ at the generic point $\mathrm{E} \in \mathrm{Rect}_2$ is

$$\mathcal{O}_{\mathrm{E}} = \mathbb{C}(w) \otimes \mathbb{C}(z).$$

2.3. *The formal rectilinear polydisc* PDisc_2

Let us denote by

$$\mathrm{PDisc}_2(a,b) = \mathrm{Disc}_1(a) \times^{\mathbf{Top}} \mathrm{Disc}_1(b)$$

the product in topological spaces of two copies of the formal disc. As a set $\mathrm{PDisc}_2(a,b)$ consists of four points:

- The closed point $(a,b) \in \mathbb{C}^2$
- The line $(w=a)$
- The line $(z=b)$
- The generic point E.

We call $\mathrm{PDisc}_2(a,b)$ the *formal rectilinear polydisc*, and

$$\mathrm{PDisc}_2^{\times}(a,b) := \mathrm{PDisc}_2(a,b) \setminus \{(a,b)\},$$

the *punctured formal rectilinear polydisc*, at (a,b). Let $\hat{\mathcal{O}}$ be the sheaf in commutative algebras on $\mathrm{PDisc}_2(a,b)$ given by

$$\begin{aligned}
\hat{\mathcal{O}}(\mathrm{PDisc}_2(a,b)) = \hat{\mathcal{O}}(\mathrm{PDisc}_2^{\times}(a,b)) &= \mathbb{C}[[w-a]] \otimes \mathbb{C}[[z-b]] \\
\hat{\mathcal{O}}(\mathrm{PDisc}_2(a,b) \setminus \overline{\{(w=a)\}}) &= \mathbb{C}((w-a)) \otimes \mathbb{C}[[z-b]] \\
\hat{\mathcal{O}}(\mathrm{PDisc}_2(a,b) \setminus \overline{\{(z=b)\}}) &= \mathbb{C}[[w-a]] \otimes \mathbb{C}((z-b)) \\
\hat{\mathcal{O}}(\{\mathrm{E}\}) &= \mathbb{C}((w-a)) \otimes \mathbb{C}((z-b))
\end{aligned} \tag{7}$$

whose stalks are

$$\begin{aligned}
\hat{\mathcal{O}}_{\mathrm{E}} &= \mathbb{C}((w-a)) \otimes \mathbb{C}((z-b)) & \hat{\mathcal{O}}_{(w=a)} &= \mathbb{C}[[w-a]] \otimes \mathbb{C}((z-b)) \\
\hat{\mathcal{O}}_{(z=b)} &= \mathbb{C}((w-a)) \otimes \mathbb{C}[[z-b]] & \hat{\mathcal{O}}_{(a,b)} &= \mathbb{C}[[w-a]] \otimes \mathbb{C}[[z-b]].
\end{aligned}$$

(It is the external tensor product of the structure sheaves on the two factors Disc_1.)

When we identify $\mathrm{PDisc}_2(a,b)$ as a subset of Rect_2 in the obvious way, there are embeddings of algebras $\mathcal{O}_x \hookrightarrow \hat{\mathcal{O}}_x$ for every point $x \in \mathrm{PDisc}_2(a,b)$, given by expanding in formal series in both the local coordinates, $w-a$ and $z-b$.

2.4. *Derived global sections and higher sheaf cohomology*

There is a vital difference between the disc Disc_1 and the polydisc PDisc_2. If we remove the closed point from the Disc_1, the algebra of global sections of the structure sheaf gets bigger, as we see in (5):

$$\Gamma(\mathrm{Disc}_1, \hat{\mathcal{O}}) = \mathbb{C}[[z]] \subsetneq \mathbb{C}((z)) = \Gamma(\mathrm{Disc}_1^{\times}, \hat{\mathcal{O}}).$$

By contrast, if we remove the closed point from the polydisc in dimension two, we don't get any more sections than we had before, as we see in (7):

$$\Gamma(\mathrm{PDisc}_2, \hat{\mathcal{O}}) = \mathbb{C}[[w]] \otimes \mathbb{C}[[z]] = \Gamma(\mathrm{PDisc}_2^{\times}, \hat{\mathcal{O}}).$$

The same is true of Rect_2 compared to the affine line $\mathbb{A}^1$. If we remove a single point $c \in \mathbb{C}$ from the affine line, the algebra of global sections of the structure sheaf goes from the polynomial algebra $\mathbb{C}[z]$ to the algebra $\mathbb{C}(z)_c = \mathbb{C}[(z-c)^{\pm 1}]$

of Laurent polynomials. By contrast, note the lack of dependence on the (c_k, d_k) in (6): we may remove any finite number of closed points as we wish and no more global sections appear.

This is a classical phenomenon which occurs also for the structure sheaf on the usual affine plane $\mathbb{A}^2 := \operatorname{Spec}\mathbb{C}[w, z]$. It has an analog in the complex-analytic setting, known as Hartog's theorem (see e.g. [36, Theorem 1.8] or [27]): in complex dimension at least two, every holomorphic function on a punctured polydisc can be analytically continued to a function on the unpunctured polydisc.

As stressed in [18,34], and in [25], one should think that the "missing" global sections over the punctured space have not vanished, but merely moved to higher cohomology. Recall that the *derived space of global sections* $R\Gamma^\bullet(\mathrm{PDisc}_2^\times, \hat{\mathcal{O}})$ of the sheaf $\hat{\mathcal{O}}$ on $\mathrm{PDisc}_2^\times$ is the cochain complex defined, up to zigzags of quasi-isomorphisms, by the requirement that its cohomology computes the sheaf cohomology of $\hat{\mathcal{O}}$,

$$H^\bullet(R\Gamma(\mathrm{PDisc}_2^\times, \hat{\mathcal{O}})) \cong H^\bullet(\mathrm{PDisc}_2^\times, \hat{\mathcal{O}}).$$

There is higher cohomology in the case of the punctured polydisc (cf. Corollary 17 below)

$$H^k(\mathrm{PDisc}_2^\times, \hat{\mathcal{O}}) = \begin{cases} \mathbb{C}[[w]] \otimes \mathbb{C}[[z]] & k = 0 \\ w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] & k = 1 \\ 0 & k \notin \{0, 1\} \end{cases}$$

One can think of $w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}]$ here as the higher analog of the negative modes $z^{-1}\mathbb{C}[z^{-1}] \subset \mathbb{C}((z))$ in the one-dimensional case.

Our aim in the present paper is to construct explicit models of spaces of derived sections, in our rectilinear setting, with the following properties:

- we want models both for the local situation, i.e. for $R\Gamma(\mathrm{PDisc}_2^\times, \hat{\mathcal{O}})$, and for the global case $R\Gamma(\mathrm{Rect}_2 \setminus \{\text{closed points}\}, \mathcal{O})$;
- we want these models to make explicit the global-to-local maps

$$R\Gamma(\mathrm{Rect}_2 \setminus \bigcup_i \{(a_i, b_i, \mathcal{O})\}) \to R\Gamma(\mathrm{PDisc}_2^\times(a_i, b_i), \hat{\mathcal{O}}),$$

  that are the higher analogs of taking formal Laurent expansions;
- ultimately we want models in dg commutative algebras, rather than just dg vector spaces.

Our motivation is that we want explicit models, in dg Lie algebras, for the higher analogs of the usual current Lie algebras $\mathfrak{g} \otimes \mathbb{C}((t)) = \mathfrak{g} \otimes \Gamma(\mathrm{Disc}_1^\times, \hat{\mathcal{O}})$, and their global analogs, and the dg Lie algebra maps between them.

Models in dg commutative algebras for derived sections can be obtained in various ways; see [34, Appendix A]. The construction we use centres on the Thom-Sullivan-Whitney functor, whose definition we recall in Section 5.

The starting point is the familiar definition of sheaf cohomology. Recall that the sheaf cohomology of $\hat{\mathcal{O}}$ on $\mathrm{PDisc}_2^\times$ is, by definition, the cohomology $H^\bullet(\mathrm{PDisc}_2^\times, \hat{\mathcal{O}}) := H^\bullet(\mathcal{F})$ of any resolution

$$0 \to \hat{\mathcal{O}} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

of $\hat{\mathcal{O}}$ by flasque sheaves. We shall construct such resolutions, of $\hat{\mathcal{O}}$ and of $\mathcal{O}$, in the spirit of the adelic complexes for schemes due to Parshin [52] and Beilinson [5], but adapted to our simpler rectilinear spaces, PDisc_2 and Rect_2 (which are not schemes).

Remark 1. For more on adelic complexes, see e.g. [31,50,49,51]. In the case of schemes, the algebras attached to flags of subschemes are defined by repeated localizations and completions – see [49], especially §3.2 and §3.3, and references therein. One of the ways in which our present rectilinear setting is simpler is that, because we just have a topological product of affine lines, the algebras we attach to flags below will be merely products of algebras appearing in that familiar case. Another is that we simply have a much smaller semisimplicial set of flags, because we need only the rectilinear flags. ◁

3. Semicosimplicial algebras associated to rectilinear flags

We must first introduce semisimplicial sets of rectilinear flags, and algebras associated to them: these will be the building blocks of our models for spaces of derived sections.

3.1. Rectilinear flags

Given any subset $U \subseteq \mathrm{Rect}_2$ (open or not), let $\mathrm{Flag}_n(U)$ for $n = 0, 1, 2$ denote the set of n-step flags in U:

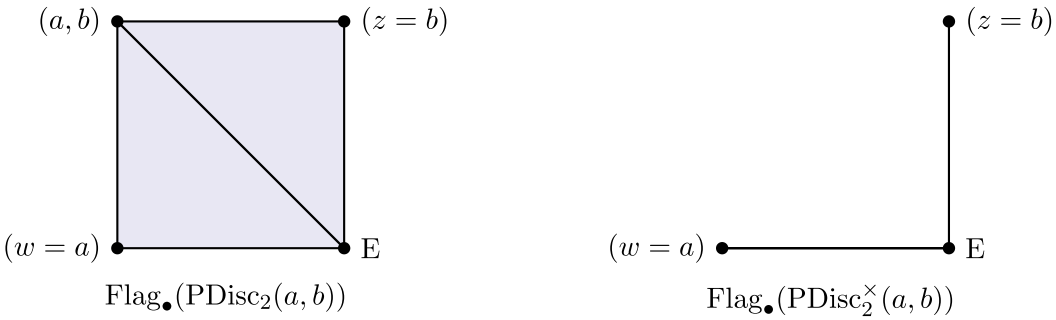
$$\mathrm{Flag}_n(U) := \left\{(a_0, \ldots, a_n) \in U^n : \overline{\{a_0\}} \subsetneq \cdots \subsetneq \overline{\{a_n\}}\right\}.$$

(Thus, $\mathrm{Flag}_1(\mathrm{Rect}_2)$ consists of points-in-lines and lines-in-the-surface, and $\mathrm{Flag}_2(\mathrm{Rect}_2)$ consists of points-in-lines-in-the-surface.)

Let

$$\begin{aligned} &\partial_i^n : \mathrm{Flag}_{n+1}(U) \to \mathrm{Flag}_n(U); \\ &\quad (a_0, \ldots, a_{n+1}) \mapsto (a_0, \ldots, \widehat{a}_i, \ldots, a_{n+1}) \end{aligned}$$

be the map given by removing one space from a flag, for $i = 0, \ldots, n+1$ and $n = 0, 1, 2$. These maps endow $\mathrm{Flag}_\bullet(U)$ with the structure of a *semisimplicial set*, as we recall below. For intuition, one should visualise the example of the finite set of flags $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2(a, b))$ for the formal rectilinear polydisc $\mathrm{PDisc}_2(a, b) \subset \mathrm{Rect}_2$ at the closed point (a, b). It consists of exactly four vertices, five edges and two 2-simplices, while the set $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times(a, b)) \subset \mathrm{Flag}_\bullet(\mathrm{PDisc}_2(a, b))$ of flags in the punctured formal rectilinear polydisc has exactly three vertices, two edges and no higher simplices:



(8)

3.2. Semisimplicial sets

Let Δ denote the category whose objects are the finite totally-ordered sets

$$[n] := \{0 < 1 < \cdots < n\}, \qquad n \in \mathbb{Z}_{\geq 0},$$

and whose morphisms are the strictly order-preserving maps $\theta : [n] \to [N]$. Such maps are injections and exist only for $n \leq N$. They are generated by the *coface maps*

$$d_i^n : [n] \to [n+1]; \quad j \mapsto \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

for $i = 0, 1, \ldots, n+1$ (together with the identity maps $\mathrm{id}_{[n]}$) for $n = 0, 1, 2, \ldots$. One thinks of the category Δ as follows:

$$\ldots [2] \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; [1] \;\substack{\longleftarrow\\\longleftarrow}\; [0].$$

A *semisimplicial object* Z in a category $\mathcal{C}$ is a functor $Z : \Delta^{\mathrm{op}} \to \mathcal{C}$. In particular, a *semisimplicial set* $S : \Delta^{\mathrm{op}} \to \mathbf{Set}$ is a semisimplicial object S in the category of sets. For each n, $S([n])$ is called the set of n-simplices of S. The maps $\partial_i^n := S(d_i^n) : S([n+1]) \to S([n])$ are the *face maps* of S.[1]

In our present case, we have the functor

$$\mathrm{Flag}(\mathrm{Rect}_2) : \Delta^{\mathrm{op}} \to \mathbf{Set}$$

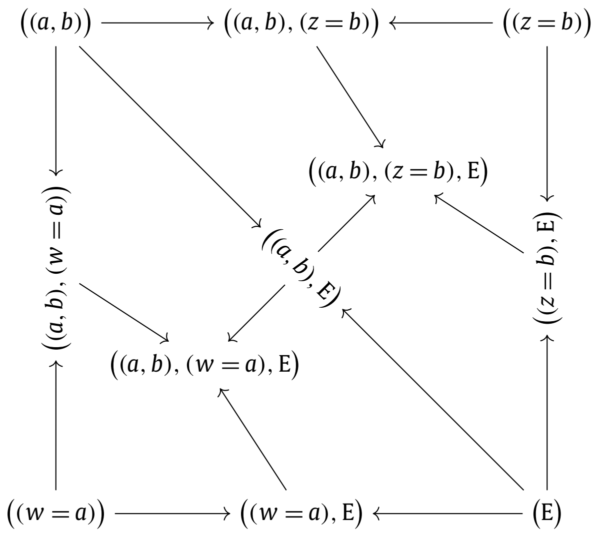
given on objects by $[n] \mapsto \mathrm{Flag}_n(\mathrm{Rect}_2)$ and on morphisms by $d_i^n \mapsto \partial_i^n$. One may think of the semisimplicial structure on $\mathrm{Flag}_\bullet(\mathrm{Rect}_2)$ as follows:

$$\mathrm{Flag}_2(\mathrm{Rect}_2) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathrm{Flag}_1(\mathrm{Rect}_2) \;\substack{\longrightarrow\\\longrightarrow}\; \mathrm{Flag}_0(\mathrm{Rect}_2).$$

[1] *Simplicial sets* are defined in the same way but with "strictly order-preserving", i.e. increasing, replaced by "weakly order-preserving", i.e. non-decreasing, in the definition of Δ. Simplicial sets have extra structure (degenerate simplices and degeneracy maps) which we shall not need here.

3.3. The comma category $\Delta\downarrow S$

Given any semisimplicial set $S : \Delta^{\mathrm{op}} \to \mathbf{Set}$, let $\Delta\downarrow S$ denote the category whose objects are the simplices of S, and in which there is a unique morphism $f \to F$ if f is a subsimplex of F, i.e. if $f = \phi(F)$ for some morphism ϕ of $S(\Delta^{\mathrm{op}})$, and no morphisms $f \to F$ otherwise. One can regard $\Delta\downarrow S$ as a partially ordered set. For example, the category $\Delta\downarrow \mathrm{Flag}_\bullet(\mathrm{PDisc}_2(a,b))$ is the partially ordered set given by



(9)

There is another useful way of regarding the category $\Delta\downarrow S$. Recall that the Yoneda embedding $\Delta \xrightarrow{\text{Yoneda}} [\Delta^{\mathrm{op}}, \mathbf{Set}]$ embeds Δ as a full subcategory of the category of semisimplicial sets, by sending $[n] \in \Delta$ to the *standard n-simplex* $\Delta^n := \mathrm{Hom}_\Delta(-,[n])$. We can then regard $\Delta\downarrow S$ as the comma category associated to the diagram of functors

$$\begin{array}{ccc} & & \mathbf{1} \\ & & \big\downarrow{\scriptstyle S} \\ \Delta & \xrightarrow{\text{Yoneda}} & [\Delta^{\mathrm{op}}, \mathbf{Set}] \end{array}$$

That is: we can think that an object of $\Delta\downarrow S$ is by definition a copy of the standard n-simplex Δ^n for some n together with a map of semisimplicial sets $\Delta^n \to S$; and a morphism from $(\Delta^n \to S)$ to $(\Delta^N \to S)$ in $\Delta\downarrow S$ is a morphism $\phi : \Delta^n \to \Delta^N$ of semisimplicial sets such that the diagram

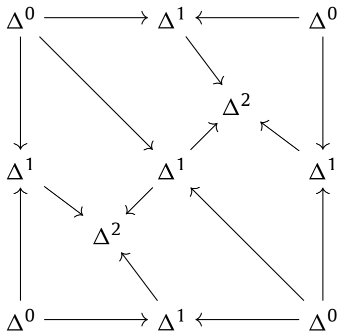
$$\begin{array}{ccc} \Delta^n & \xrightarrow{\phi} & \Delta^N \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

commutes. An advantage of this perspective on $\Delta\downarrow S$ is that we get the functor

$$\Delta\downarrow S \xrightarrow{\text{Forgetful}} \Delta \xrightarrow{\text{Yoneda}} [\Delta^{\mathrm{op}}, \mathbf{Set}]$$

which forgets about the maps of the simplices into S. We can think of this functor as a diagram in the category $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ of semisimplicial sets. (Its colimit is S itself.)

For example the image of $\Delta\downarrow \mathrm{Flag}_\bullet(\mathrm{PDisc}_2(a,b))$ in the category of semisimplicial sets is the diagram



(10)

Intuitively speaking, this tells us how to build $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2(a,b))$ by sewing together standard simplices.

3.4. Semicosimplicial algebras

A *semicosimplicial object* A in a category $\mathcal{A}$ is a functor $A:\Delta\to\mathcal{A}$, i.e. an object of the functor category

$$[\Delta,\mathcal{A}].$$

The relevant categories $\mathcal{A}$ for us are commutative algebras, Lie algebras, and the differential graded analogs of these. When it is not necessary to be more precise, we shall refer to objects of $\mathcal{A}$ as algebras and to semicosimplicial objects in $\mathcal{A}$ as *semicosimplicial algebras*.

Thus, a semicosimplicial algebra A has, by definition, an *algebra* $A([n])\in\mathcal{A}$ of n-cosimplices, for each $n\in\mathbb{Z}_{\geq 0}$, and coface morphisms $d_i^n : A([n])\to A([n+1])$, $i=0,1,\ldots,n+1$ between them:

$$\ldots A([2]) \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; A([1]) \;\substack{\longleftarrow\\[-2pt]\longleftarrow}\; A([0]).$$

Actually, our semicosimplicial algebras will have a finer structure than this: they will arise from semisimplicial sets (of flags) by first attaching algebras to *individual simplices* (i.e. individual flags) of a given semisimplicial set. It is useful to keep track of this structure, and to that end we make the following definition.

3.5. S-algebras

Given a semisimplicial set S, define the category of *$S-objects$ in $\mathcal{A}$* or *S-algebras* to be the functor category

$$[\Delta\downarrow S,\mathcal{A}],$$

i.e. the category whose objects are functors $\Delta\downarrow S\to\mathcal{A}$ and whose morphisms are natural transformations between such functors. Given an S-object in $\mathcal{A}$ there is a natural way to recover a semicosimplicial object in $\mathcal{A}$. Namely, given a functor $A:\Delta\downarrow S\to\mathcal{A}$, we may define a functor $\Pi A:\Delta\to\mathcal{A}$ as follows. We set

$$(\Pi A)\,([n]) := \prod_{f\in S([n])} A_f$$

and if $\phi:[n]\to[N]$ in Δ then the morphism $(\Pi A)\,(\phi):(\Pi A)\,([n])\to(\Pi A)\,([N])$ is given by its restrictions to the factors A_f:

$$(\Pi A)\,(\phi)|_{A_f} = \prod_{F\in S([N])\,:\,S(\phi)(F)=f} A(f\to F)$$

(Recall that $A(f\to F):A_f\to A_F$. A given n-simplex f may belong to the boundary of infinitely many N-simplices of S, and for that reason we need the direct product $\prod$ rather than the direct sum $\bigoplus$. On the other hand, any given N-simplex F has only finitely many boundary n-simplices, so there are only finitely many factors A_f such that $f\to F$ is a morphism of $\Delta\downarrow S$, and thus the restriction of $(\Pi A)(\phi)\big(\Pi A([n])\big)$ to the factor A_F is a well-defined sum of finitely many terms.)

Lemma 2. *This Π defines the action on objects of a functor*

$$\Pi=\Pi_S:[\Delta\downarrow S,\mathcal{A}]\to[\Delta,\mathcal{A}]$$

from S-objects in $\mathcal{A}$ to semicosimplicial objects in $\mathcal{A}$. □

Remark 3. This functor Π is left adjoint,

$$[\Delta, \mathcal{A}] \underset{\Pi}{\overset{u^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\Delta \downarrow S, \mathcal{A}] \ ,$$

to the pull-back $u^* : B \mapsto u^*(B) = B \circ u$ of the forgetful functor $u : \Delta \downarrow S \to \Delta$. $\triangleleft$

3.6. Semisimplicial subsets and the restriction morphism

The other fact we need concerns semisimplicial *subsets* of S. Suppose R is a semisimplicial subset of S, by which we mean that for each n the set of n-simplices of R is a subset of the set of n-simplices of S, $R([n]) \subset S([n])$ and that the face maps respect these embeddings of sets. That is, the embedding maps $i([n]) : R([n]) \hookrightarrow S([n])$ define a morphism of semisimplicial sets $i : R \to S$, i.e. they are the components of a natural transformation i

$$\Delta^{\mathrm{op}} \underset{S}{\overset{R}{\Downarrow i}} \mathbf{Set}$$

between the functors R and S. We get a functor $\Delta \downarrow i : \Delta \downarrow R \to \Delta \downarrow S$ between the corresponding comma categories. Given an S-object A in $\mathcal{A}$ we have then also its restriction $A|_R$, an R-object in $\mathcal{A}$. Namely, $A|_R$ is the composition

$$\Delta \downarrow R \to \Delta \downarrow S \xrightarrow{A} \mathcal{A}.$$

This defines a functor $[\Delta \downarrow S, \mathcal{A}] \to [\Delta \downarrow R, \mathcal{A}]$; $A \mapsto A|_R$. (That is, $A|_R := (\Delta \downarrow i)^* A$.) We can then form two semicosimplicial objects in $\mathcal{A}$, namely $\Pi_S A$ and $\Pi_R A|_R$.

Lemma 4. *There is a morphism of semicosimplicial objects in $\mathcal{A}$,*

$$\pi : \Pi_S A \to \Pi_R A|_R$$

given by

$$\pi|_{A_f} = \begin{cases} \mathrm{id}_{A_f} & f \in R \\ 0 & f \notin R. \end{cases} \quad \square$$

Remark 5. Note that while we also have the obvious embedding maps $(\Pi_R A|_R)([n]) \hookrightarrow \Pi_S A([n])$, these do not in general define a morphism of semicosimplicial algebras $\Pi_R A|_R \to \Pi_S A$. Indeed, we get failures of naturality whenever $f \in S([n])$ and $F \in S([N])$ are such that $f \in R([n])$ and yet $F \notin R([N])$. $\triangleleft$

3.7. First example

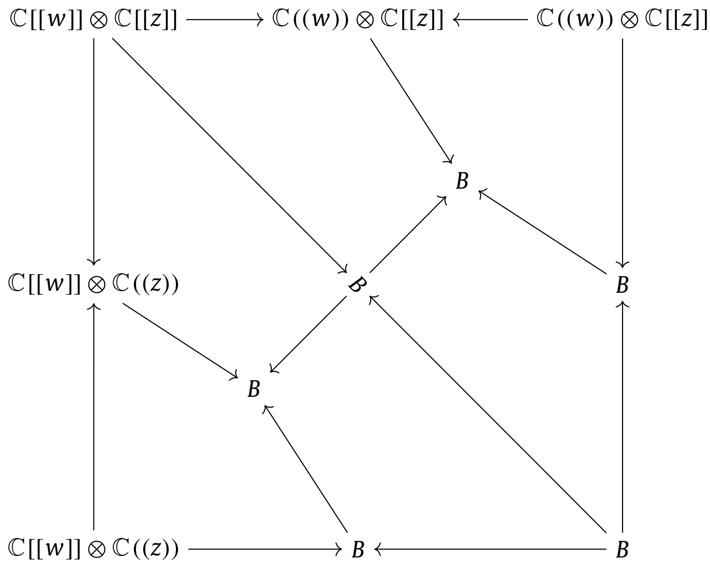
We now turn to an example which will play a central role. Let $\mathbf{CAlg}^{\mathrm{emb}}$ denote the category whose objects are commutative ($\mathbb{C}$-)algebras and whose morphisms are embeddings of commutative algebras.

Let $\mathrm{PDisc}_2 := \mathrm{PDisc}_2(0,0)$ be the formal rectilinear polydisc at the point $(0,0)$. To give a $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2)$-object in $\mathbf{CAlg}^{\mathrm{emb}}$, i.e. a functor

$$\Delta \downarrow \mathrm{Flag}_\bullet(\mathrm{PDisc}_2) \to \mathbf{CAlg}^{\mathrm{emb}},$$

is by definition to give a certain commuting diagram of commutative algebras and embeddings between them, cf. (9) and (10). Let us define such an algebra, A_{PDisc_2}, as follows.

$A_{\mathrm{PDisc}_2} :=$



(11)

where B is the commutative algebra

$$B := \mathbb{C}((z)) \otimes \mathbb{C}((w)).$$

That is, $A_{\mathrm{PDisc}_2}(f) := B$ for all simplices f with only the following exceptions:

$$A_{\mathrm{PDisc}_2}((0,0)) := \mathbb{C}[[w]] \otimes \mathbb{C}[[z]]$$
$$A_{\mathrm{PDisc}_2}((w=0)) := A_{\mathrm{PDisc}_2}((0,0),(w=0)) := \mathbb{C}[[w]] \otimes \mathbb{C}((z))$$
$$A_{\mathrm{PDisc}_2}((z=0)) := A_{\mathrm{PDisc}_2}((0,0),(z=0)) := \mathbb{C}((w)) \otimes \mathbb{C}[[z]].$$

Let $A_{\mathrm{PDisc}_2^\times}$ denote the restriction, in the sense of Lemma 4, of the $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2)$-algebra A_{PDisc_2} to a $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times)$-algebra:

$$A_{\mathrm{PDisc}_2^\times} := A_{\mathrm{PDisc}_2}|_{\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times)}.$$

3.8. The associated cochain complex of a cosimplicial algebra

Let $\mathbf{CCh}(\mathcal{A})$ denote the category of cochain complexes in $\mathcal{A}$. There is a functor

$$\mathcal{C} : [\Delta, \mathcal{A}] \to \mathbf{CCh}(\mathcal{A}); \qquad A \mapsto (\mathcal{C}^\bullet(A), \mathrm{d})$$

which assigns to any semicosimplicial object A in $\mathcal{A}$ a cochain complex $(\mathcal{C}^\bullet(A), \mathrm{d})$ concentrated in nonnegative degrees, its *associated complex*. (See e.g. [55, §8.2.1 and §8.4.3].) For each $n \geq 0$ the space $\mathcal{C}^n(A)$ is a copy of $A([n])$ put into cohomological degree n,

$$\mathcal{C}^n(A) := \mathsf{s}^{-n} A([n])$$

and the differential $\mathrm{d}_\mathcal{C} = \sum_n \mathrm{d}_\mathcal{C}^n$, $\mathrm{d}_\mathcal{C}^n : \mathcal{C}^n(A) \to \mathcal{C}^{n+1}(A)$, is given by the alternating sum of the coface maps,

$$\mathrm{d}_\mathcal{C}^n = \mathsf{s}^{-1} \circ \left(A(d_0^n) - A(d_1^n) + \cdots + (-1)^{n+1} A(d_{n+1}^n) \right).$$

Here we use the standard notation

$$\mathsf{s}^n : \mathbf{CCh}(\mathcal{A}) \to \mathbf{CCh}(\mathcal{A}); \quad \mathsf{s}^n V := [n] \otimes V$$

where $[n]$ is the one-dimensional graded vector space concentrated in cohomological degree $-n$. In particular if V is concentrated in degree 0 then $\mathsf{s}^{-n} V$ is concentrated in degree n.

3.9. Homotopy equivalences and deformation retracts

Recall that a map $f : V \to W$ of cochain complexes $V, W \in \mathbf{CCh}(\mathcal{A})$ is a *homotopy equivalence* if it is invertible up to homotopies, in the sense that there exists a map of cochain complexes $g : W \to V$ in the opposite direction such that

$$g \circ f \simeq \mathrm{id}_V \qquad \text{and} \qquad f \circ g \simeq \mathrm{id}_W .$$

Here we used $\simeq$ to indicate that two cochain maps are *homotopic*, meaning that there exists a cochain homotopy between them. That is, in this case, there are maps in $\mathcal{A}$

$$h : V^n \to V^{n-1} \qquad \text{and} \qquad k : W^n \to W^{n-1}$$

for each n, such that

$$g \circ f - \mathrm{id}_V = [h, \mathrm{d}_V] := h \circ \mathrm{d}_V + \mathrm{d}_V \circ h$$
$$f \circ g - \mathrm{id}_W = [k, \mathrm{d}_W] := k \circ \mathrm{d}_W + \mathrm{d}_W \circ k.$$

This situation is often denoted

$$h \circlearrowright V \underset{g}{\overset{f}{\rightleftarrows}} W \circlearrowleft k .$$

As a special case, if $g \circ f = \mathrm{id}_V$ holds exactly then V is a *deformation retract* of W:

$$V \underset{g}{\overset{f}{\rightleftarrows}} W \circlearrowleft h$$

See for example [41, §1.5.5].

Every homotopy equivalence is a quasi-isomorphism of cochain complexes, i.e. it gives rise to an isomorphism in cohomology. For cochain complexes in vector spaces, i.e. for dg vector spaces, the converse is also true. To see this, it is enough to note $H^\bullet(V)$ is always a deformation retract of V. See e.g. [41, §9.4.3]. (Recall we work over $\mathbb{C}$, here and throughout.)

4. The rectilinear adelic complex

We are now in a position to define the complex which will model the derived sections of the sheaf $\mathcal{O}$ on rectilinear space Rect_2.

The main result of this section is Theorem 6. Let us remark that the subsequent sections of the paper are self-contained and can be read independently of this section.

Recall that $\mathrm{Flag}_\bullet(U)$ is the semisimplicial set of flags in a subset $U \subset \mathrm{Rect}_2$ of rectilinear space, as in §3.1. There is manifestly an embedding of semisimplicial sets $\mathrm{Flag}_\bullet(U) \hookrightarrow \mathrm{Flag}_\bullet(V)$, cf. §3.6, whenever $U \subset V$, and these embeddings compose correctly, i.e. we get a functor

$$\mathrm{Flag}_\bullet(-) : \mathbf{Open}_{\mathrm{Rect}_2} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{emb}}_{/\,\mathrm{Flag}_\bullet(\mathrm{Rect}_2)}; \quad U \mapsto \mathrm{Flag}_\bullet(U)$$

to the category of embedded semisimplicial subsets of $\mathrm{Flag}_\bullet(\mathrm{Rect}_2)$.[2] We recognize the (inverse) limit

$$\varprojlim_{U \ni (a,b)} \mathrm{Flag}_\bullet(U) = \mathrm{Flag}_\bullet(\mathrm{PDisc}_2(a,b))$$

as the semisimplicial set $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2(a,b))$ of flags in the formal rectilinear polydisc at the point $(a,b) \in \mathbb{C}^2$.

Now let us define a $\mathrm{Flag}_\bullet(\mathrm{Rect}_2)$-object $\mathbf{A}$ in commutative algebras, i.e. a functor

$$\mathbf{A} : \Delta \downarrow \mathrm{Flag}_\bullet(\mathrm{Rect}_2) \to \mathbf{CAlg}$$

as follows: every flag, i.e. every simplex, is sent to $\mathbb{C}(w) \otimes \mathbb{C}(z)$ with only the following exceptions:

$$\mathbf{A}((a,b)) := S_a^{-1}\mathbb{C}[w] \otimes S_b^{-1}\mathbb{C}[z] \tag{12a}$$

[2] Here, given any semisimplicial set S, we let $[\Delta^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{emb}}_{/S}$ denote the category of its embedded semisimplicial subsets, i.e. the category whose objects are tuples $(R, R \hookrightarrow S)$ consisting of a semisimplicial set R and an embedding of R into S, and whose morphisms $(R, R \hookrightarrow S) \to (T, T \hookrightarrow S)$ are morphisms $R \to T$ such that the diagram $R \to T$, $R \hookrightarrow S$, $T \hookrightarrow S$ commutes.

$$\mathbf{A}((a,b),(w=a)) := \mathbf{A}((w=a)) := S_a^{-1}\mathbb{C}[w]\otimes\mathbb{C}(z) \tag{12b}$$
$$\mathbf{A}((a,b),(z=b)) := \mathbf{A}((z=b)) := \mathbb{C}(w)\otimes S_b^{-1}\mathbb{C}[z] \tag{12c}$$

for all $a,b\in\mathbb{C}$. By restriction, we obtain also a $\mathrm{Flag}_\bullet(U)$-object $\mathbf{A}_U$ in commutative algebras for every subset (open or not) $U\subset\mathrm{Rect}_2$.

At this point, we would like to apply the functor Π of §3.5 to obtain a semicosimplicial algebra $\Pi\mathbf{A}_U$ for each open U. There is however a crucial subtlety. To get the resolution we seek of the sheaf $\mathcal{O}$, it is necessary to modify the definitions of the algebras of 1-simplices and 2-simplices:

$$\Pi'\mathbf{A}_U([0]) := \prod_{F\in\mathrm{Flag}_0(U)}\mathbf{A}(F)$$
$$\Pi'\mathbf{A}_U([1]) := \left\{\mathbf{x}=(x_F)\in\prod_{F\in\mathrm{Flag}_1(U)}\mathbf{A}(F): \begin{array}{l}\text{for all but finitely many flags of the form } F=(\{\text{pt.}\}\subset(\mathit{line})),\\ \quad x_F \text{ actually belongs to } \mathbf{A}(\{\text{pt.}\}), \text{ and}\\ \text{for all but finitely many flags of the form } F=((\mathit{line})\subset \mathrm{E}),\\ \quad x_F \text{ actually belongs to } \mathbf{A}((\mathit{line}))\end{array}\right\} \tag{13}$$
$$\Pi'\mathbf{A}_U([2]) := \left\{\mathbf{x}=(x_F)\in\prod_{F\in\mathrm{Flag}_2(U)}\mathbf{A}(F): \begin{array}{l}\text{for all but finitely many flags } F=(\{\text{pt.}\}\subset(\mathit{line})\subset\mathrm{E}),\\ \quad x_F \text{ actually belongs to } \mathbf{A}(\{\text{pt.}\}).\end{array}\right\}$$

We obtain a sheaf in semicosimplicial algebras $U\mapsto\Pi'\mathbf{A}_U$. The restriction maps $\Pi'\mathbf{A}_V\to\Pi'\mathbf{A}_U$ for $U\subset V$ just consist in throwing away some terms in the products and are manifestly surjective. Thus this sheaf is flasque. On taking the associated cochain complexes we obtain a flasque sheaf in cochain complexes in commutative algebras

$$U\mapsto\mathcal{C}^\bullet(\Pi'\mathbf{A}_U).$$

Theorem 6. *This sheaf $U\mapsto\mathcal{C}^\bullet(\Pi'\mathbf{A}_U)$ on Rect_2 is a flasque resolution of $\mathcal{O}$.*

Thus, $\mathcal{C}^\bullet(\Pi'\mathbf{A})$ is a model for the derived sections of $\mathcal{O}$:

$$R\Gamma^\bullet(U,\mathcal{O})\simeq\mathcal{C}^\bullet(\Pi'\mathbf{A}_U),$$

for each open $U\subset\mathrm{Rect}_2$.

Proof. The proof is given in Appendix A. □

The Thom-Whitney construction, Section 5, provides another model, $\mathrm{Th}^\bullet(\Pi'\mathbf{A})$, which comes equipped with the structure of a dg commutative algebra.

For completeness, we note also the following. Let A_{PDisc_2} be the $\mathrm{Flag}(\mathrm{PDisc}_2)$-algebra from §3.7. It restricts to a $\mathrm{Flag}(U)$-algebra A_U for each open $U\subset\mathrm{PDisc}_2$ and this defines a sheaf $U\mapsto\Pi A_U$ in semicosimplicial commutative algebras on PDisc_2.

Theorem 7. *This sheaf $U\mapsto\mathcal{C}^\bullet(\Pi A_U)$ on PDisc_2 is a flasque resolution of $\hat{\mathcal{O}}$.*

Thus, $\mathcal{C}^\bullet(\Pi\mathbf{A})$ is a model for the derived sections of $\hat{\mathcal{O}}$:

$$R\Gamma^\bullet(U,\hat{\mathcal{O}})\simeq\mathcal{C}^\bullet(\Pi A_U),$$

for each open $U\subset\mathrm{PDisc}_2$. □

4.1. Remark on completed local rings

In place of the definition (12) of $\mathbf{A}$, we could make the following alternative choice:

$$\hat{\mathbf{A}}((a,b)) := \mathbb{C}[[w-a]]\otimes\mathbb{C}[[z-b]] \tag{14a}$$
$$\hat{\mathbf{A}}((a,b),(w=a)) := \mathbb{C}[[w-a]]\otimes\mathbb{C}((z-b)) \qquad \hat{\mathbf{A}}((w=a)) := \mathbb{C}[[w-a]]\otimes\mathbb{C}(z)$$
$$\hat{\mathbf{A}}((a,b),(z=b)) := \mathbb{C}((w-a))\otimes\mathbb{C}[[z-b]] \qquad \hat{\mathbf{A}}((z=b)) := \mathbb{C}(w)\otimes\mathbb{C}[[z-b]]$$

$$\hat{\mathbf{A}}((a,b),(z=b),\mathrm{E}) := \mathbb{C}((w-a))\otimes\mathbb{C}((z-b)) \qquad \hat{\mathbf{A}}((z=b),\mathrm{E}) := \mathbb{C}(w)\otimes\mathbb{C}((z-b))$$
$$\hat{\mathbf{A}}((a,b),(w=a),\mathrm{E}) := \mathbb{C}((w-a))\otimes\mathbb{C}((z-b)) \qquad \hat{\mathbf{A}}((w=a),\mathrm{E}) := \mathbb{C}((w-a))\otimes\mathbb{C}(z)$$
$$\hat{\mathbf{A}}(\mathrm{E}) := \mathbb{C}(w)\otimes\mathbb{C}(z) \tag{14b}$$

(The proof of Theorem 6 in Appendix A goes through with an additional step: one checks that $\mathcal{C}\Pi'\mathbf{A}_V \simeq \mathcal{C}\Pi'\hat{\mathbf{A}}_V$ for suitably small open sets V.)

This choice of $\mathbf{A}$ vs. $\hat{\mathbf{A}}$ has an analog in the familiar case of the affine line $\mathbb{A}^1$, where, at the level of global sections, one has the usual short exact sequence,

$$0 \to \mathbb{C}[x] \to \mathbb{C}(x) \oplus \prod_{a\in\mathbb{C}} \mathbb{C}[[x-a]] \to \prod_{a\in\mathbb{C}}{}' \mathbb{C}((x-a)) \to 0,$$

but also the following one,

$$0 \to \mathbb{C}[x] \to \mathbb{C}(x) \oplus \prod_{a\in\mathbb{C}} S_a^{-1}\mathbb{C}[x] \to \prod_{a\in\mathbb{C}}{}' \mathbb{C}(x) \to 0.$$

(The elements of $\prod'_{a\in\mathbb{C}}\mathbb{C}(x)$ are called *rational* or *non-complete* adeles. See e.g. [24].)

4.2. Global sections as the homotopy kernel

In the familiar case of adeles for complex dimension one, we may also consider puncturing the affine line $\mathbb{A}^1$ at only a prescribed finite collection of closed points $\{a_1,\dots,a_N\}$ of our choice, and we get the following short exact sequence

$$0 \to \mathbb{C}[x] \to \mathbb{C}(x)_{a_1,\dots,a_N} \oplus \bigoplus_{i=1}^{N} \mathbb{C}[[x-a_i]] \to \bigoplus_{i=1}^{N} \mathbb{C}((x-a_i)) \to 0 \tag{15}$$

which, more conceptually, is

$$0 \to \Gamma(\mathbb{A}^1,\mathcal{O}) \to \Gamma(\mathbb{A}^1\setminus\{a_1,\dots,a_N\},\mathcal{O}) \oplus \bigoplus_{i=1}^{N}\hat{\mathcal{O}}_{a_i} \to \bigoplus_{i=1}^{N}\Gamma(\mathrm{Disc}_1^\times(a_i,\hat{\mathcal{O}})) \to 0.$$

One way to interpret this exact sequence is to say that the space of global sections $\Gamma(\mathbb{A}^1,\mathcal{O})$ is the kernel of the map into the tuples of sections over the punctured discs:

$$\Gamma(\mathbb{A}^1,\mathcal{O}) = \ker\left(\Gamma(\mathbb{A}^1\setminus\{a_1,\dots,a_N\},\mathcal{O}) \oplus \bigoplus_{i=1}^{N}\hat{\mathcal{O}}_{a_i} \to \bigoplus_{i=1}^{N}\Gamma(\mathrm{Disc}_1^\times(a_i,\hat{\mathcal{O}}))\right)$$

(and this is true whether we choose to complete the local rings $\mathcal{O}_a$ or not, cf. §4.1).

As explained in [18], this statement has a derived analog: the space of global sections becomes a certain *homotopy kernel*. In our case, the statement is Proposition 8 below. This is nothing but a restatement of [18, Proposition 1.1.4] in our rectilinear setting, and the argument below is merely an expanded version of the argument given there[3]

As we saw in Theorem 6, the cochain complex

$$C := \mathcal{C}^\bullet(\Pi'(\mathbf{A})) \tag{16}$$

models the derived space of global sections of $\mathcal{O}$ on Rect_2

$$R\Gamma^\bullet(\mathrm{Rect}_2,\mathcal{O}) \simeq \mathcal{C}^\bullet(\Pi'(\mathbf{A})).$$

Now let

$$\mathbf{x} = \{(a_1,b_1),\dots,(a_N,b_N)\}$$

denote a finite collection of marked points in $\mathbb{C}\times\mathbb{C}$. Flags in the semisimplicial set of rectilinear flags $\mathrm{Flag}_\bullet(\mathrm{Rect}_2)$ either start at one of the marked points, or they do not; thus, by definition of the unnormalized cochains functor $\mathcal{C}^\bullet$, we have

[3] Though any errors which have appeared in it are of course due to the present authors.

$$C^n = \prod'_{f\in \mathrm{Flag}_n(\mathrm{Rect}_2)} \mathbf{A}(f) = \prod'_{f\in \mathrm{Flag}_n(\mathrm{Rect}_2\setminus \mathbf{x})} \mathbf{A}(f) \oplus \prod'_{\substack{f\in \mathrm{Flag}_n(\mathrm{Rect}_2)\\ :f_0\subset \mathbf{x}}} \mathbf{A}(f) \tag{17}$$

Here, we recognise the first of the two summands on the right as the space C_1^n of the complex

$$C_1^\bullet := \mathcal{C}^\bullet(\Pi'(\mathbf{A}_{\mathrm{Rect}_2\setminus\mathbf{x}})) \simeq R\Gamma(\mathrm{Rect}_2\setminus\mathbf{x},\mathcal{O}) \tag{18}$$

which we know models the derived space of sections of $\mathcal{O}$ on $\mathrm{Rect}_2\setminus\mathbf{x}$. We want to interpret the second summand on the right in (17). When $n=0$, it is nothing but

$$\bigoplus_{(a,b)\in\mathbf{x}} \mathbf{A}(\{(a,b)\}) = \bigoplus_{(a,b)\in\mathbf{x}} \mathcal{O}_{(a,b)} =: C_2^0, \tag{19}$$

which we may choose to think of as the degree zero space of a complex $0\to C_2^0\to 0$. For $n\geq 1$, we need the following observation about the definition (12): for any flag[4] $f_0\subset f_1\subset\dots$ whose first space is a point, we have

$$\mathbf{A}(f_0\subset f_1\subset\dots) = \mathbf{A}(f_1\subset\dots).$$

(The only cases to check are those in (12b) and (12c) and $\mathbf{A}((a,b)\subset \mathrm{E}) = \mathbf{A}(\mathrm{E})$.) If $f_0=(a,b)$ is one of the marked points then the truncated flags $f_1\subset\dots$ on the right here belong to our semisimplicial set of flags $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times(a,b))$ in the punctured polydisc at this marked point. Using this fact, we get that, for $n\geq 1$,

$$C_3^{n-1} := \prod'_{\substack{f\in \mathrm{Flag}_n(\mathrm{Rect}_2)\\ :f_0\in \mathbf{x}}} \mathbf{A}(f) = \bigoplus_{(a,b)\in\mathbf{x}} \mathcal{C}^{n-1}(\Pi(\mathbf{A}|_{\mathrm{Flag}(\mathrm{PDisc}_2^\times(a,b))})) \tag{20}$$

(note the degree shift). We know the complexes appearing on the right here, namely

$$\mathcal{C}^\bullet(\Pi(\mathbf{A}|_{\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times(a,b))})) \simeq R\Gamma^\bullet(\mathrm{PDisc}_2^\times(a,b),\mathcal{O}),$$

model the derived algebra of sections of $\mathcal{O}$ on the punctured formal rectilinear polydiscs at the marked points $(a,b)\in\mathbf{x}$.

The 0-step flags in C_2 form a semisimplicial subset of $\mathrm{Flag}_\bullet(\mathrm{Rect}_2)$, i.e. $\Delta\downarrow C_2$ is a full subcategory of $\Delta\downarrow\mathrm{Flag}_\bullet(\mathrm{Rect}_2)$. (It just consists of isolated points.) We get the C_2-object in commutative algebras $\mathbf{A}|_{C_2}$, which just sends, cf. (12),

$$\mathbf{A}: \{(a,b)\}\mapsto S_a^{-1}\mathbb{C}[w]\otimes S_b^{-1}\mathbb{C}[z] = \mathcal{O}_{(a,b)}$$

for each marked point $(a,b)\in\mathbf{x}$.

Thus, as a graded vector space, we have that

$$C^\bullet = \left(C_1\oplus C_2\oplus \mathrm{s}^{-1}C_3\right)^\bullet, \qquad \text{i.e.} \qquad C^n = C_1^n\oplus C_2^n\oplus C_3^{n-1} \tag{21}$$

for each n, for these complexes C, C_1, C_2 and C_3 we defined in (16), (18), (19) and (20). The differential of the complex $C^\bullet$ is given, in matrix form, by

$$\mathrm{d}_C = \begin{pmatrix} \mathrm{d}_{C_1}+\mathrm{d}_{C_2} & 0 \\ d & -\mathrm{d}_{C_3}\end{pmatrix} \tag{22}$$

where d is a (degree zero) cochain map

$$d: C_1\oplus C_2\to C_3$$

defined by our choices above. Conceptually, $d|_{C_2}$ is the sum of the maps

$$\mathcal{O}_{(a,b)} = \Gamma(\mathrm{PDisc}_2(a,b),\mathcal{O}) \to R\Gamma(\mathrm{PDisc}_2^\times(a,b),\mathcal{O})$$

while $d|_{C_1}$ is the diagonal map

$$R\Gamma(\mathrm{Rect}_2\setminus\mathbf{x},\mathcal{O}) \to R\Gamma(\mathrm{PDisc}_2^\times(a,b),\mathcal{O}).$$

But presented as in (21), (22), one recognises the complex $(C^\bullet,\mathrm{d}_C)$ as the *mapping cocone* $\mathrm{Cocone}(d)$ of the cochain map d. (See e.g. [48], or [55, Chapter 10], and note that $\mathrm{Cocone}(d) = \mathrm{s}^{-1}\mathrm{Cone}(d)$.) In turn the mapping cocone represents the *homotopy kernel* $\mathrm{hoker}(d)$ of the map d, so we arrive at the following statement.

[4] Here and occasionally elsewhere, we use a suggestive but rather loose notation: the flag is strictly-speaking the tuple $(f_0,f_1,\dots)$ with $\overline{\{f_0\}}\subset\overline{\{f_1\}}\subset\dots$.

Proposition 8 *(Following [18] Proposition 1.1.4). The space of global sections of $\mathcal{O}$ on rectilinear space* Rect_2 *is the homotopy kernel of the map d:*

$$\Gamma(\mathrm{Rect}_2, \mathcal{O}) = \mathrm{hoker}\left(R\Gamma(\mathrm{Rect}_2 \setminus \mathbf{x}, \mathcal{O}) \oplus \bigoplus_{(a,b)\in\mathbf{x}} \mathcal{O}_{(a,b)} \xrightarrow{d} \bigoplus_{(a,b)\in\mathbf{x}} R\Gamma(\mathrm{PDisc}_2^{\times}(a,b), \mathcal{O}) \right), \quad \square$$

5. The Thom-Whitney-Sullivan functor

Now we describe the tool we use to produce models of derived spaces of sections which come equipped with the structure of differential graded commutative or Lie algebras. In the literature it goes by the name of the Thom-Sullivan [30,6,34] or Thom-Whitney [19] construction.

5.1. Polynomial forms on the standard algebro-geometric simplex

There is a semisimplicial[5] commutative differential graded algebra

$$\Omega : \Delta^{\mathrm{op}} \to \mathbf{dgCAlg}$$

defined as follows. For each $n \geq 0$, $\Omega([n])$ is the commutative differential graded algebra

$$\Omega([n]) := \mathbb{C}[t_0, \dots, t_n; \mathrm{d}t_0, \dots \mathrm{d}t_n] \big/ \langle \sum_{i=0}^{n} t_i - 1, \sum_{i=0}^{n} \mathrm{d}t_i \rangle$$

with t_i in degree 0 and $\mathrm{d}t_i$ in degree 1, for each i, and equipped with the usual de Rham differential. One should think of $\Omega([n])$ as the complex of polynomial differential forms on the standard algebro-geometric n-simplex. For any map $\phi : [n] \to [N]$ of Δ,

$$\Omega(\phi) : \Omega([N]) \to \Omega([n])$$

is the map of dg commutative algebras defined by $t_i \mapsto \sum_{j\in\phi^{-1}(i)} t_j$.

5.2. The functor Th

Suppose $\mathfrak{g} : \Delta \to \mathbf{dgLieAlg}$ is a semicosimplicial differential graded Lie algebra. We can construct the bigraded vector space $C^{\bullet,\bullet}$ whose spaces are

$$C^{p,q} = \Big\{ \mathbf{a} = (a_m)_{m\geq 0} \in \prod_{m\geq 0} \Omega([m])^p \otimes \mathfrak{g}([m])^q : \\ (\mathrm{id} \otimes \mathfrak{g}(\phi))\, a_n = (\Omega(\phi) \otimes \mathrm{id})\, a_N \quad \text{in} \quad \Omega([n])^p \otimes \mathfrak{g}([N])^q \\ \text{for all } n \leq N \text{ and all maps } \phi : [n] \to [N] \text{ of } \Delta \Big\}. \tag{23}$$

It is a bicomplex, with the de Rham differential $\mathrm{d}^p : C^{p,q} \to C^{p+1,q}$ and the internal differential of $\mathfrak{g}$, $\mathrm{d}^q_{\mathfrak{g}} : C^{p,q} \to C^{p,q+1}$. The Thom-Whitney complex $(\mathrm{Th}^\bullet(\mathfrak{g}), \mathrm{d}_{\mathrm{Th}})$ is by definition the corresponding total complex

$$\mathrm{Th}^n(\mathfrak{g}) = \bigoplus_{p+q=n} C^{p,q}$$

with differential

$$\mathrm{d}_{\mathrm{Th}} := \mathrm{d} + \mathrm{d}_{\mathfrak{g}}.$$

That is,

$$\mathrm{d}_{\mathrm{Th}}(\omega \otimes a) = \mathrm{d}\omega \otimes a + (-1)^{\mathrm{gr}\,\omega} \omega \otimes \mathrm{d}_{\mathfrak{g}} a.$$

[5] It is actually simplicial, i.e. it has degeneracy as well as face maps, but we shall not need this.

As a cochain complex $\mathrm{Th}^\bullet(\mathfrak{g})$ is quasi-isomorphic to the total complex $\mathrm{Tot}^\bullet(\mathfrak{g})$, $\mathrm{Tot}^n(\mathfrak{g}) = \bigoplus_{p+q=n} C^p(\mathfrak{g}^q)$, of the unnormalized cochain complex, cf. §3.8, of the dg Lie algebra $\mathfrak{g}$ [56] [29, §4]. A quasi-isomorphism $\int : \mathrm{Th}(\mathfrak{g}) \to \mathrm{Tot}(\mathfrak{g})$ is defined by integrating over the simplices; see [30, §5.2.6]. In fact an explicit deformation retracts

$$\circlearrowright \mathrm{Th}(\mathfrak{g}) \underset{E}{\overset{\int}{\rightleftarrows}} \mathrm{Tot}(\mathfrak{g}) \ ,$$

is known [13]; see [19, §6] and references therein.

The great advantage of the Thom-Whitney complex is that it comes with the structure of a differential graded Lie algebra. The graded Lie bracket is given by

$$[\omega \otimes a, \tau \otimes b] := (-1)^{\mathrm{gr}\, a \,\mathrm{gr}\, \tau} \omega \wedge \tau \otimes [a, b]$$

for all $a, b \in \mathfrak{g}([n])$ and $\omega, \tau \in \Omega([n])$, for each n. One obtains a functor, the Thom-Whitney functor, from semicosimplicial differential graded Lie algebras to differential graded Lie algebras,

$$\mathrm{Th} : [\Delta, \mathbf{dgLieAlg}] \to \mathbf{dgLieAlg}.$$

Entirely analogously, one has a functor

$$\mathrm{Th} : [\Delta, \mathbf{dgCAlg}] \to \mathbf{dgCAlg}$$

(which we also denote Th) from semicosimplicial dg commutative algebras to dg commutative algebras.

Any Lie algebra can be regarded as a differential graded Lie algebra concentrated in degree zero, and any commutative algebra can be regarded as a differential graded commutative algebra concentrated in degree zero. So the functors above restrict to functors

$$\mathrm{Th} : [\Delta, \mathbf{LieAlg}] \to \mathbf{dgLieAlg}, \quad \text{and} \quad \mathrm{Th} : [\Delta, \mathbf{CAlg}] \to \mathbf{dgCAlg}$$

which will actually be all that we need here.

5.3. *Thom-Whitney complex of an S-algebra*

Suppose S is a semisimplicial set and

$$\mathfrak{g} : \Delta \downarrow S \to \mathbf{dgLieAlg}$$

an S-object in differential graded Lie algebras, in the sense of §3.5. On composing the functor $\Pi : [\Delta \downarrow S, \mathbf{dgLieAlg}] \to [\Delta, \mathbf{dgLieAlg}]$ with the Thom-Whitney functor, we get a functor

$$\mathrm{Th} \circ \Pi : [\Delta \downarrow S, \mathbf{dgLieAlg}] \to \mathbf{dgLieAlg}.$$

There is an intuitively clear geometrical interpretation of the differential graded Lie algebra $\mathrm{Th}(\mathfrak{g})$. Recall that an S-algebra assigns an algebra to each simplex of the semisimplicial set S, and specifies maps between them. We can realize S geometrically and consider polynomial differential forms on S, with the form on each simplex valued in the corresponding algebra. It is natural to consider forms compatible with the maps between these algebras in the obvious sense. And indeed we see $C^{p,q}$ of (23) becomes

$$\begin{aligned} C^{p,q} = \Bigg\{ \mathbf{a} = (a_x)_{x \in \bigsqcup_n S([n])} \in \prod_{x \in \bigsqcup_n S([n])} \Omega([\dim x])^p \otimes \mathfrak{g}(x)^q : \\ (\mathrm{id} \otimes \mathfrak{g}(\phi))\, a_x = (\Omega(\phi) \otimes \mathrm{id})\, a_X \quad \text{in} \quad \Omega([\dim x])^p \otimes \mathfrak{g}(X)^q \\ \text{for all maps } \phi : x \to X \text{ of } \Delta \downarrow S \Bigg\}. \end{aligned} \tag{24}$$

That is, an element of $C^{p,q}$ consists of an $\mathfrak{g}(x)$-valued polynomial differential form a_x on each simplex of x of S, such that whenever x is a simplex of S on the boundary of another simplex X in S, then the pullback to x of the form a_X agrees with the image, under the map $\mathfrak{g}(x) \to \mathfrak{g}(X)$, of the form a_x.

If R is a semisimplicial subset of S then we have the morphism of semicosimplicial algebras $\pi : \Pi A \to \Pi(A|_R)$ of Lemma 4 and hence, by functoriality of Th, a map

$$\mathrm{Th}(\pi) : \mathrm{Th}(A) \to \mathrm{Th}(A|_R) \tag{25}$$

This is just the map which pulls back a differential form on S to one on R.

5.4. Example: the dg Lie algebra $\mathfrak{g}_{\mathrm{PDisc}_2^\times}$

It is helpful to see an example of the functor $\mathrm{Th} \circ \Pi$ in action.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Recall from (8) the semisimplicial set $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times)$, and from (11) our definition of the $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times)$-algebra $A_{\mathrm{PDisc}_2^\times}$:

$$\mathbb{C}[[w]] \otimes \mathbb{C}((z)) \longrightarrow B \longleftarrow B \longrightarrow B \longleftarrow \mathbb{C}((w)) \otimes \mathbb{C}[[z]]$$

where

$$B = \mathbb{C}((w)) \otimes \mathbb{C}((z)).$$

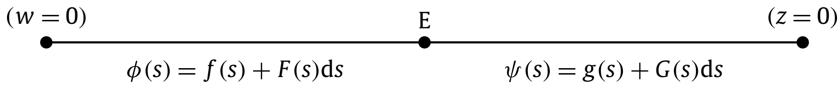
We shall write

$$\mathfrak{g}_{\mathrm{PDisc}_2^\times} := \mathrm{Th}(\mathfrak{g} \otimes A_{\mathrm{PDisc}_2^\times})$$

for the resulting differential graded Lie algebra of Thom-Whitney forms. Explicitly, $\mathfrak{g}_{\mathrm{PDisc}_2^\times}$ consists of pairs of differential forms:

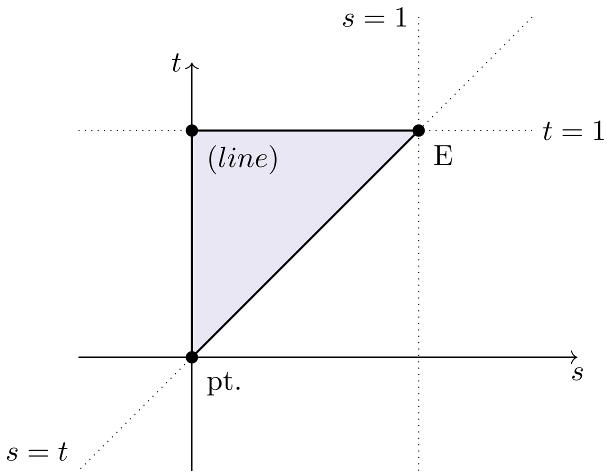
$$\begin{aligned} \mathfrak{g}_{\mathrm{PDisc}_2^\times} = \Big\{ \big(\phi(s) = f(s) + F(s)\mathrm{d}s,\ \psi(s) = g(s) + G(s)\mathrm{d}s\big) :& \\ f(s), F(s), g(s), G(s) \in \mathfrak{g} \otimes \mathbb{C}((w)) \otimes \mathbb{C}((z)) \quad &\text{for all } s, \\ f(1) = g(1),& \\ f(0) \in \mathfrak{g} \otimes \mathbb{C}[[w]] \otimes \mathbb{C}((z)), \quad g(0) \in \mathfrak{g} \otimes \mathbb{C}((w)) \otimes \mathbb{C}[[z]] \Big\}.& \end{aligned} \tag{26}$$

We should think of these forms as painted onto the edges of the semisimplicial set $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times)$, cf. §5.5:



5.5. Conventions for coordinates on simplices

For us, every 2-simplex corresponds to a flag of the form point pt. $\subset$ (*line*) $\subset$ E. (Compare (8).) On each individual such simplex, we choose coordinates (s, t) as follows.



Thus, on each simplex:

- $\mathrm{d}s$ is a nonzero constant one-form that vanishes on the edge pt. $\subset$ (*line*)
- $\mathrm{d}t$ is a nonzero constant one-form that vanishes on the edge (*line*) $\subset$ E
- these one-forms agree, $\mathrm{d}s = \mathrm{d}t$, on the edge pt. $\subset$ E.

6. Homotopy Manin triples

As we discussed in the introduction, our main goal in the present work is to give higher generalizations of certain Manin triples which are important in the theory of integrable systems.

To that end we must first clarify what such a generalization of a Manin triple should mean. We begin by expressing the usual definition of a Manin triple of Lie algebras in a form amenable to generalization. Recall that we work over $\mathbb{C}$, here and throughout. A *Manin triple* $(\mathfrak{a}, \mathfrak{a}_\pm, \iota_\pm, \langle - | - \rangle)$ is the data of

(1) Lie algebras $\mathfrak{a}$, $\mathfrak{a}_+$, $\mathfrak{a}_-$,
(2) Lie algebra maps $\mathfrak{a}_+ \xrightarrow{\iota_+} \mathfrak{a} \xleftarrow{\iota_-} \mathfrak{a}_-$, and
(3) a map of vector spaces $\langle - | - \rangle : \mathfrak{a} \otimes \mathfrak{a} \to \mathbb{C}$,

subject to the following conditions:

(i) the map of vector spaces $(\iota_+, \iota_-) : \mathfrak{a}_+ \oplus \mathfrak{a}_- \to \mathfrak{a}$ is an isomorphism.
(ii) the map $\langle - | - \rangle : \mathfrak{a} \otimes \mathfrak{a} \to \mathbb{C}$ is
  - symmetric: $\langle x | y \rangle = \langle y | x \rangle$ for all $x, y \in \mathfrak{a}$.
  - invariant: $\langle [x, y] | z \rangle + \langle y | [x, z] \rangle = 0$ for all $x, y, z \in \mathfrak{a}$.
(iii) the map $\langle - | - \rangle : \mathfrak{a} \otimes \mathfrak{a} \to \mathbb{C}$ is non-degenerate: If $\langle x | - \rangle = 0$ as maps $\mathfrak{a} \to \mathbb{C}$ then $x = 0$.
(iv) both $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are isotropic, i.e. the maps

$$\langle \iota_\pm(-) | \iota_\pm(-) \rangle : \mathfrak{a}_\pm \otimes \mathfrak{a}_\pm \to \mathbb{C}$$

are zero.

Having expressed the definition this way, it seems natural to make the following generalization to dg Lie algebras:

Definition 9. A *homotopy Manin triple (of dg Lie algebras)* $(\mathfrak{a}, \mathfrak{a}_\pm, \iota_\pm, \langle - | - \rangle, n)$ is the data of

(1) dg Lie algebras $\mathfrak{a}$, $\mathfrak{a}_+$ and $\mathfrak{a}_-$
(2) dg Lie algebra maps $\mathfrak{a}_+ \xrightarrow{\iota_+} \mathfrak{a} \xleftarrow{\iota_-} \mathfrak{a}_-$, and
(3) a (degree zero) map of dg vector spaces $\langle - | - \rangle : \mathfrak{a} \otimes \mathfrak{a} \to \mathsf{s}^{-n}\mathbb{C}$

subject to the following conditions:

(i) the map of dg vector spaces $(\iota_+, \iota_-) : \mathfrak{a}_+ \oplus \mathfrak{a}_- \to \mathfrak{a}$ is a homotopy equivalence
(ii) the map $\langle - | - \rangle : \mathfrak{a} \otimes \mathfrak{a} \to \mathsf{s}^{-n}\mathbb{C}$ is
  - (graded) symmetric: $\langle x | y \rangle = (-1)^{\mathrm{gr}\, x\, \mathrm{gr}\, y} \langle y | x \rangle$ for all $x \in \mathfrak{a}^{\mathrm{gr}\, x}$, $y \in \mathfrak{a}^{\mathrm{gr}\, y}$.
  - invariant:

$$\langle [x, y] | z \rangle + (-1)^{\mathrm{gr}\, x\, \mathrm{gr}\, y} \langle y | [x, z] \rangle = 0$$
$$\langle \mathrm{d}_{\mathfrak{a}} x | y \rangle + (-1)^{\mathrm{gr}\, x} \langle x | \mathrm{d}_{\mathfrak{a}} y \rangle = 0$$

  for all $x \in \mathfrak{a}^{\mathrm{gr}\, x}$, $y \in \mathfrak{a}^{\mathrm{gr}\, y}$ and $z \in \mathfrak{a}^{\mathrm{gr}\, z}$.
(iii) the map $\langle - | - \rangle : \mathfrak{a} \otimes \mathfrak{a} \to \mathsf{s}^{-n}\mathbb{C}$ is non-degenerate up to homotopy: If $\langle x | - \rangle \simeq 0$ then $x \simeq 0$ (i.e. x is exact).
(iv) both $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are isotropic, i.e. the maps

$$\langle \iota_\pm(-) | \iota_\pm(-) \rangle : \mathfrak{a}_\pm \otimes \mathfrak{a}_\pm \to \mathsf{s}^{-n}\mathbb{C}$$

are homotopic to zero.

Let us make several remarks about this definition.

Remark 10. Recall that in the category of dg vector spaces, every quasi-isomorphism is a homotopy equivalence (cf. §3.9). Consequently, conditions (i) (iii) and (iv) respectively could be replaced with the following equivalent demands:

(i') the map (ι_+, ι_-) induces an isomorphism of graded vector spaces

$$H(\mathfrak{a}_+) \oplus H(\mathfrak{a}_-) \cong_{\mathbf{grVect}} H(\mathfrak{a})$$

(iii') the map of graded vector spaces

$$H(\mathfrak{a}) \otimes H(\mathfrak{a}) \to \mathrm{s}^{-n}\mathbb{C}$$

induced by $\langle - \mid - \rangle$ is non-degenerate (on the nose).

(iv') both $H(\mathfrak{a}_+)$ and $H(\mathfrak{a}_-)$ are isotropic (on the nose) as subspaces of $H(\mathfrak{a})$. ◁

Remark 11. For a suitable notion of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$, condition (iii) is equivalent to

(iii") The map $\mathfrak{a} \to \mathrm{s}^{-n}\mathfrak{a}^*$ induced by $\langle - \mid - \rangle$ is a homotopy equivalence.

(For us $\mathfrak{a}$ is often a cochain complex in *topological* vector spaces, and the appropriate dual cochain complex has spaces $\mathrm{Hom}^{\mathrm{cts}}(\mathfrak{a}^n, \mathbb{C})$ consisting of the *continuous* linear maps to $\mathbb{C}$ with its discrete topology. For example, in the case of $\mathbb{C}[[t]]$ with its t-adic topology, one has $\mathrm{Hom}^{\mathrm{cts}}(\mathbb{C}[[t]], \mathbb{C}) \cong t^{-1}\mathbb{C}[t^{-1}]$, as one wants.) ◁

Remark 12. Let $\mathbf{dgLieAlg}^\circ$ denote the full subcategory of **dgLieAlg** whose objects are those dg Lie algebras that are both fibrant and cofibrant in some model structure (for example, in the standard projective model structure on **dgLieAlg** induced from the standard projective model structure on **dgVect** [28]). Recall that in $\mathbf{dgLieAlg}^\circ$ every quasi-isomorphism is a homotopy equivalence, i.e. is invertible up to homotopies. One sees that if $\mathfrak{a}, \mathfrak{a}_\pm$ and $\mathfrak{b}, \mathfrak{b}_\pm$ are objects in $\mathbf{dgLieAlg}^\circ$ and $\mathfrak{a} \simeq \mathfrak{b}$ and $\mathfrak{a}_\pm \simeq \mathfrak{b}_\pm$, then a Manin triple structure for $\mathfrak{a}, \mathfrak{a}_\pm$ induces a unique Manin triple structure for $\mathfrak{b}, \mathfrak{b}_\pm$. ◁

Remark 13. Kravchenko gives a definition of a Manin L_∞-triple, or strongly homotopy Manin triple, in [37]. We discuss the relationship between that definition and Definition 9 in Appendix C. ◁

7. Local homotopy Manin triple

With the above definition of a homotopy Manin triple in place, we are ready in this section to give our first main example of such a structure. Namely, we define a homotopy Manin triple associated to the punctured formal rectilinear polydisc $\mathrm{PDisc}_2^\times$. See Theorem 14 below. It can be seen as a higher analog of the Manin triple (1) from the introduction.

Recall first the dg Lie algebra $\mathfrak{g}_{\mathrm{PDisc}_2^\times}$ we defined in §5.4. It plays the role of the Lie algebra $\mathfrak{g} \otimes \mathbb{C}((z)) = \mathfrak{g} \otimes \Gamma(\mathrm{Disc}_1^\times, \widehat{\mathcal{O}})$ in the Manin triple in (1). Now we define the two dg Lie algebras which will be analogs of $\mathfrak{g} \otimes \mathbb{C}[[z]]$ and $\mathfrak{g} \otimes z^{-1}\mathbb{C}[z^{-1}]$.

7.1. The dg Lie algebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$

Let

$$\mathfrak{g}_+ := \mathfrak{g} \otimes \mathbb{C}[[w]] \otimes \mathbb{C}[[z]],$$

regarded as a dg Lie algebra concentrated in degree zero. Let

$$\mathfrak{g}_- := \mathrm{Th}(\mathfrak{g} \otimes A^{--}_{\mathrm{PDisc}_2^\times}),$$

where we introduce another $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times)$-algebra $A^{--}_{\mathrm{PDisc}_2^\times}$, given by

$$0 \longrightarrow w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] \longleftarrow w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] \longrightarrow w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] \longleftarrow 0$$

Explicitly then, $\mathfrak{g}_-$ consists of pairs of differential forms:

$$\mathfrak{g}_- = \Big\{ \big(\phi(s) = f(s) + F(s)\mathrm{d}s,\ \psi(s) = g(s) + G(s)\mathrm{d}s\big) : \\ f(s), F(s), g(s), G(s) \in \mathfrak{g} \otimes w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] \quad \text{for all } s, \\ f(1) = g(1), \quad f(0) = 0, \quad g(0) = 0 \Big\},$$

cf. (26). We have the maps of dg Lie algebras

$$\mathfrak{g}_+ \xrightarrow{i_+} \mathfrak{g}_{\mathrm{PDisc}_2^\times} \xleftarrow{i_-} \mathfrak{g}_-$$

where $i_+ : \mathfrak{g}_+ \to \mathfrak{g}_{\mathrm{PDisc}_2^\times}$ sends $f \in \mathfrak{g}_+$ to the constant function $f \in \mathfrak{g}_{\mathrm{PDisc}_2^\times}$, and $i_- : \mathfrak{g}_- \to \mathfrak{g}_{\mathrm{PDisc}_2^\times}$ is the map of dg Lie algebras coming from the canonical embeddings, using Lemma 2 and functoriality of Th.

7.2. The bilinear form

Now we define on $\mathfrak{g}_{\mathrm{PDisc}_2^\times}$ a symmetric invariant form

$$\langle - \mid - \rangle : \mathfrak{g}_{\mathrm{PDisc}_2^\times} \otimes \mathfrak{g}_{\mathrm{PDisc}_2^\times} \to \mathsf{s}^{-1}\mathbb{C}.$$

(Recall $\mathsf{s}^{-1}\mathbb{C}$ is a copy of $\mathbb{C}$ put into cohomological degree 1.) We pick an orientation of $\mathrm{Flag}(\mathrm{PDisc}_2^\times)$: let

$$\Sigma = \big((w=0), \mathrm{E}\big) - \big((z=0), \mathrm{E}\big)$$

be the 1-chain whose boundary is $\partial\Sigma = (w=0) - (z=0)$. Observe that, in the notation of §5.4, we have

$$\int_\Sigma (\mathrm{d}s, -\mathrm{d}s) = \int_{\Delta^1} \mathrm{d}s + \int_{\Delta^1} \mathrm{d}s = 2.$$

For $a, b \in \mathfrak{g}$ and $\omega, \lambda \in \mathrm{Th}(A_{\mathrm{PDisc}_2^\times})$, we set

$$\langle a \otimes \omega \mid b \otimes \lambda \rangle := \mathsf{s}^{-1} \frac{1}{2} \kappa\big(a|b\big) \int_\Sigma \mathrm{res}_w \, \mathrm{res}_z \, \omega \wedge \lambda$$

where $\kappa\big(-|-\big)$ denotes the standard invariant symmetric bilinear form on the simple Lie algebra $\mathfrak{g}$, and where

$$\mathrm{res}_t : \mathbb{C}((t)) \to \mathbb{C}; \quad \sum_k f_k t^k \mapsto f_{-1}$$

is the residue map. Then we extend $\langle - \mid - \rangle$ by linearity to all of $\mathfrak{g}_{\mathrm{PDisc}_2^\times} \otimes \mathfrak{g}_{\mathrm{PDisc}_2^\times}$.

7.3. Manin triple

The main result of this section is then the following.

Theorem 14. *These data*

$$(\mathfrak{g}_{\mathrm{PDisc}_2^\times}, \mathfrak{g}_+, \mathfrak{g}_-, i_+, i_-, \langle - \mid - \rangle)$$

constitute a homotopy Manin triple in dg Lie algebras, in the sense of Definition 9.

Proof. Condition (i) is Proposition 15 below. For condition (ii), graded symmetry is clear, and invariance is Proposition 19. For the nondegeneracy and isotropy conditions (iii) and (iv), it is convenient to establish the equivalent statements about cohomologies from Remark 10. We do so in Proposition 20. □

We start with the following, which is fundamental for us.

Proposition 15. *At the level of dg vector spaces, $\mathfrak{g}_- \oplus \mathfrak{g}_+$ is a deformation retract of $\mathfrak{g}_{\mathrm{PDisc}_2^\times}$:*

$$\mathfrak{g}_- \oplus \mathfrak{g}_+ \underset{P}{\overset{I}{\rightleftarrows}} \mathfrak{g}_{\mathrm{PDisc}_2^\times} \circlearrowleft h$$

Proof. The maps i_+ and i_- of dg Lie algebras define the map of dg vector spaces

$$I = i_+ \oplus i_- : \mathfrak{g}_- \oplus \mathfrak{g}_+ \to \mathfrak{g}_{\mathrm{PDisc}_2^\times}.$$

(Note that it is not a map of dg Lie algebras: the images of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ in $\mathfrak{g}_{\mathrm{PDisc}_2^\times}$ are not mutually commuting.)

We must define a map

$$P : \mathfrak{g}_{\mathrm{PDisc}_2^\times} \to \mathfrak{g}_- \oplus \mathfrak{g}_+$$

and a homotopy $h : \mathfrak{g}_{\mathrm{PDisc}_2^\times} \to \mathfrak{g}_{\mathrm{PDisc}_2^\times}$.

Let

$$\omega(s) = (\phi(s), \psi(s)) = (f(s) + F(s)\mathrm{d}s, g(s) + G(s)\mathrm{d}s)$$

be the element of $\mathfrak{g}_{\mathrm{PDisc}_2^\times}$ we introduced above, cf. (26). To define the map P and the homotopy h, we first note that we get a unique decomposition

$$\omega = \omega^{++} + \omega^{+-} + \omega^{-+} + \omega^{--} \tag{27}$$

coming from the direct sum decomposition of vector spaces

$$\begin{aligned}&\mathbb{C}((w)) \otimes \mathbb{C}((z))\\ &\cong_{\mathbb{C}} \mathbb{C}[[w]] \otimes \mathbb{C}[[z]] \oplus \mathbb{C}[[w]] \otimes z^{-1}\mathbb{C}[z^{-1}] \oplus w^{-1}\mathbb{C}[w^{-1}] \otimes \mathbb{C}[[z]] \oplus w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}].\end{aligned}$$

We observe that ω^{--} may be interpreted as an element of $\mathfrak{g}_-$, and define

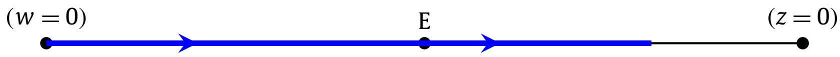
$$P(\omega) := \left(\omega^{--}\,,\ \omega^{++}|_{s=1}\right). \tag{28}$$

We should then also set $h(\omega^{--}) = 0$, for indeed $\omega^{--} = I \circ P(\omega^{--})$ holds exactly.

Next we define

$$h(\omega^{-+})(s) = \left(\int_0^s F^{-+}(s')\mathrm{d}s'\,,\ \int_0^1 F^{-+}(s')\mathrm{d}s' + \int_1^s G^{-+}(s')\mathrm{d}s'\right)$$

which we may sketch as



The choice of base point for these integrals is fixed by the following consideration. The coefficient function $F^{-+}(s)$ of the one-form $F^{-+}(s)\mathrm{d}s$ obeys no conditions at $s = 0$ in general, and yet we need the function $h(F^{-+}(s)\mathrm{d}s)$ to vanish there, since $w^{-1}\mathbb{C}[w^{-1}] \otimes \mathbb{C}[[z]] \cap \mathbb{C}[[w]] \otimes \mathbb{C}((z)) = 0$.

We then indeed have that

$$\begin{aligned}[\mathrm{d}, h](f^{-+}(s), g^{-+}(s)) = h \circ \mathrm{d}(f^{-+}(s), g^{-+}(s)) &= \left(f^{-+}(s), g^{-+}(s) - g^{-+}(1) + f^{-+}(1)\right)\\ &= (f^{-+}(s), g^{-+}(s))\end{aligned}$$

and

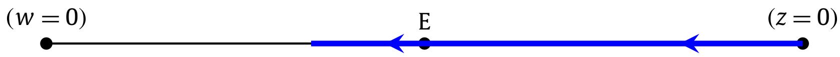
$$[\mathrm{d}, h](F^{-+}(s)\mathrm{d}s, G^{-+}(s)\mathrm{d}s) = \mathrm{d} \circ h(F^{-+}(s)\mathrm{d}s, G^{-+}(s)\mathrm{d}s) = (F^{-+}(s)\mathrm{d}s, G^{-+}(s)\mathrm{d}s).$$

That is, $[\mathrm{d}, h]\omega^{-+} = \omega^{-+}$.

For the same reasons, for the component ω^{+-} we are forced to integrate from the other end. We set

$$h(\omega^{+-})(s) = \left(\int_0^1 G^{+-}(s')\mathrm{d}s' + \int_1^s F^{+-}(s')\mathrm{d}s'\,,\ \int_0^s G^{+-}(s')\mathrm{d}s'\,,\right)$$

which we may sketch as

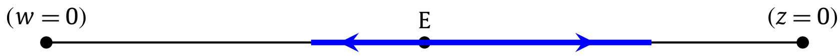


and then we have $[\mathrm{d}, h]\omega^{+-} = \omega^{+-}$.

For the component ω^{++} the new feature is that $f^{++}(s)$ can have a non-zero constant term. We set

$$h(\omega^{++})(s) = \left(\int_1^s F^{++}(s')\mathrm{d}s'\,,\ \int_1^s G^{++}(s')\mathrm{d}s'\right)$$

which we may sketch as



We then have

$$\begin{aligned}[\mathrm{d},h](f^{++}(s),g^{++}(s)) &= h\circ \mathrm{d}(f^{++}(s),g^{++}(s))\\ &= (f^{++}(s)-f^{++}(1),g^{++}(s)-g^{++}(1))\\ &= (f^{++}(s),g^{++}(s))-(f^{++}(1),g^{++}(1))\\ &= (\mathrm{id}-I\circ P)(f^{++}(s),g^{++}(s))\end{aligned}$$

and

$$[\mathrm{d},h](F^{++}(s)\mathrm{d}s,G^{++}(s)\mathrm{d}s)=\mathrm{d}\circ h(F^{++}(s)\mathrm{d}s,G^{++}(s)\mathrm{d}s)=(F^{++}(s)\mathrm{d}s,G^{++}(s)\mathrm{d}s).$$

This completes the proof that $[\mathrm{d},h]=\mathrm{id}-I\circ P$. It is clear on inspection that the relation $\mathrm{id}=P\circ I$ holds exactly. □

The next statement shows that the cohomology of $\mathfrak{g}_-$ is a copy of $\mathfrak{g}\otimes w^{-1}z^{-1}\mathbb{C}[w^{-1},z^{-1}]$ in cohomological degree one.

Proposition 16. *There is a deformation retract dg vector spaces*

$$\mathsf{s}^{-1}\mathfrak{g}\otimes w^{-1}z^{-1}\mathbb{C}[w^{-1},z^{-1}] \underset{p}{\overset{i}{\rightleftarrows}} \mathfrak{g}_- \circlearrowleft h$$

Proof. We let

$$i(\mathsf{s}^{-1}a):=\frac{1}{2}a(\mathrm{d}s,-\mathrm{d}s),$$

$$p\big((f(s)+F(s)\mathrm{d}s,g(s)+G(s)\mathrm{d}s)\big):=\mathsf{s}^{-1}\int_0^1(F(s')-G(s'))\mathrm{d}s'$$

and finally

$$h\big((f(s)+F(s)\mathrm{d}s,g(s)+G(s)\mathrm{d}s)\big)$$
$$:=\left(\int_0^s F(s')\mathrm{d}s'+\frac{1}{2}s\int_0^1(F(s')-G(s')\mathrm{d}s'),\ \int_0^s G(s')\mathrm{d}s'-\frac{1}{2}s\int_0^1(F(s')-G(s')\mathrm{d}s')\right).$$

Then $p(i(\mathsf{s}^{-1}a))=\mathsf{s}^{-1}\int_0^1 a\mathrm{d}s'=\mathsf{s}^{-1}a\int_0^1\mathrm{d}s'=\mathsf{s}^{-1}a$, so $p\circ i=\mathrm{id}$, and

$$\begin{aligned}&(h\circ\mathrm{d})\big((f(s)+F(s)\mathrm{d}s,g(s)+G(s)\mathrm{d}s)\big)\\ &=h\big((f'(s)\mathrm{d}s,g'(s)\mathrm{d}s)\big)\\ &=\Big(f(s)+\frac{1}{2}s(f(1)-f(0)-g(1)+g(0)),\\ &\qquad g(s)-\frac{1}{2}s(f(1)-f(0)-g(1)+g(0))\Big)=(f(s),g(s)),\end{aligned}$$

since $f(1)=g(1)$ and $f(0)=0=g(0)$, while

$$\begin{aligned}&(\mathrm{d}\circ h)\big((f(s)+F(s)\mathrm{d}s,g(s)+G(s)\mathrm{d}s)\big)\\ &=\mathrm{d}\left(\int_0^s F(s')\mathrm{d}s'+\frac{1}{2}s\int_0^1(F(s')-G(s')\mathrm{d}s'),\ \int_0^s G(s')\mathrm{d}s'-\frac{1}{2}s\int_0^1(F(s')-G(s')\mathrm{d}s'\right).\\ &=\big(F(s)\mathrm{d}s,G(s)\mathrm{d}s)+\frac{1}{2}\int_0^1(F(s')-G(s')\mathrm{d}s')(\mathrm{d}s,-\mathrm{d}s)\end{aligned}$$

and here we recognize the last term as $i(p(\big((f(s)+F(s)\mathrm{d}s,g(s)+G(s)\mathrm{d}s)\big)))$, so that

$$\mathrm{id}-i\circ p=[\mathrm{d},h]$$

as required. □

On combining this with Proposition 15, we get the following corollary.

Corollary 17. *At the level of graded vector spaces*

$$H^k(\mathfrak{g}_{\mathrm{PDisc}_2^\times}) = \begin{cases} \mathfrak{g} \otimes \mathbb{C}[[w]] \otimes \mathbb{C}[[z]] & k = 0 \\ \mathfrak{g} \otimes w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] & k = 1 \\ 0 & k \notin \{0, 1\} \end{cases}$$

Remark 18. Since $H^\bullet(\mathfrak{g}_{\mathrm{PDisc}_2^\times})$ is a retract in of $\mathfrak{g}_{\mathrm{PDisc}_2^\times}$ in **dgVect**, it is possible to endow it with the structure of an L_∞ algebra via the homotopy transfer theorem, cf. Appendix C. ◁

Proposition 19. *The symmetric bilinear form $\langle - \mid - \rangle$ is $\mathfrak{g}_{\mathrm{PDisc}_2^\times}$-invariant, i.e.*

$$\langle [x, y] \mid z\rangle + (-1)^{\mathrm{gr}\, x\, \mathrm{gr}\, y} \langle y \mid [x, z]\rangle = 0$$
$$\langle \mathrm{d}x \mid y\rangle + (-1)^{\mathrm{gr}\, x} \langle x \mid \mathrm{d}y\rangle = 0$$

for all $x, y, z \in \mathfrak{g}_{\mathrm{PDisc}_2^\times}$.

Proof. The first part is clear, given the $\mathfrak{g}$-invariance of $\kappa(-|-)$. For the second part, we recall that $\mathrm{d}(a \otimes \omega) = a \otimes \mathrm{d}\omega$ in our setting, and we have

$$\begin{aligned} &\langle a \otimes \mathrm{d}\omega \mid b \otimes \lambda\rangle + (-1)^{\mathrm{gr}\, a\otimes\omega} \langle a \otimes \omega \mid b \otimes \mathrm{d}\lambda\rangle \\ &= \mathsf{s}^{-1}\frac{1}{2}\kappa(a|b)\int_\Sigma \mathrm{res}_w\, \mathrm{res}_z\, \mathrm{d}\,(\omega \wedge \lambda) \\ &= \mathsf{s}^{-1}\frac{1}{2}\kappa(a|b)\, \mathrm{res}_w\, \mathrm{res}_z\, ((\omega \wedge \lambda)|_{w=0} - (\omega \wedge \lambda)|_{z=0})\,. \end{aligned}$$

Obviously both the pullbacks here $(\omega \wedge \lambda)|_{w=0}$ and $(\omega \wedge \lambda)|_{z=0}$ receive contributions only from the degree zero components of ω and λ. Both vanish after taking the residues $\mathrm{res}_w\, \mathrm{res}_z$, because ω and λ obey the boundary conditions in (26). □

Proposition 20. *The pairing $\langle - \mid - \rangle$ induces a non-degenerate pairing*

$$H(\mathfrak{g}_{\mathrm{PDisc}_2^\times}) \otimes H(\mathfrak{g}_{\mathrm{PDisc}_2^\times}) \to \mathsf{s}^{-1}\mathbb{C},$$

with respect to which both $H(\mathfrak{g}_+)$ and $H(\mathfrak{g}_-)$ are isotropic.

Proof. Recall that $H^0(\mathfrak{g}_{\mathrm{PDisc}_2^\times}) = \mathfrak{g}_+ := \mathfrak{g} \otimes \mathbb{C}[[w, z]]$ and the map $i_+ : \mathfrak{g}_+ \to \mathfrak{g}_{\mathrm{PDisc}_2^\times}$ of Proposition 15 maps elements of $\mathfrak{g}_+$ to constant $\mathfrak{g}_+$-valued 0-forms on $\mathrm{Flag}\,\mathrm{PDisc}_2^\times$. Similarly, the map

$$H^1(\mathfrak{g}_{\mathrm{PDisc}_2^\times}) \cong \mathsf{s}^{-1}\mathfrak{g} \otimes w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] \xrightarrow{i} \mathfrak{g}_- \xrightarrow{i_-} \mathfrak{g}_{\mathrm{PDisc}_2^\times}$$

from Proposition 16 and Proposition 15 maps elements of $H^1(\mathfrak{g}_{\mathrm{PDisc}_2^\times})$ to multiples of the 1-form $(\mathrm{d}s, -\mathrm{d}s)$. The pairing $\langle - \mid - \rangle$ restricts to a manifestly non-degenerate pairing between them:

$$\begin{aligned} H^0(\mathfrak{g}_{\mathrm{PDisc}_2^\times}) \otimes H^1(\mathfrak{g}_{\mathrm{PDisc}_2^\times}) &\to \mathsf{s}^{-1}\mathbb{C} \\ (x, y) &\mapsto \langle i_+(x) \mid i_-(i(y))\rangle \end{aligned}$$

while $H^0(\mathfrak{g}_{\mathrm{PDisc}_2^\times})$ and $H^1(\mathfrak{g}_{\mathrm{PDisc}_2^\times})$ are isotropic. (This establishes the result, because $H(\mathfrak{g}_+) = H^0(\mathfrak{g}_{\mathrm{PDisc}_2^\times})$ and $H(\mathfrak{g}_-) = H^1(\mathfrak{g}_{\mathrm{PDisc}_2^\times})$.) □

This completes the proof of Theorem 14.

7.4. The unpunctured polydisc

Now we turn to the dg Lie algebra for the unpunctured formal rectilinear polydisc:

$$\mathfrak{g}_{\mathrm{PDisc}_2} := \mathrm{Th}(\mathfrak{g} \otimes A_{\mathrm{PDisc}_2}).$$

The next statement implies in particular that $\mathfrak{g}_{\mathrm{PDisc}_2}$ and $\mathfrak{g}_+ := \mathfrak{g} \otimes \mathbb{C}[[w]] \otimes \mathbb{C}[[z]]$ are quasi-isomorphic. The proof is an instructive warm-up for the global cases in the next section.

Proposition 21. *There is a deformation retract of dg vector spaces*

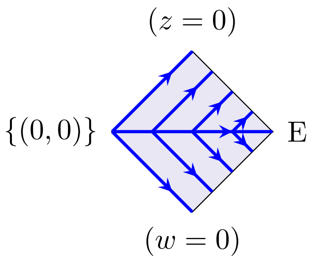
$$\mathfrak{g}_+ \underset{P}{\overset{I}{\rightleftarrows}} \mathfrak{g}_{\mathrm{PDisc}_2} \circlearrowleft h$$

in which both I and P are maps of dg Lie algebras.

Moreover, we may choose the map P to be given by pull-back of the non-singular part to the 0-simplex E*:*

$$P(\omega) := \omega^{++}|_{\mathrm{E}}$$

Proof. It is helpful to sketch the idea first:



An element of $\mathfrak{g}_{\mathrm{PDisc}_2}$ is given by a pair of forms, one on each of the two 2-simplices of PDisc_2,

$$\omega = (\Phi, \Psi)$$

valued in $\mathfrak{g} \otimes \mathbb{C}((w)) \otimes \mathbb{C}((z))$. We give the definition of h on the first of the two simplices as they are ordered here; the situation on the second is similar. In the coordinates s, t from §5.5, we have

$$\Phi(s,t) = f(s,t) + f_s(s,t)\mathrm{d}s + f_t(s,t)\mathrm{d}t + f_{st}(s,t)\mathrm{d}s \wedge \mathrm{d}t \tag{29}$$

for some coefficient functions, and we define

$$h(\Phi) := a(\Phi) + b(\Phi)$$

with

$$a(\Phi)(s,t) := \left(\int_s^t f_t(s,t')\mathrm{d}t'\right) - \left(\int_s^t f_{st}(s,t')\mathrm{d}t'\right)\mathrm{d}s \tag{30}$$

and

$$b(\Phi)(s,t) := \left(\int_0^s \big(f_s(s',s') + f_t(s',s')\big)\,\mathrm{d}s'\right)$$

Observe that the definition of h is compatible with the boundary conditions. In particular, on the "far edge" $t = 1$, i.e. $(z = 0) \subset \mathrm{E}$, the boundary condition is empty since the space attached to this edge, $\mathfrak{g} \otimes \mathbb{C}((w)) \otimes \mathbb{C}((z))$, is the same as that attached to the 2-simplex it borders.

We then compute, first,

$$(\mathrm{d} \circ a(\Phi))(s,t) = \left(\int_s^t f_{t,1}(s,t')\mathrm{d}t'\right)\mathrm{d}s + f_t(s,t)\mathrm{d}t - f_t(s,s)\mathrm{d}s + f_{st}(s,t)\mathrm{d}s \wedge \mathrm{d}t$$

and

$$\begin{aligned}(a \circ \mathrm{d}(\Phi))(s,t) &= a\Big(f_{,1}(s,t)\mathrm{d}s + f_{,2}(s,t)\mathrm{d}t + \big(f_{t,1}(s,t) - f_{s,2}(s,t)\big)\,\mathrm{d}s \wedge \mathrm{d}t\Big) \\ &= \left(\int_s^t f_{,2}(s,t')\mathrm{d}t'\right) - \left(\int_s^t (f_{t,1}(s,t') - f_{s,2}(s,t'))\mathrm{d}t'\right)\mathrm{d}s \\ &= f(s,t) - f(s,s) - \left(\int_s^t f_{t,1}(s,t')\mathrm{d}t'\right)\mathrm{d}s + f_s(s,t)\mathrm{d}s - f_s(s,s)\mathrm{d}s,\end{aligned}$$

and so find that

$$([\mathrm{d}, a](\Phi))(s,t) = \Phi(s,t) - f(s,s) - f_t(s,s)\mathrm{d}s - f_s(s,s)\mathrm{d}s.$$

(Here we recognize $f(s,s) + (f_s(s,s) + f_t(s,s))\,\mathrm{d}s$ as the pullback of the form Φ to the 1-simplex pt. $\subset$ E on which $s = t$.)

Next we compute

$$(\mathrm{d} \circ b(\Phi))(s,t) = f_s(s,s)\mathrm{d}s + f_t(s,s)\mathrm{d}s$$

and

$$\begin{aligned}(b \circ \mathrm{d}(\Phi))(s,t) &= b\Big(f_{,1}(s,t)\mathrm{d}s + f_{,2}(s,t)\mathrm{d}t + \big(f_{t,1}(s,t) - f_{s,2}(s,t)\big)\,\mathrm{d}s \wedge \mathrm{d}t\Big) \\ &= \int_0^s \big(f_{,1}(s',s') + f_{,2}(s',s')\big)\,\mathrm{d}s' = \int_0^s \partial_{s'} f(s',s')\mathrm{d}s' \\ &= f(s,s) - f(0,0)\end{aligned}$$

which means we have

$$([\mathrm{d}, b](\Phi))(s,t) = f(s,s) + f_s(s,s)\mathrm{d}s + f_t(s,s)\mathrm{d}s - f(0,0)$$

Thus, in total, we see that

$$([\mathrm{d}, h](\Phi))(s,t) = \Phi(s,t) - f(0,0).$$

Then we define $P : \mathfrak{g}_{\mathrm{PDisc}_2} \to \mathfrak{g}_+$ to be the map $\omega \mapsto f(0,0) = \omega|_{\{(0,0)\}}$, picking out the pullback of ω to the vertex $\{(0,0)\}$, and $I : \mathfrak{g}_+ \to \mathfrak{g}_{\mathrm{PDisc}_2}$ to be the embedding of an element of $\mathfrak{g}_+$ as a constant function. This ensures that $P \circ I = \mathrm{id}$ and that the equality above becomes

$$[\mathrm{d}, h] = \mathrm{id} - I \circ P,$$

which completes the proof that $\mathfrak{g}_+$ is a retract of $\mathfrak{g}_{\mathrm{PDisc}_2}$.

It remains to establish the "moreover" part of the proposition. Above, we chose to set $P(\omega) = \omega|_{\{(0,0)\}}$. In what follows, it will be helpful to note the following alternative choice for the maps I, P, h. We keep the definition of I. We let $P : \mathfrak{g}_{\mathrm{PDisc}_2} \to \mathfrak{g}_+$ be the map $\omega \mapsto f(s=1, t=1)^{++} = \omega^{++}|_{\mathrm{E}}$, picking out the pullback of the regular part of ω at the vertex E. And we let h be given by

$$h_{new}(\Phi) = h_{old}(\Phi) + \int_1^0 \big(f_s(s',s')^{++} + f_t(s',s')^{++}\big)\,\mathrm{d}s'$$

Then $(\mathrm{d} \circ h(\Phi))(s,t)$ is unaltered (we have added a constant), while $(h \circ \mathrm{d}(\Phi))(s,t)$ receives the extra term

$$\int_1^0 \partial_{s'} f(s',s')^{++}\mathrm{d}s' = f(0,0)^{++} - f(1,1)^{++} = f(0,0) - f(1,1)^{++}$$

so that

$$([\mathrm{d}, h](\Phi))(s,t) = \Phi(s,t) - f(s=1, t=1)^{++}$$

as we now need with our new definition of P. □

7.5. Pictorial notation for homotopies and retracts

The proofs above are prototypes of the sort of computation we shall need in many places below. The general strategy remains the same: given a polynomial form ω on some semisimplicial set, we shall pick a decomposition, much as we did in (27), chosen to ensure that each summand has empty boundary conditions on at least some boundaries. Then, summand by summand, we shall retract away from those boundaries, in a sense we now discuss.

Recall our conventions for coordinates from §5.5.

Let $V_\triangle$ denote the complex consisting of $\mathbb{C}$-valued polynomial differential forms on a 2-simplex subject to the boundary condition that they must vanish on pullback to some given choice of (none, one, two, or all three) of the edges $s = 0$ and $s = t$ and $t = 1$.

Let $V_{s=t}$ denote the complex of $\mathbb{C}$-valued polynomial differential forms on the boundary 1-simplex at $s = t$, and with the boundary condition inherited from $V_\triangle$ (e.g. unconstrained, required to vanish at one or both boundary vertices, or required to vanish identically).

In the proof of Proposition 21 we actually established part (i) of the following lemma. The other parts are very similar.

Lemma 22 *(Retracting from a triangle to an edge).*

(i) *If the boundary condition on the edge $t = 1$ is empty, then there is a deformation retract*

$$V_{s=t} \underset{p}{\overset{i_s}{\rightleftarrows}} V_\triangle \circlearrowleft a$$

where $p : V_\triangle \to V_{s=t}$ is the restriction map pulling back the form to this boundary, where $i_s : V_{s=t} \hookrightarrow V_\triangle$ is given by

$$i_s : f(s) + f_s(s)\mathrm{d}s \mapsto f(s) + f_s(s)\mathrm{d}s \tag{31}$$

and where the homotopy $a : V_\triangle \to V_\triangle$ is given in (30).

(ii) *if the boundary condition on the edge $s = 0$ is empty, then there is a deformation retract*

$$V_{s=t} \underset{p}{\overset{i_t}{\rightleftarrows}} V_\triangle \circlearrowleft \widetilde{a}$$

where p is as in part (i), where $i_t : V_{s=t} \hookrightarrow V_\triangle$ is given by

$$i_t : f(t) + f_t(t)\mathrm{d}t \mapsto f(t) + f_t(t)\mathrm{d}t \tag{32}$$

and where the homotopy $\widetilde{a} : V_\triangle \to V_\triangle$ is given by, cf. (29),

$$\widetilde{a}(\Phi)(s,t) := \left(\int_t^s f_s(s',t)\mathrm{d}s'\right) + \left(\int_t^s f_{st}(s',t)\mathrm{d}s'\right)\mathrm{d}t$$

(iii) *if the boundary condition on the edge $s = t$ is empty, then there are deformation retracts*

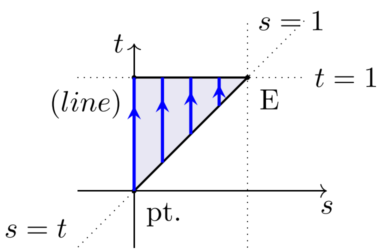
$$V_{t=1} \underset{p}{\overset{i_s}{\rightleftarrows}} V_\triangle \circlearrowleft c \quad \text{and} \quad V_{s=0} \underset{p}{\overset{i_t}{\rightleftarrows}} V_\triangle \circlearrowleft \widetilde{c}$$

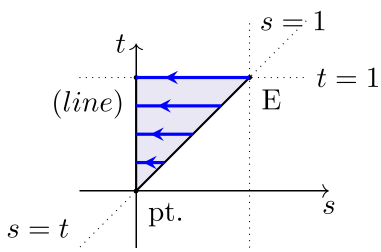
where p are the relevant restriction maps, i_s and i_t are as in (31) and (32) respectively, and where the homotopies are given by

$$c(\Phi)(s,t) := \left(\int_1^t f_t(s,t')\mathrm{d}t'\right) - \left(\int_1^t f_{st}(s,t')\mathrm{d}t'\right)\mathrm{d}s.$$

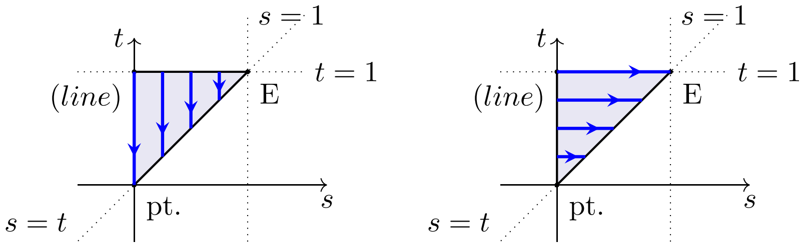
$$\widetilde{c}(\Phi)(s,t) := \left(\int_0^s f_s(s',t)\mathrm{d}s'\right) + \left(\int_0^s f_{st}(s',t)\mathrm{d}s'\right)\mathrm{d}t. \quad \square$$

This deformation retracts, and in particular their homotopies, can be conveniently encoded pictorially. Cases (i) and (ii) are





while the cases in (iii) are



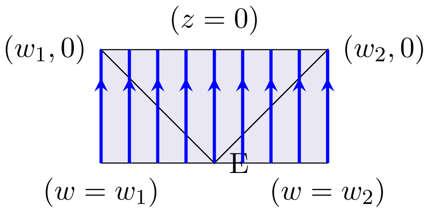
In the proof of Proposition 21 we also established the following.

Lemma 23 *(Retracting from an edge to a point). If the boundary condition at the vertex $s = t = 1$ is empty, then there is a deformation retract*

$$V_{s=t=0} \underset{p}{\overset{i}{\rightleftarrows}} V_{s=t} \circlearrowleft b$$

(where $V_{s=t=0}$ is either 0 or $\mathbb{C}$, depending on the boundary conditions). □

Deformation retracts compose, so we obtain deformation retracts corresponding to sequences of such moves. The proof of Proposition 21 contains one example of such a sequence: first two triangles retract to their common edge; then that edge retracts to a point. In what follows we need a variety of similar but more intricate cases. For example consider the picture



Provided the boundary condition on the edges $(w_1, 0) \subset (z = 0) \supset (w_2, 0)$ is empty, this picture defines a deformation retract from the complex of $\mathbb{C}$-valued polynomial differential forms on this semisimplicial set to the complex of $\mathbb{C}$-valued polynomial differential forms on the boundary $(w = w_1) \subset \mathrm{E} \supset (w = w_2)$.

In what follows, we shall use this pictorial notation freely. A representative example of the explicit calculations such pictures represent is given, in full detail, in Appendix B.

8. Global homotopy Manin triple

In this section we give the second of our two main examples of homotopy Manin triples: see Theorem 24. We shall use throughout the pictorial notation for homotopies introduced in §7.5 above.

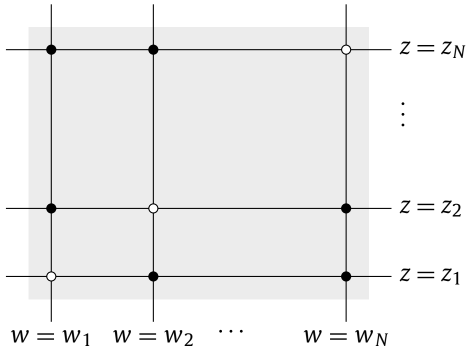
The Manin triples of this section are defined by a collection of marked points in rectilinear space, as we now describe.

8.1. Marked points

We continue to let $w, z : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be the Cartesian coordinates. Pick $N \geq 1$. Let $\mathbf{z} = z_1, \dots, z_N$ be pairwise distinct points in $\mathbb{C}$. Let $\mathbf{w} = w_1, \dots, w_N$ be pairwise distinct points in $\mathbb{C}$. Let $\mathrm{Rect}_2(N)$ denote the subset of Rect_2 consisting of

- The closed points (w_i, z_j) for all $i \neq j$, $i, j \in \{1, \dots, N\}$
- The lines $(w = w_i)$ for $i \in \{1, \dots, N\}$
- The lines $(z = z_i)$ for $i \in \{1, \dots, N\}$
- The generic point E.

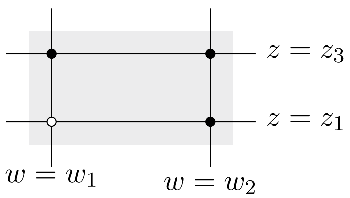
We may sketch these data as follows:

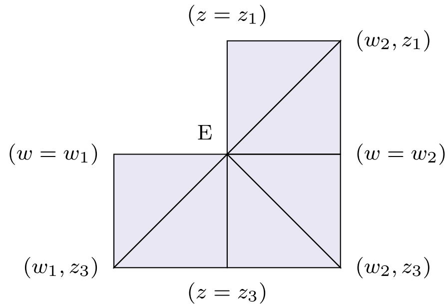


This gives rise to the semisimplicial subset $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$ of $\mathrm{Flag}_\bullet(\mathrm{Rect}_2)$. In contrast to the latter, it has finite sets of n-simplices, for each n.

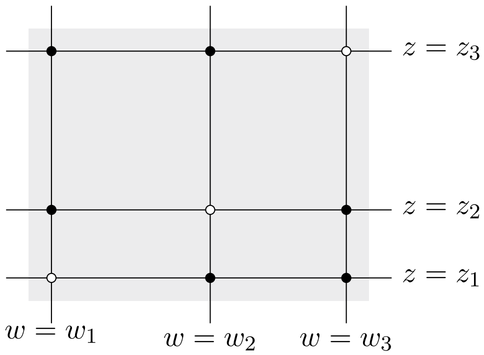
8.2. *Sketches of* $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$

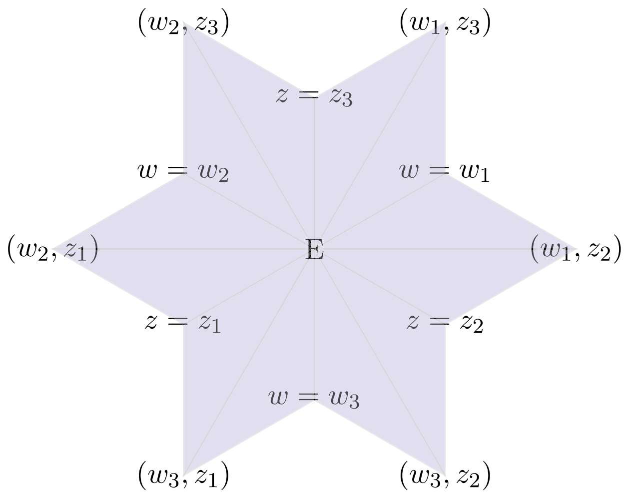
It is helpful to be able to visualise at least parts of the semisimplicial set $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$. For example we may restrict our attention to the following lines and points, and draw the corresponding simplices of $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$:





At least in the case of three marked points, it is actually possible draw the whole semisimplicial set $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(3))$:





8.3. *Global dg Lie algebra* $\mathfrak{g}_{\mathrm{Global}}$

Let $A_{\mathrm{Rect}_2(N)}$ be the $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$-object in commutative algebras given as follows. Let us write $\mathbb{C}(w)^\infty_{\mathbf{w}}$ for the $\mathbb{C}$-algebra

$$\mathbb{C}(w)^\infty_{\mathbf{w}} := \begin{Bmatrix} \text{rational expressions in } w \text{ vanishing at } \infty \text{ and with poles at most at the points} \\ \mathbf{w} = \{w_1, \ldots, w_N\} \end{Bmatrix}.$$

We assign, to every simplex of $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$, the commutative algebra

$$\mathbb{C}(w)^\infty_{\mathbf{w}} \otimes \mathbb{C}(z)^\infty_{\mathbf{z}}$$

with only the following exceptions:

$$A_{\mathrm{Rect}_2(N)}(\{(w_i, z_j)\}) := \mathbb{C}(w)^\infty_{\mathbf{w}\setminus\{w_i\}} \otimes \mathbb{C}(z)^\infty_{\mathbf{z}\setminus\{z_j\}}$$
$$A_{\mathrm{Rect}_2(N)}(\{(w_i, z_j)\} \subset \{z = z_j\}) := A_{\mathrm{Rect}_2(N)}(\{z = z_j\}) := \mathbb{C}(w)^\infty_{\mathbf{w}} \otimes \mathbb{C}(z)^\infty_{\mathbf{z}\setminus\{z_j\}}$$
$$A_{\mathrm{Rect}_2(N)}(\{(w_i, z_j)\} \subset \{w = w_i\}) := A_{\mathrm{Rect}_2(N)}(\{w = w_i\}) := \mathbb{C}(w)^\infty_{\mathbf{w}\setminus\{w_i\}} \otimes \mathbb{C}(z)^\infty_{\mathbf{z}}$$

for all i, j such that the given flag belongs to $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$.

We continue to let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Let $\mathfrak{g}_{\mathrm{Global}}$ denote the dg Lie algebra

$$\mathfrak{g}_{\mathrm{Global}} := \mathrm{Th}(\mathfrak{g} \otimes A_{\mathrm{Rect}_2(N)}).$$

Note that the exceptions above are precisely the simplices on the boundary of $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$. Thus, concretely, $\mathfrak{g}_{\mathrm{Global}}$ is the dg algebra of polynomial differential forms on $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$, valued in $\mathfrak{g} \otimes \mathbb{C}(w)^\infty_{\mathbf{w}} \otimes \mathbb{C}(z)^\infty_{\mathbf{z}}$, subject to these boundary conditions.

8.4. Local dg Lie algebras

Let us introduce the dg Lie algebras

$$\mathfrak{g}_{\mathrm{PDiscs}} := \bigoplus_{i=1}^N \mathfrak{g}_+(w_i, z_i) = \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathbb{C}[[w - w_i]] \otimes \mathbb{C}[[z - z_i]], \qquad \mathfrak{g}_{\mathrm{PDiscs}^\times} := \bigoplus_{i=1}^N \mathfrak{g}_{\mathrm{PDisc}_2^\times(w_i, z_i)}$$

and define two maps of dg Lie algebras,

$$I_{\mathrm{Global}} : \mathfrak{g}_{\mathrm{Global}} \to \mathfrak{g}_{\mathrm{PDiscs}^\times}, \qquad I_{\mathrm{Discs}} : \mathfrak{g}_{\mathrm{PDiscs}} \to \mathfrak{g}_{\mathrm{PDiscs}^\times}.$$

Summand by summand, the map I_{Discs} is defined in the same way as the embedding $\mathfrak{g}_+ \to \mathfrak{g}_{\mathrm{PDisc}_2^\times}$ from §7.1. The map I_{Global} is given by taking formal Laurent series around $(w, z) = (w_i, z_i)$ and restricting to the relevant semisimplicial subset, for each i. More precisely, to define I_{Global}, observe that $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times(w_i, z_i))$ is a semisimplicial subset of $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$, for each $i = 1, \ldots, N$. We get the restriction of $A_{\mathrm{Rect}_2(N)}$ to a $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times(w_i, z_i))$-algebra $A_{\mathrm{Rect}_2(N)}|_{\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times(w_i, z_i))}$, and there is an evident map of $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times(w_i, z_i))$-algebras

$$A_{\mathrm{Rect}_2(N)}|_{\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times(w_i, z_i))} \to A_{\mathrm{PDisc}_2^\times(w_i, z_i)}$$

given by taking formal Laurent expansions. Hence, by Lemma 2 and Lemma 4 we get the map of semicosimplicial algebras

$$\Pi A_{\mathrm{Rect}_2(N)} \to \Pi A_{\mathrm{Rect}_2(N)}|_{\mathrm{Flag}_\bullet(\mathrm{PDisc}_2^\times(w_i, z_i))} \to \Pi A_{\mathrm{PDisc}_2^\times(w_i, z_i)}$$

and therefore, by the functoriality of Th, a map of dg Lie algebras

$$I^i_{\mathrm{Global}} : \mathfrak{g}_{\mathrm{Global}} \to \mathfrak{g}_{\mathrm{PDisc}_2^\times(w_i, z_i)} \tag{33}$$

for each marked point (w_i, z_i). The resulting diagonal map is I_{Global}:

$$I_{\mathrm{Global}} := \bigoplus_{i=1}^N I^i_{\mathrm{Global}}.$$

8.5. The pairing

For each marked point (w_i, z_i), we have an invariant bilinear form $\langle - | - \rangle^i$ on $\mathfrak{g}_{\mathrm{PDisc}_2^\times(w_i, z_i)}$ defined as in §7.2 and Proposition 20. Let us use the same notation $\langle - | - \rangle$ for the resulting diagonal pairing

$$\langle A | B \rangle := \sum_{i=1}^N \langle A_i | B_i \rangle^i$$

between two elements $A = (A_i)$ and $B = (B_i)$ of $\mathfrak{g}_{\mathrm{PDiscs}^\times} = \bigoplus_{i=1}^N \mathfrak{g}_{\mathrm{PDisc}_2^\times(w_i, z_i)}$.

8.6. Manin triple

The main result of this section is the following.

Theorem 24. *The data*

$$(\mathfrak{g}_{\mathrm{PDiscs}^\times}, \mathfrak{g}_{\mathrm{PDiscs}}, \mathfrak{g}_{\mathrm{Global}}, I_{\mathrm{Discs}}, I_{\mathrm{Global}}, \langle - \mid - \rangle)$$

constitute a homotopy Manin triple in dg Lie algebras, in the sense of Definition 9.

Proof. Most of the rest of this section is devoted to establishing condition (i), the homotopy equivalence of $\mathfrak{g}_{\mathrm{Global}} \oplus \mathfrak{g}_{\mathrm{PDiscs}} \simeq \mathfrak{g}_{\mathrm{PDiscs}^\times}$, which is Theorem 25 below. For condition (ii) and (iii) there is essentially nothing new to check, given Proposition 19 and Proposition 20. For the isotropy condition (iv), we once more choose to establish the equivalent statements about cohomologies from Remark 10. We do so in §8.7. □

We have the map from the direct sum of $\mathfrak{g}_{\mathrm{Global}}$ and $\mathfrak{g}_{\mathrm{PDiscs}}$ as dg vector spaces:

$$I = (I_{\mathrm{Global}}, I_{\mathrm{Discs}}) : \mathfrak{g}_{\mathrm{Global}} \oplus \mathfrak{g}_{\mathrm{PDiscs}} \to \mathfrak{g}_{\mathrm{PDiscs}^\times}.$$

Let us stress that this is not a map of dg Lie algebras from the direct sum of $\mathfrak{g}_{\mathrm{Global}}$ and $\mathfrak{g}_{\mathrm{PDiscs}}$ as dg Lie algebras, for the same reason that the analogous map in the usual one-dimensional case is not a map of Lie algebras: the images of $\mathfrak{g}_{\mathrm{Global}}$ and $\mathfrak{g}_{\mathrm{PDiscs}}$ in $\mathfrak{g}_{\mathrm{PDiscs}^\times}$ are not mutually commuting.

The following statement justifies our definition of $\mathfrak{g}_{\mathrm{Global}}$: it establishes that, up to homotopies, $\mathfrak{g}_{\mathrm{Global}}$ provides a dg vector space complement to $\mathfrak{g}_{\mathrm{PDiscs}}$ in $\mathfrak{g}_{\mathrm{PDiscs}^\times}$.

Theorem 25. *This map I is a homotopy equivalence of dg vector spaces. That is, there is a homotopy equivalence of dg vector spaces:*

$$h_{\mathrm{Global}}+h_{\mathrm{offdiag.}} \circlearrowright \mathfrak{g}_{\mathrm{Global}} \oplus \mathfrak{g}_{\mathrm{PDiscs}} \underset{P}{\overset{I}{\rightleftarrows}} \mathfrak{g}_{\mathrm{PDiscs}^\times} \circlearrowleft h_{\mathrm{Discs}^\times}$$

Proof. We shall construct a map of cochain complexes

$$P = P_{\mathrm{Global}} \oplus P_{\mathrm{Discs}} : \mathfrak{g}_{\mathrm{PDiscs}^\times} \to \mathfrak{g}_{\mathrm{Global}} \oplus \mathfrak{g}_{\mathrm{PDiscs}}$$

inverse to I up to homotopies. We shall first define P_{Global} and then check that $P_{\mathrm{Global}} \circ I_{\mathrm{Global}}$ is homotopic to the identity on $\mathfrak{g}_{\mathrm{Global}}$. Then we shall define P_{Discs} and check the remaining homotopy relations.

Our first step is to define the map P_{Global}. Morally, the idea here in our case in dimension two is the same as in the case of dimension one from, e.g., [16]: we want to use the “singular parts” of an element of $\mathfrak{g}_{\mathrm{PDiscs}^\times}$ to construct an element of $\mathfrak{g}_{\mathrm{Global}}$.

An element $\omega \in \mathfrak{g}_{\mathrm{PDiscs}^\times}$ is a tuple $\omega = (\omega_i)_{i=1}^N$, $\omega_i \in \mathfrak{g}_{\mathrm{PDisc}^\times_2(w_i,z_i)}$, and we shall define $P_{\mathrm{Global}}(\omega)$ summand by summand,

$$P_{\mathrm{Global}}(\omega) := \sum_{i=1}^N P^i_{\mathrm{Global}}(\omega_i), \qquad P^i_{\mathrm{Global}} : \mathfrak{g}_{\mathrm{PDisc}^\times_2(w_i,z_i)} \to \mathfrak{g}_{\mathrm{Global}}.$$

Given $\omega_i \in \mathfrak{g}_{\mathrm{PDisc}^\times_2(w_i,z_i)}$ we have, just as in (27), the decomposition

$$\omega_i = \omega_i^{++} + \omega_i^{-+} + \omega_i^{+-} + \omega_i^{--} \tag{34}$$

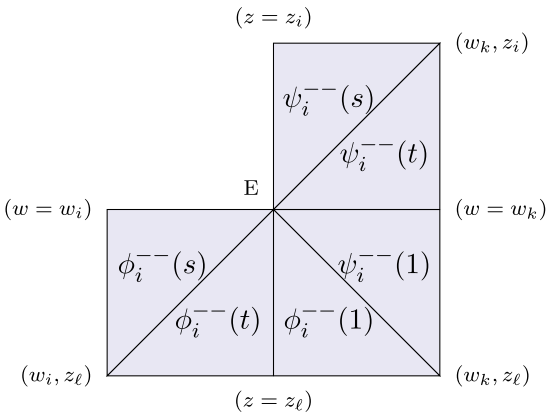
coming from the decomposition of the vector space $\mathbb{C}((w-w_i)) \otimes \mathbb{C}((z-z_i))$ into the polar and regular parts with respect to each of the local coordinates, $w-w_i$ and $z-z_i$. In particular, the $^{--}$ part can be interpreted as a rational function vanishing at infinity, via the embedding of commutative algebras

$$(w-w_i)^{-1}(z-z_i)^{-1}\mathbb{C}[(w-w_i)^{-1}, (z-z_i)^{-1}] \hookrightarrow \mathbb{C}(w)^\infty_{\mathbf{w}} \otimes \mathbb{C}(z)^\infty_{\mathbf{z}}.$$

Now, we have

$$\omega_i^{--}(s) = (\phi_i^{--}(s), \psi_i^{--}(s))$$

as in (26), and we define $P^i_{\mathrm{Global}}(\omega_i)$ to be the element given as follows,

$P^i_{\text{Global}}(\omega_i) :=$ 

(35)

for each $k, \ell \neq i$. If $k = \ell$ then the lower right square is absent.

Observe that this obeys the continuity conditions on internal edges: in particular, along the edges of the form $\big((w_k, z_\ell), \mathrm{E}\big)$ it is continuous because $\phi^{--}(1) = f^{--}(1) = g^{--}(1) = \psi^{--}(1)$. Note also why $P_{\text{Global}}(\omega_i)$ obeys the boundary conditions: since ω_i^{--} has poles only at $w = w_i$ and $z = z_i$, the boundary conditions are non-trivial only on the edges of the form $\big((w_k, z_i), (z = z_i)\big)$ and $\big((w_i, z_\ell), (w = w_i)\big)$, and here they are obeyed because $\phi^{--}(0) = 0$ and $\psi^{--}(0) = 0$, by the boundary conditions on ω_i^{--} itself.

Let us stress that while $I_{\text{Global}} : \mathfrak{g}_{\text{Global}} \to \mathfrak{g}_{\text{PDiscs}^\times}$ is a map of dg Lie algebras, $P_{\text{Global}} : \mathfrak{g}_{\text{PDiscs}^\times} \to \mathfrak{g}_{\text{Global}}$ is a map of dg vector spaces only.

Lemma 26. *There exists a homotopy* $h_{\text{Global}} : \mathfrak{g}_{\text{Global}} \to \mathfrak{g}_{\text{Global}}$ *such that*

$$\mathrm{id}_{\mathfrak{g}_{\text{Global}}} - P_{\text{Global}} \circ I_{\text{Global}} = \mathrm{d} \circ h_{\text{Global}} + h_{\text{Global}} \circ \mathrm{d}$$

holds as an equality of cochain maps $\mathfrak{g}_{\text{Global}} \to \mathfrak{g}_{\text{Global}}$.

Proof. We have the decomposition of vector spaces

$$\mathbb{C}(w)^\infty_{\mathbf{w}} \otimes \mathbb{C}(z)^\infty_{\mathbf{z}} \cong_{\mathbb{C}} \bigoplus_{i,j} (w - w_i)^{-1}(z - z_j)^{-1}\mathbb{C}[(w - w_i)^{-1}, (z - z_j)^{-1}]$$

coming from taking partial fractions in each global coordinate, z and w. In this way an element $\mu \in \mathfrak{g}_{\text{Global}}$ has partial fraction decomposition

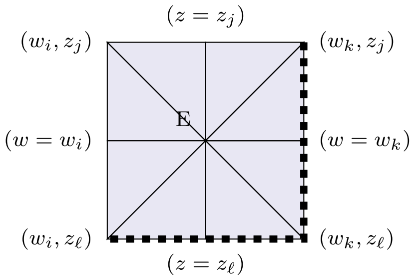
$$\mu = \sum_{i,j=1}^{N} \mu_{ij} \tag{36}$$

where μ_{ij} is a polynomial differential form on $A_{\text{Rect}_2(N)}$ with coefficents in $(w - w_i)^{-1}(z - z_j)^{-1}\mathbb{C}[(w - w_i)^{-1}, (z - z_j)^{-1}]$. We shall define $h_{\text{Global}}(\mu)$ summand by summand,

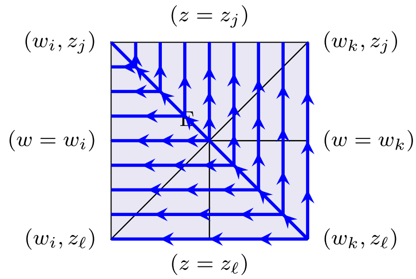
$$h_{\text{Global}}(\mu) := \sum_{i,j=1}^{N} h^{ij}_{\text{Global}}(\mu_{ij}) \tag{37}$$

The argument is similar to that in the proof of Proposition 21, and we shall use the pictorial notation from §7.5.

First consider a summand $\mu_{k\ell}$ with $k \neq \ell$. By definition of P_{Global}, $P_{\text{Global}} \circ I_{\text{Global}}(\mu_{k\ell}) = 0$ vanishes, so we must arrange our homotopy to contract $\mathrm{id}(\mu_{k\ell}) = \mu_{k\ell}$ to zero (just as, in Proposition 15, we had to contract $\omega^{\pm\mp}$ to zero). The boundary conditions mean that $\mu_{k\ell}$ must vanish when pulled back to any of the edges drawn as thick dashed lines in the sketch at the left below, and we define $h^{k\ell}_{\text{Global}}$ to be the map drawn on the right:



$h^{k\ell}_{\mathrm{Global}}(\mu) :=$

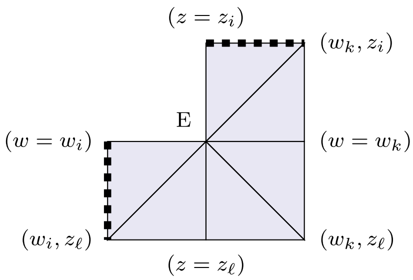


Here we have sketched part of the semisimplicial set $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$ corresponding to some i, j with $i \neq j$ and $i, j \notin \{k, \ell\}$. Special cases occur when $k = j$ or $i = \ell$ or both, and when $i = j$, but these special cases just correspond to omitting one or more of the squares, and in a way which, one sees, does not obstruct us in retracting back to the point (w_k, z_ℓ) as above. One may then verify by direct calculations that

$$[\mathrm{d}, h_{\mathrm{Global}}](\mu_{k\ell}) = \mu_{k\ell} - \mu_{k,\ell}|_{(w_k, z_\ell)} = \mu_{k\ell} - 0 = \mu_{k\ell}$$

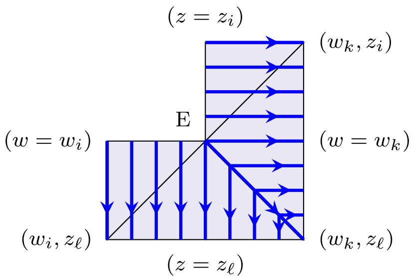
as we wanted. The details of this calculation are given in Appendix B.

It remains to consider the diagonal summands μ_{ii}. The new feature is that the semisimplicial set of edges of $\mathrm{Flag}_\bullet(\mathrm{Rect}_2(N))$ on which μ_{ii} vanishes is no longer contractible to a single vertex:



(if $k = \ell$ here then the lower right square is absent). We set

$h^{ii}_{\mathrm{Global}}(\mu) :=$



Again, one may verify by direct calculation that

$$[\mathrm{d}, h_{\mathrm{Global}}](\mu_{ii}) = \mu_{ii} - P_{\mathrm{Global}}(I_{\mathrm{Global}}(\mu_{ii}))$$

with P_{Global} as we defined it in (35) above. □

We continue with the proof of Theorem 25. The next step is to construct a map of dg vector spaces

$$P_{\mathrm{Discs}} : \mathfrak{g}_{\mathrm{PDiscs}^\times} \to \mathfrak{g}_{\mathrm{PDiscs}}$$

such that

$$\mathrm{id}_{\mathfrak{g}_{\mathrm{PDiscs}^\times}} - I_{\mathrm{Global}} \circ P_{\mathrm{Global}} \simeq I_{\mathrm{Discs}} \circ P_{\mathrm{Discs}}, \tag{38}$$

i.e. such that for any $\omega \in \mathfrak{g}_{\mathrm{PDiscs}^\times}$, the difference

$$\widetilde{\omega} := \omega - I_{\mathrm{Global}} \circ P_{\mathrm{Global}}(\omega) \tag{39}$$

is equal to $I_{\mathrm{Discs}} \circ P_{\mathrm{Discs}}(\omega)$ up to homotopy.

We define P_{Discs} by

$$P_{\text{Discs}}(\omega)_i := (\omega_i - I_{\text{Global}}(P_{\text{Global}}(\omega))_i)^{++}|_{s=1} = \left(\widetilde{\omega}_i^{++}|_{s=1}\right) \tag{40}$$

– cf. (28) and (34). By construction, we have $\widetilde{\omega}_i^{--} = 0$. It remains to find a homotopy $h_{\text{Discs}^\times} : \mathfrak{g}_{\text{PDiscs}^\times} \to \mathfrak{g}_{\text{PDiscs}^\times}$ contracting $\widetilde{\omega}_i^{+-}$, $\widetilde{\omega}_i^{-+}$, and the non-constant terms in $\widetilde{\omega}_i^{++}$. This can be done much as in the proof of Proposition 15. We may write $\widetilde{\omega}$ as

$$\widetilde{\omega} = (\widetilde{\omega}_i)_{1\le i\le N} = \left(\widetilde{\phi}_i(s) = \widetilde{f}_i(s) + \widetilde{F}_i(s)\mathrm{d}s,\ \widetilde{\psi}_i(s) = \widetilde{g}_i(s) + \widetilde{G}_i(s)\mathrm{d}s\right)_{1\le i\le N}$$

and set

$$h_{\text{Discs}^\times}(\omega_i^{++})(s) = \left(\int_1^s \widetilde{F}_i^{++}(s')\mathrm{d}s'\,,\ \int_1^s \widetilde{G}_i^{++}(s')\mathrm{d}s'\right), \tag{41}$$

where we stress that we are defining $h_{\text{Discs}^\times}(\omega)$ and it is the components of $\widetilde{\omega}$ that appear on the right. Using the fact that $I_{\text{Global}} \circ P_{\text{Global}}$ is a cochain map, and so commutes with d, we then have

$$\begin{aligned}[\mathrm{d}, h_{\text{Discs}^\times}](f_i^{++}(s), g_i^{++}(s)) &= (h_{\text{Discs}^\times} \circ \mathrm{d})(f_i^{++}(s), g_i^{++}(s)) \\ &= \left(\widetilde{f}_i^{++}(s) - \widetilde{f}_i^{++}(1), \widetilde{g}_i^{++}(s) - \widetilde{g}_i^{++}(1)\right) \\ &= (\widetilde{f}_i^{++}(s), \widetilde{g}_i^{++}(s)) - (1,1)\widetilde{f}_i^{++}(1)\end{aligned}$$

and

$$\begin{aligned}[\mathrm{d}, h_{\text{Discs}^\times}](F_i^{++}(s)\mathrm{d}s, G_i^{++}(s)\mathrm{d}s) &= (\mathrm{d} \circ h_{\text{Discs}^\times})(F_i^{++}(s)\mathrm{d}s, G_i^{++}(s)\mathrm{d}s) \\ &= (\widetilde{F}_i^{++}(s)\mathrm{d}s, \widetilde{G}_i^{++}(s)\mathrm{d}s).\end{aligned}$$

That is,

$$[\mathrm{d}, h_{\text{Discs}^\times}](\omega_i^{++}) = \widetilde{\omega}_i^{++} - \widetilde{\omega}_i^{++}|_{s=1} = (\mathrm{id} - I_{\text{Global}} \circ P_{\text{Global}} - I_{\text{Discs}} \circ P_{\text{Discs}})(\omega_i^{++})$$

as we want.

The argument for ω_i^{+-} and ω_i^{-+} is similar, again following the prototype in the proof of Proposition 15. (In fact, observe that $\omega_i^{+-} = \widetilde{\omega}_i^{+-}$ and $\omega_i^{-+} = \widetilde{\omega}_i^{-+}$.)

This establishes that (38) holds, as we wanted. That is, $I \circ P$ is homotopic to the identity map $\mathrm{id}_{\mathfrak{g}_{\text{PDiscs}^\times}}$.

To complete the proof of Theorem 25 it remains to check that $P \circ I$ is homotopic to the identity map $\mathrm{id}_{\mathfrak{g}_{\text{Global}} \oplus \mathfrak{g}_{\text{PDiscs}}}$. We already defined the required homotopy for the global part $P_{\text{Global}} \circ I_{\text{Global}}$, in Lemma 26. The restriction of $P \circ I$ to $\mathfrak{g}_{\text{PDiscs}}$ is the identity map on the nose, i.e. we have

$$P \circ I|_{\mathfrak{g}_{\text{PDiscs}}} = P_{\text{Discs}} \circ I_{\text{Discs}} = \mathrm{id}_{\mathfrak{g}_{\text{PDiscs}}},$$

and it is manifest that $P_{\text{Global}} \circ I_{\text{Discs}} = 0$ vanishes. However, the map $P \circ I$ does have a nonzero off-diagonal component

$$P_{\text{Discs}} \circ I_{\text{Global}} : \mathfrak{g}_{\text{Global}} \to \mathfrak{g}_{\text{PDiscs}}$$

which we must show is homotopic to zero.

Lemma 27. *For each $i = 1, \ldots, N$, we have*

$$P_{\text{Discs}}(I_{\text{Global}}(\mu))_i = \iota_{w-w_i, z-z_i} \sum_{\substack{k,\ell=1 \\ k\neq\ell\neq i\neq k}}^{N} \mu_{k\ell}|_{\mathrm{E}},$$

where $\mu = \sum_{i,j=1}^N \mu_{ij}$ is the partial fraction decomposition of an element $\mu \in \mathfrak{g}_{\text{Global}}$ as in (36).

Proof. First, note that the operation of restricting (i.e. pulling back) a polynomial differential form on the semisimplicial set $\mathrm{Flag}(\mathrm{Rect}_2(N))$ to the vertex E factors through the operation of first restricting it to $\mathrm{Flag}(\mathrm{PDisc}_2^\times(w_i, z_i))$ for any one of the punctured formal rectilinear polydiscs $\mathrm{PDisc}_2^\times(w_i, z_i)$.

We have $I_{\text{Global}}(\mu)_i^{--} = \mu_{ii}$. Therefore the restriction of the polynomial differential form $P_{\text{Global}}(I_{\text{Global}}(\mu)) \in \mathfrak{g}_{\text{Global}}$ to the vertex E of $\mathrm{Flag}(\mathrm{Rect}_2(N))$ is given by $\sum_{j=1}^N \mu_{jj}|_{\mathrm{E}}$, according to our definition (35) of P_{Global}. Thus the restriction of

$I_{\rm Global}(P_{\rm Global}(I_{\rm Global}(\mu)))_i$ to the vertex E of Flag(PDisc$_2^\times(w_i,z_i)$) is given by the Laurent expansion $\iota_{w-w_i,z-z_i}\sum_{j=1}^N \mu_{jj}|_{\rm E}$. By definition (40) of $P_{\rm Discs}$, we find, for each $i=1,\dots,N$, that

$$P_{\rm Discs}(I_{\rm Global}(\mu))_i = (I_{\rm Global}(\mu)_i - I_{\rm Global}(P_{\rm Global}(I_{\rm Global}(\mu)))_i)^{++}|_{\rm E}$$
$$= \iota_{w-w_i,z-z_i}\left(\sum_{\substack{k,\ell=1\\ k,\ell\neq i}}^N \mu_{k\ell}|_{\rm E} - \sum_{\substack{j=1\\ j\neq i}}^N \mu_{jj}|_{\rm E}\right) = \iota_{w-w_i,z-z_i}\sum_{\substack{k,\ell=1\\ k\neq\ell\neq i\neq k}}^N \mu_{k\ell}|_{\rm E}$$

as required. □

We may then define

$$h_{\rm offdiag.}:\mathfrak{g}_{\rm Global}\to\mathfrak{g}_{\rm PDiscs}$$

to be the map given in terms of the partial fraction decomposition (36) of $\mu\in\mathfrak{g}_{\rm Global}$ by

$$h_{\rm offdiag.}(\mu)_i := \iota_{w-w_i,z-z_i}\sum_{\substack{k,\ell=1\\ k\neq\ell\neq i\neq k}}^N h^{k\ell}_{\rm Global}(\mu_{k\ell})|_{\rm E}$$
$$= \iota_{w-w_i,z-z_i}\sum_{\substack{k,\ell=1\\ k\neq\ell\neq i\neq k}}^N \int_{(w_k,z_\ell)}^{\rm E} \mu_{k\ell}|_{((w_k,z_\ell),{\rm E})} \tag{42}$$

(the integral is over the edge of Flag(Rect$_2(N)$) joining (w_k,z_ℓ) to E). Then indeed $\mathrm{d}(h_{\rm offdiag.}(\mu))_i=0$ and

$$h_{\rm offdiag.}(\mathrm{d}(\mu))_i = P_{\rm Discs}(I_{\rm Global}(\mu))_i - 0 = P_{\rm Discs}(I_{\rm Global}(\mu))_i.$$

Thus we have shown that $P_{\rm Discs}\circ I_{\rm Global}\simeq 0$, as we wanted.

This completes the proof of Theorem 25. □

8.7. *The cohomology and the pairing*

As in the proof of Proposition 20, the pairing $\langle - | - \rangle$ restricts to a non-degenerate pairing

$$H^0(\mathfrak{g}_{\rm PDiscs^\times})\otimes H^1(\mathfrak{g}_{\rm PDiscs^\times})\to \mathsf{s}^{-1}\mathbb{C}$$

between $H^0(\mathfrak{g}_{\rm PDiscs^\times})=\mathfrak{g}_{\rm PDiscs}$ and $H^1(\mathfrak{g}_{\rm PDiscs^\times})$, and these subspaces are again both manifestly isotropic. Therefore it is enough to establish the following.

Proposition 28. *There is a deformation retract of dg vector spaces*

$$H^1(\mathfrak{g}_{\rm PDiscs^\times}) \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathfrak{g}_{\rm Global} \circlearrowleft h$$

(We stress that neither π *nor* ι *here are maps of dg Lie algebras.)*

Proof. Let $\mathfrak{g}^i_- := \mathrm{Th}(\mathfrak{g}\otimes A^{--}_{{\rm PDisc}_2^\times(w_i,z_i)})$ denote the copy of the dg Lie algebra $\mathfrak{g}_-$, cf. §7.1, associated to the punctured formal disc at the marked point (w_i,z_i). As in Proposition 16 we have the deformation retract of dg vector spaces

$$H^1(\mathfrak{g}_{{\rm PDisc}_2^\times(w_i,z_i)}) \rightleftarrows \mathfrak{g}^i_- \circlearrowleft$$

for each i. Deformation retracts compose. So to prove the result it is enough to show that there is a deformation retract of dg vector spaces

$$\bigoplus_{i=1}^N \mathfrak{g}^i_- \underset{g}{\overset{f}{\rightleftarrows}} \mathfrak{g}_{\rm Global} \circlearrowleft h_{\rm Global}$$

And indeed, we have a dg Lie algebra map $I^i_{\mathfrak{g}_-}:\mathfrak{g}^i_-\to\mathfrak{g}_{{\rm PDisc}_2^\times(w_i,z_i)}$ defined as in §7.1, and the dg vector space map $P^i_{\rm Global}:\mathfrak{g}_{{\rm PDisc}_2^\times(w_i,z_i)}\to\mathfrak{g}_{\rm Global}$ from (35). We let

$$f := \bigoplus_{i=1}^{N} f^i, \qquad f^i := P^i_{\mathrm{Global}} \circ I^i_{\mathfrak{g}_-} : \mathfrak{g}^i_- \to \mathfrak{g}_{\mathrm{Global}}.$$

In the other direction, let

$$g = \bigoplus_{i=1}^{N} g^i, \qquad g^i : \mu \mapsto I^i_{\mathrm{Global}}(\mu_{ii})$$

where I^i_{Global} was defined in (33) and μ_{ii} is as in (36). On inspection, one sees that $g^i \circ f^i = \mathrm{id}_{\mathfrak{g}^i_-}$ for each i (and that $g^i \circ f^j = 0$ for $i \neq j$). The homotopy h_{Global} here is the same as in Lemma 26. As there, it contracts all the off-diagonal pole terms μ_{ij} in the partial fraction decomposition of μ,

$$[\mathrm{d}, h_{\mathrm{Global}}](\mu_{ij}) = \mu_{ij}, \qquad i \neq j$$

and retracts each diagonal term μ_{ii} to the semisimplicial subset $\mathrm{Flag}(\mathrm{PDisc}^\times_2(w_i, z_i)$. By direct calculation one checks that

$$[\mathrm{d}, h_{\mathrm{Global}}](\mu_{ii}) = \mu_{ii} - f^i(g^i(\mu_{ii})). \quad \square$$

9. Triangular decompositions of enveloping algebras

The main results of this section are Corollary 29 and Corollary 34.

9.1. Local case

In this section we establish the following corollary of Proposition 15.

Corollary 29. *There is a deformation retract of $(U(\mathfrak{g}_-), U(\mathfrak{g}_+))$-bimodules*

$$U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+) \underset{U(P)}{\overset{U(I)}{\rightleftarrows}} U(\mathfrak{g}_{\mathrm{PDisc}^\times_2}) \circlearrowleft \tilde{h}$$

Proof. Consider the deformation retract

$$\mathfrak{g}_- \oplus \mathfrak{g}_+ \underset{P}{\overset{I}{\rightleftarrows}} \mathfrak{g}_{\mathrm{PDisc}^\times_2} \circlearrowleft h$$

of Proposition 15. The cochain map $I \circ P : \mathfrak{g}_{\mathrm{PDisc}^\times_2} \to \mathfrak{g}_{\mathrm{PDisc}^\times_2}$ is a projector (because $(I \circ P) \circ (I \circ P) = I \circ (P \circ I) \circ I = I \circ \mathrm{id}_{\mathfrak{g}_- \oplus \mathfrak{g}_+} \circ P = I \circ P$). Its image is a dg subspace which we can and shall regard as an embedded copy of the dg vector space $\mathfrak{g}_- \oplus \mathfrak{g}_+$. We get also the projector $\mathrm{id}_{\mathfrak{g}_{\mathrm{PDisc}^\times_2}} - I \circ P$ onto a dg subspace

$$\mathfrak{g}^\perp := (\mathrm{id}_{\mathfrak{g}_{\mathrm{PDisc}^\times_2}} - I \circ P)(\mathfrak{g}_{\mathrm{PDisc}^\times_2}),$$

and we obtain the direct sum decomposition of cochain complexes,

$$\mathfrak{g}_{\mathrm{PDisc}^\times_2} = \mathfrak{g}_- \oplus \mathfrak{g}^\perp \oplus \mathfrak{g}_+ \tag{43}$$

To be concrete, recall the decomposition (27) of an element $\omega \in \mathfrak{g}_{\mathrm{PDisc}^\times_2}$. The decomposition above is

$$\omega = \left(\omega^{--}, \omega^{+-} + \omega^{-+} + \omega^{++} - \omega^{++}|_{s=1}, \omega^{++}|_{s=1}\right)$$

We are therefore in the setting of the following lemma, which is essentially the dg analog of Proposition 2.2.7 and (a special case of) Proposition 2.2.9 of [11].

Lemma 30. *Suppose $\mathfrak{a}^- \hookrightarrow \mathfrak{a}$ and $\mathfrak{a}^+ \hookrightarrow \mathfrak{a}$ are embeddings of dg Lie algebras, and $\mathfrak{a}^\perp \hookrightarrow \mathfrak{a}$ an embedding of dg vector spaces, such that*

$$\mathfrak{a} \cong \mathfrak{a}^- \oplus \mathfrak{a}^\perp \oplus \mathfrak{a}^+$$

as dg vector spaces. Then

$$U(\mathfrak{a}) \cong U(\mathfrak{a}^-) \otimes \mathrm{Sym}(\mathfrak{a}^\perp) \otimes U(\mathfrak{a}^+)$$

as dg vector spaces and, moreover, as $(U(\mathfrak{a}^-), U(\mathfrak{a}^+))$-bimodules. $\square$

In view of this lemma, we have the isomorphism of $(U(\mathfrak{g}_-), U(\mathfrak{g}_+))$-bimodules

$$U(\mathfrak{g}_{\mathrm{PDisc}_2^\times}) \cong U(\mathfrak{g}_-) \otimes \mathrm{Sym}(\mathfrak{g}^\perp) \otimes U(\mathfrak{g}_+).$$

To prove Corollary 29 it remains to show that our homotopy h from Proposition 15 gives rise to a retract of $(U(\mathfrak{g}_-), U(\mathfrak{g}_+))$-bimodules

$$U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+) \underset{U(P)}{\overset{U(I)}{\rightleftarrows}} U(\mathfrak{g}_-) \otimes \mathrm{Sym}(\mathfrak{g}^\perp) \otimes U(\mathfrak{g}_+) \circlearrowright \widetilde{h}$$

To do that, we adapt an argument taken from the proof of [26, Proposition 2.5.5]. The natural embedding and projection maps of dg vector spaces

$$\mathbb{C} \cong \mathrm{Sym}^0(\mathfrak{g}^\perp) \rightleftarrows \mathrm{Sym}(\mathfrak{g}^\perp)$$

give rise to maps $U(I)$ and $U(P)$ of free $(U(\mathfrak{g}_-), U(\mathfrak{g}_+)$-bimodules

$$U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+) \underset{U(P)}{\overset{U(I)}{\rightleftarrows}} U(\mathfrak{g}_-) \otimes \mathrm{Sym}(\mathfrak{g}^\perp) \otimes U(\mathfrak{g}_+).$$

Now, our homotopy $h : \mathfrak{g}_{\mathrm{PDisc}_2^\times} \to \mathfrak{g}_{\mathrm{PDisc}_2^\times}$ from Proposition 15 by construction preserves the decomposition (43), and is zero on the summands $\mathfrak{g}_+$ and $\mathfrak{g}_-$. So it defines a map $\mathfrak{g}^\perp \to \mathfrak{g}^\perp$ which, abusively, we continue to call h. (In the language of [26], our homotopy obeys the <u>*side conditions*</u> and our retract is thus a <u>*strong deformation retract*</u> or <u>*contraction*</u>.) As maps $\mathfrak{g}^\perp \to \mathfrak{g}^\perp$, we have

$$[\mathrm{d}, h] = \mathrm{id}.$$

Recall that any map of dg vector spaces $V \to V$ extends uniquely to a derivation of the free dg commutative algebra $\mathrm{Sym}(V)$. In particular, we may extend h and id to derivations of $\mathrm{Sym}(\mathfrak{g}^\perp)$. But the extension of id to a derivation is just the map which multiplies each element of the dg subspace $\mathrm{Sym}^n(\mathfrak{g}^\perp)$ by a factor of n, for each $n \geq 0$. So we obtain that

$$[\mathrm{d}, h]|_{\mathrm{Sym}^n(\mathfrak{g}^\perp)} = n\, \mathrm{id}_{\mathrm{Sym}^n(\mathfrak{g}^\perp)}$$

for each $n \geq 0$. We may therefore define the required homotopy

$$\widetilde{h} : U(\mathfrak{g}_-) \otimes \mathrm{Sym}(\mathfrak{g}^\perp) \otimes U(\mathfrak{g}_+) \to U(\mathfrak{g}_-) \otimes \mathrm{Sym}(\mathfrak{g}^\perp) \otimes U(\mathfrak{g}_+)$$

to be

$$\widetilde{h} := \mathrm{id} \otimes \left(\sum_{n \geq 1} \frac{1}{n} h|_{\mathrm{Sym}^n \mathfrak{g}^\perp} \right) \otimes \mathrm{id}$$

for then $[\mathrm{d}, \widetilde{h}]$ is indeed the identity on each subspace $U(\mathfrak{g}_-) \otimes \mathrm{Sym}^n(\mathfrak{g}^\perp) \otimes U(\mathfrak{g}_+)$ with $n \geq 1$, and is by definition zero on $U(\mathfrak{g}_-) \otimes \mathrm{Sym}^0(\mathfrak{g}^\perp) \otimes U(\mathfrak{g}_+)$. Thus finally we get that

$$[\mathrm{d}, \widetilde{h}] = \mathrm{id} - U(I) \circ U(P),$$

as required. It is evident that $\widetilde{h}$ is a map of $(U(\mathfrak{g}_-), U(\mathfrak{g}_+))$-bimodules. This completes the proof of Corollary 29. □

Next, we would like to do something similar in the global case, while staying in the world of dg algebras. For that purpose, we would really wish to have a deformation retract, rather than merely a homotopy equivalence as we have in Theorem 25.

We shall obtain a result in this direction in Theorem 33 below. There are two steps:

First, we shall introduce, on both sides of the map I, additional summands associated to all unpunctured discs $\mathrm{PDisc}_2(w_i, z_j)$ at the "off-diagonal" points (w_i, z_j), $i \neq j$. Indeed, arguing as for Theorem 25 it is not hard to see that there is also a homotopy equivalence

$$h_{\mathrm{Global}} \circlearrowright \mathfrak{g}_{\mathrm{Global}} \oplus \bigoplus_{i,j=1}^{N} \mathfrak{g}_+(w_i, z_j) \underset{J}{\overset{I}{\rightleftarrows}} \bigoplus_{i=1}^{N} \mathfrak{g}_{\mathrm{PDisc}_2^\times(w_i,z_i)} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{\mathrm{PDisc}_2(w_i,z_j)} \circlearrowright h$$

(Note that by paying the price of introducing these extra points, we get rid of the off-diagonal terms in the homotopy on the left.)

Second, we introduce new models for all the summands on the right (at both the punctured and unpunctured discs). These models are chosen to be sufficiently large that we can reconstruct an element of $\mathfrak{g}_{\text{Global}}$ on the nose, rather than merely up to homotopy, from its image under I.

We turn to these enlarged models now.

9.2. Big models for the local algebras

Now we model the disc algebras as $\text{Flag}_\bullet(\text{Rect}_2(N))$-objects.

Pick any $(i, j) \in \{1, \ldots, N\}^2$. We set $A^{ij}_{\text{Rect}_2(N)}$ to be the $\text{Flag}_\bullet(\text{Rect}_2(N))$-object in commutative algebras given as follows. We assign, to every flag/simplex of $\text{Flag}_\bullet(\text{Rect}_2(N))$, the commutative algebra

$$\mathbb{C}((w - w_i)) \otimes \mathbb{C}((z - z_j)),$$

with only the following exceptions:

$$A^{ij}_{\text{Rect}_2(N)}(\{\text{pt.}\} \subset \{w = w_i\}) := A^{ij}_{\text{Rect}_2(N)}(\{w = w_i\}) := \mathbb{C}[[w - w_i]] \otimes \mathbb{C}((z - z_j))$$
$$A^{ij}_{\text{Rect}_2(N)}(\{\text{pt.}\} \subset \{z = z_j\}) := A^{ij}_{\text{Rect}_2(N)}(\{z = z_j\}) := \mathbb{C}((w - w_i)) \otimes \mathbb{C}[[z - z_j]]$$

and (when $i \neq j$)

$$A^{ij}_{\text{Rect}_2(N)}(\{(w_i, z_j)\}) := \mathbb{C}[[w - w_i]] \otimes \mathbb{C}[[z - z_j]].$$

Let $\mathfrak{g}^{ij}_{\text{Global}}$ denote the dg Lie algebra

$$\mathfrak{g}^{ij}_{\text{Global}} := \text{Th}(\mathfrak{g} \otimes A^{ij}_{\text{Rect}_2(N)}) \tag{44}$$

Thus, $\mathfrak{g}^{ij}_{\text{Global}}$ is the dg algebra of polynomial differential forms on $\text{Flag}_\bullet(\text{Rect}_2(N))$, valued in $\mathfrak{g} \otimes \mathbb{C}((w - w_i)) \otimes \mathbb{C}((z - z_j))$, and subject to boundary conditions above – which now, in contrast to $\mathfrak{g}_{\text{Global}}$ in §8.3), are only on the boundary simplices corresponding to the flags intersecting the point (w_i, z_j).

The next lemma says that $\mathfrak{g}^{ij}_{\text{Global}}$ provides another model for $\mathfrak{g}_{\text{PDisc}_2(w_i,z_j)}$ (for $i \neq j$) and that $\mathfrak{g}^{ii}_{\text{Global}}$ provides another model for $\mathfrak{g}_{\text{PDisc}^\times_2(w_i,z_i)}$.

Lemma 31. *There are deformation retracts of dg vector spaces,*

$$\mathfrak{g}_{\text{PDisc}^\times_2(w_i,z_i)} \underset{f_{ii}}{\overset{g_{ii}}{\rightleftarrows}} \mathfrak{g}^{ii}_{\text{Global}} \circlearrowleft h_{ii}$$

and

$$\mathfrak{g}_{\text{PDisc}_2(w_i,z_j)} \underset{f_{ij}}{\overset{g_{ij}}{\rightleftarrows}} \mathfrak{g}^{ij}_{\text{Global}} \circlearrowleft h_{ij}$$

for $i \neq j$, where the maps $g_{ij}, f_{ij}, g_{ii}, f_{ii}$ are maps of dg Lie algebras.

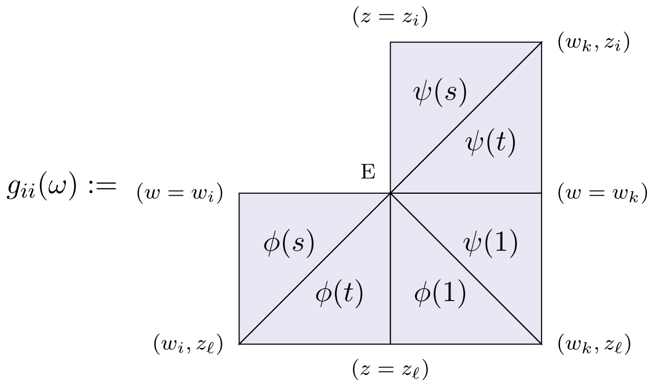
Proof. We consider first $\mathfrak{g}^{ii}_{\text{Global}}$. Since $\text{Flag}_\bullet(\text{PDisc}^\times_2(w_i, z_i))$ is a semisimplicial subset of $\text{Flag}_\bullet(\text{Rect}_2(N))$, we get the restriction of $A_{\text{Rect}_2(N)}$ to a $\text{Flag}_\bullet(\text{PDisc}^\times_2(w_i, z_i))$-algebra, and we recognise the latter as a copy of $A_{\text{PDisc}^\times_2(w_i,z_i)}$. Hence, by Lemma 2 and Lemma 4 we get the map of semicosimplicial algebras $\Pi A_{\text{Rect}_2(N)} \to \Pi A_{\text{PDisc}^\times_2(w_i,z_i)}$ and therefore, by the functoriality of Th, a map of dg Lie algebras

$$f_{ii} : \mathfrak{g}^{ii}_{\text{Global}} \to \mathfrak{g}_{\text{PDisc}^\times_2(w_i,z_i)}.$$

Now we define the map g_{ii}. Consider an element

$$\omega(s) = (\phi(s), \psi(s)) \in \mathfrak{g}_{\text{PDisc}^\times_2(w_i,z_i)}$$

cf. (26). Let $g_{ii}(\omega) \in \mathfrak{g}^{ii}_{\text{Global}}$ be the polynomial differential form on $\text{Flag}_\bullet(\text{Rect}_2(N))$ given as follows:



(If $k = \ell$ then the lower right square is absent.)

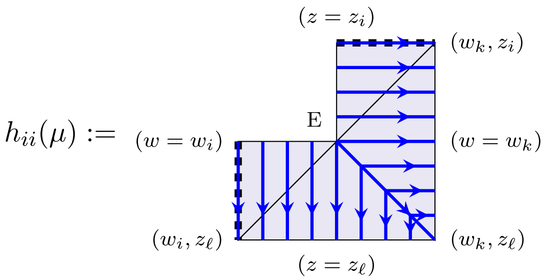
First, observe that this does define a map of dg Lie algebras $\mathfrak{g}_{\mathrm{PDisc}_2^\times(w_i,z_i)} \to \mathfrak{g}^{ii}_{\mathrm{Global}}$. And it is evident that

$$f_{ii} \circ g_{ii}(\omega) = \omega.$$

It remains to show that, for all $\mu \in \mathfrak{g}^{ii}_{\mathrm{Global}}$,

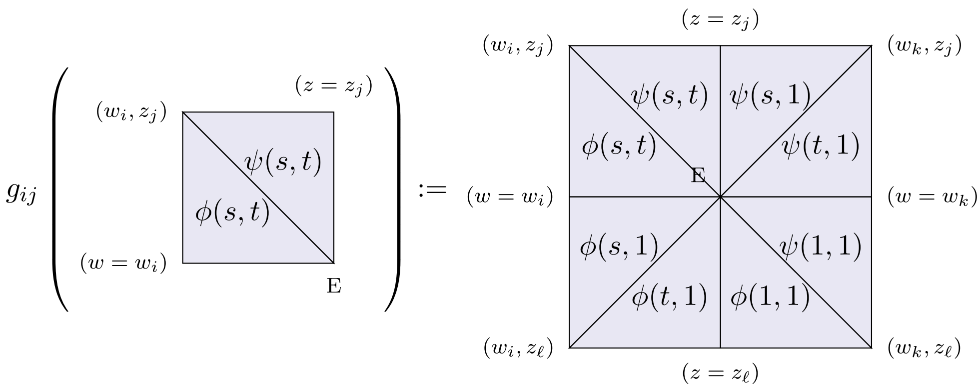
$$g_{ii} \circ f_{ii}(\mu) = \mu + [h_{ii}, \mathrm{d}](\mu)$$

for some suitable homotopy $h_{ii} : \mathfrak{g}^{ii}_{\mathrm{Global}} \to \mathfrak{g}^{ii}_{\mathrm{Global}}$. And indeed, by direct calculation, one checks that a suitable homotopy is given as follows, in the pictorial notation we introduced in §7.5:



(Here we have highlighted the edges on which the boundary conditions are non-trivial.)

Now we turn to $\mathfrak{g}^{ij}_{\mathrm{Global}}$ for $i \neq j$. This case is very similar to the previous case of $\mathfrak{g}^{ii}_{\mathrm{Global}}$, with the following modification to the definition of the map g_{ij}:



(where, again, squares can be absent in special cases). □

Deformation retracts compose, so in view of Proposition 21, we get the following corollary. Let

$$\mathfrak{g}^{ij}_+ := \mathfrak{g} \otimes \mathbb{C}[[w - w_i]] \otimes \mathbb{C}[[z - z_j]]. \tag{45}$$

Corollary 32. *Pick any $i \neq j$. There is a deformation retract*

$$\mathfrak{g}_+^{ij} \underset{P}{\overset{I}{\rightleftarrows}} \mathfrak{g}^{ij}_{\text{Global}} \circlearrowleft h_{ij}$$

in which both I and P are maps of dg Lie algebras.

Moreover, we may choose the map P to be given by pull-back of the non-singular part to the 0*-simplex* E*:*

$$P(\omega) := \omega^{++}|_{\mathrm{E}} \quad \square$$

9.3. Global deformation retract

Let

$$\widetilde{\mathfrak{g}_{\text{PDiscs}}} := \bigoplus_{i,j=1}^{N} \mathfrak{g}_+^{ij} \qquad \text{and} \qquad \widetilde{\mathfrak{g}_{\text{PDiscs}^\times}} := \bigoplus_{i,j=1}^{N} \mathfrak{g}^{ij}_{\text{Global}}$$

cf. (45) and (44). We have the dg Lie algebra maps

$$I_{\text{Discs}} : \widetilde{\mathfrak{g}_{\text{PDiscs}}} \to \widetilde{\mathfrak{g}_{\text{PDiscs}^\times}} \qquad \text{and} \qquad I_{\text{Global}} : \mathfrak{g}_{\text{Global}} \to \widetilde{\mathfrak{g}_{\text{PDiscs}^\times}}.$$

Theorem 33. *There is a deformation retract of dg vector spaces*

$$\mathfrak{g}_{\text{Global}} \oplus \widetilde{\mathfrak{g}_{\text{PDiscs}}} \underset{P=P_{\text{Global}} \oplus P_{\text{Discs}}}{\overset{I=(I_{\text{Global}}, I_{\text{Discs}})}{\rightleftarrows}} \widetilde{\mathfrak{g}_{\text{PDiscs}^\times}} \circlearrowleft h$$

Proof. We first define the map P_{Global}. An element of $\omega \in \widetilde{\mathfrak{g}_{\text{PDiscs}^\times}}$ is a tuple

$$\omega = (\omega_{ij})_{i,j=1}^N, \qquad \omega_{ij} \in \mathfrak{g}^{ij}_{\text{Global}}.$$

Given $\omega_{ij} \in \mathfrak{g}^{ij}_{\text{Global}}$ we have, just in (27), the decomposition

$$\omega_{ij} = \omega_{ij}^{++} + \omega_{ij}^{-+} + \omega_{ij}^{+-} + \omega_{ij}^{--}$$

coming from the direct sum decomposition of the vector space $\mathbb{C}((w-w_i)) \otimes \mathbb{C}((z-z_j))$ into the parts polar/regular parts with respect to each of the local coordinates, $w - w_i$ and $z - z_j$. In particular, the $^{--}$ part can be interpreted as a rational function vanishing at infinity, via the embedding of commutative algebras

$$(w-w_i)^{-1}(z-z_j)^{-1}\mathbb{C}[(w-w_i)^{-1}, (z-z_j)^{-1}] \hookrightarrow \mathbb{C}(w)^\infty_{\mathbf{w}} \otimes \mathbb{C}(z)^\infty_{\mathbf{z}}.$$

Making implicit use of these embeddings, we set

$$P_{\text{Global}}(\omega) := \sum_{i,j=1}^{N} \omega_{ij}^{--}.$$

It is then manifest that

$$P_{\text{Global}} \circ I_{\text{Global}} = \mathrm{id}_{\mathfrak{g}_{\text{Global}}}.$$

(Indeed, we have the decomposition of vector spaces

$$\mathbb{C}(w)^\infty_{\mathbf{w}} \otimes \mathbb{C}(z)^\infty_{\mathbf{z}} \cong_{\mathbb{C}} \bigoplus_{i,j} (w-w_i)^{-1}(z-z_j)^{-1}\mathbb{C}[(w-w_i)^{-1}, (z-z_j)^{-1}]$$

coming from taking partial fractions in each global coordinate, z and w. In this way an element $\mu \in \mathfrak{g}_{\text{Global}}$ has partial fraction decomposition $\mu = \sum_{i,j=1}^N \mu_{ij}$ where μ_{ij} is a polynomial differential form on $A_{\text{Rect}_2(N)}$ with coefficents in $(w - w_i)^{-1}(z-z_j)^{-1}\mathbb{C}[(w-w_i)^{-1}, (z-z_j)^{-1}]$. We see that $I_{\text{Global}}(\mu)_{ij}^{--} = \mu_{ij}$ for each i, j, and so we have $(P_{\text{Global}} \circ I_{\text{Global}})(\mu) = \sum_{i,j=1}^N I_{\text{Global}}(\mu)_{ij}^{--} = \sum_{i,j=1}^N \mu_{ij} = \mu$.)

Observe that

$$P_{\text{Global}} \circ I_{\text{Discs}} = 0.$$

Next we define the map $P_{\mathrm{Discs}} : \widetilde{\mathfrak{g}_{\mathrm{PDiscs}^\times}} \to \widetilde{\mathfrak{g}_{\mathrm{PDiscs}}}$ of dg vector spaces. Let us define

$$\widetilde{\omega} := \omega - I_{\mathrm{Global}} \circ P_{\mathrm{Global}}(\omega)$$

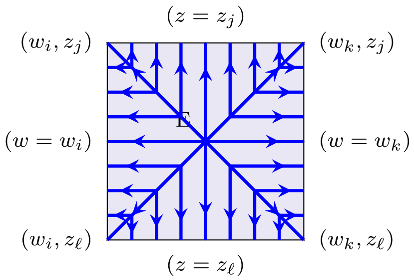
much as we did in (39), and then set

$$P_{\mathrm{Discs}}(\omega)_{ij} := \widetilde{\omega}_{ij}^{++}|_{\mathrm{E}}$$

for each i, j. Our goal is now to show that $\mathrm{id}_{\widetilde{\mathfrak{g}_{\mathrm{PDiscs}^\times}}} - I \circ P$ is homotopic to zero. For any $\omega \in \widetilde{\mathfrak{g}_{\mathrm{PDiscs}^\times}}$, we have

$$\begin{aligned}(\mathrm{id}_{\widetilde{\mathfrak{g}_{\mathrm{PDiscs}^\times}}} - I \circ P)(\omega)_{ij} &= \omega_{ij} - (I_{\mathrm{Global}} \circ P_{\mathrm{Global}}(\omega))_{ij} - (I_{\mathrm{Discs}} \circ P_{\mathrm{Discs}}(\omega))_{ij} \\ &= \widetilde{\omega}_{ij} - \widetilde{\omega}_{ij}^{++}|_{\mathrm{E}}\end{aligned}$$

It follows from our definitions that $\widetilde{\omega}_{ij}^{--} = 0$. Thus we have again to find a homotopy contracting $\widetilde{\omega}_{ij}^{\pm\mp}$ and the non-constant terms in $\widetilde{\omega}_{ij}^{++}$. We define the homotopy $h : \widetilde{\mathfrak{g}_{\mathrm{PDiscs}^\times}} \to \widetilde{\mathfrak{g}_{\mathrm{PDiscs}^\times}}$ as follows. Consider first $h(\omega)_{ij}^{++}$. It needs obey no nontrivial boundary conditions, and we define it to be given by the integrals of $(\widetilde{\omega})_{ij}^{++}$ encoded in the following picture:

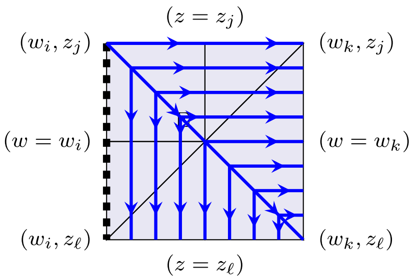
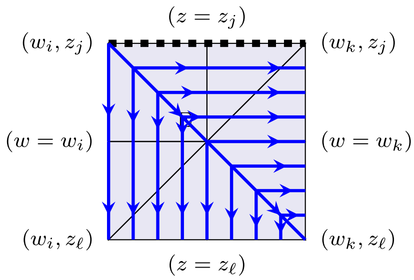


(We stress the integrals here and below are over $\widetilde{\omega}$ not ω. The situation is analogous to that in (41).)

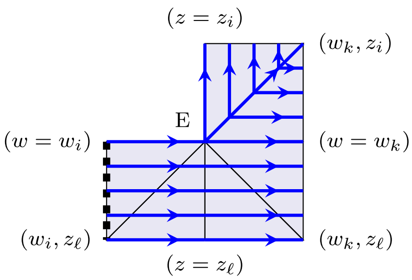
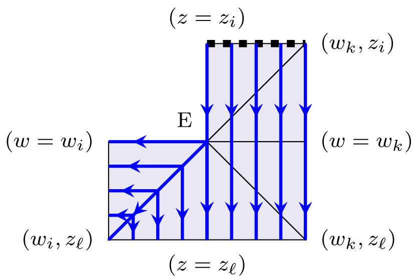
Again, in this picture, the squares at the top left, top right, lower left, and lower right are absent in the special cases $i = j$, $k = j$, $i = \ell$ and $k = \ell$ respectively.

Now we consider $h(\omega)_{ij}^{\pm\mp}$. Here we must distinguish the cases $i \neq j$ and $i = j$.

First suppose $i \neq j$. Consider $h(\omega)_{ij}^{-+}$. It must vanish on the edges $\big(\text{pt.}, (w = w_i)\big)$, since $(w - w_i)^{-1}\mathbb{C}[(w - w_i)^{-1}] \cap \mathbb{C}[[w - w_i]] = 0$. Similarly, $h(\omega)_{ij}^{+-}$ must vanish on the edges $\big(\text{pt.}, (z = z_j)\big)$. In either case we may define the action of the homotopy in the same way:

$h(\omega)_{ij}^{-+} :=$  $h(\omega)_{ij}^{+-} :=$ 

Next suppose $i = j$. Then the top left square in the pictures above is absent, but on the other hand we are guaranteed that the lower left and upper right squares are present, i.e. $i \neq \ell$ and $k \neq j$. We retract back to a boundary where the form vanishes as follows:

$h(\omega)_{ij}^{-+} :=$  $h(\omega)_{ij}^{+-} :=$ 

Finally, we set of course $h(\omega)_{ij}^{--} = 0$.

With these definitions, we obtain

$$\omega - I \circ P(\omega) = [\mathrm{d}, h](\omega)$$

for each i, j, as required. This completes the construction of the homotopy $h : \widetilde{\mathfrak{g}_{\mathrm{PDiscs}^\times}} \to \widetilde{\mathfrak{g}_{\mathrm{PDiscs}^\times}}$, and hence the proof of Theorem 33. □

Corollary 34. *There is a deformation retract of* $(U(\mathfrak{g}_{\mathrm{Global}}), U(\widetilde{\mathfrak{g}_{\mathrm{PDiscs}}}))$*-bimodules*

$$U(\mathfrak{g}_{\mathrm{Global}}) \otimes U(\widetilde{\mathfrak{g}_{\mathrm{PDiscs}}}) \underset{U(P)}{\overset{U(I)}{\rightleftarrows}} U(\widetilde{\mathfrak{g}_{\mathrm{PDiscs}^\times}}) \circlearrowleft \tilde{h}$$

Proof. This follows from Theorem 33 in the same way that Corollary 29 followed from Proposition 15. □

Data availability

No data was used for the research described in the article.

Appendix A. Proof of Theorem 6

We must show $\mathcal{C}^\bullet(\Pi'\mathbf{A})$ is a flasque resolution of $\mathcal{O}$. As we noted before the statement of Theorem 6, $\mathcal{C}^\bullet(\Pi'\mathbf{A})$ is certainly flasque. What remains is to check that the stalks of $\mathcal{C}^\bullet(\Pi'\mathbf{A})$ resolve the stalks of $\mathcal{O}$.

Let us consider the stalks at a point $p \in \mathrm{Rect}_2$. It is enough to show that, given any open $U \ni p$, there exists an open V with $p \in V \subset U$ such that $\Gamma(V, \mathcal{C}^\bullet(\Pi'\mathbf{A}) = \mathcal{C}^\bullet(\Pi'\mathbf{A}_V)$ is a resolution of $\mathcal{O}(V)$.

We may suppose V is of the form

$$V = \mathrm{Rect}_2 \setminus \bigcup_{i=1}^m \overline{\{(w = a_i)\}} \setminus \bigcup_{j=1}^m \overline{\{(z = b_j)\}}$$

i.e. that it has no isolated missing points, only missing lines.

(Indeed, every open neighbourhood of p is of the form $U = \mathrm{Rect}_2 \setminus \bigcup_{i=1}^m \overline{\{(w = a_i')\}} \setminus \bigcup_{j=1}^n \overline{\{(z = b_j')\}} \setminus \bigcup_{k=1}^p \{(c_k', d_k')\}$, as in (6). Suppose p is a closed point $(a, b) \in \mathbb{C} \times \mathbb{C}$. Then given such a U we may take $V = \mathrm{Rect}_2 \setminus \bigcup_{i=1}^n \overline{\{(w = a_i')\}} \setminus \bigcup_{j=1}^m \overline{\{(z = b_j')\}} \setminus \bigcup_{\substack{k=1 \\ c_k' \neq a}}^p \overline{\{(w = c_k')\}} \setminus \bigcup_{\substack{k=1 \\ d_k' \neq b}}^p \overline{\{(z = d_k')\}}$, which is an open subset of U, containing the point (a, b), and of the form we wanted. If instead p is a line $(w = a)$ or $(z = b)$, or the generic point E, the argument is similar.)

Let us regard the space

$$\mathcal{O}(V) = \mathbb{C}(w)_{a_1,\dots,a_m} \otimes \mathbb{C}(z)_{b_1,\dots,b_n}$$

of sections of $\mathcal{O}$ over V as a complex concentrated in degree zero:

$$0 \to \mathcal{O}(V) \to 0$$

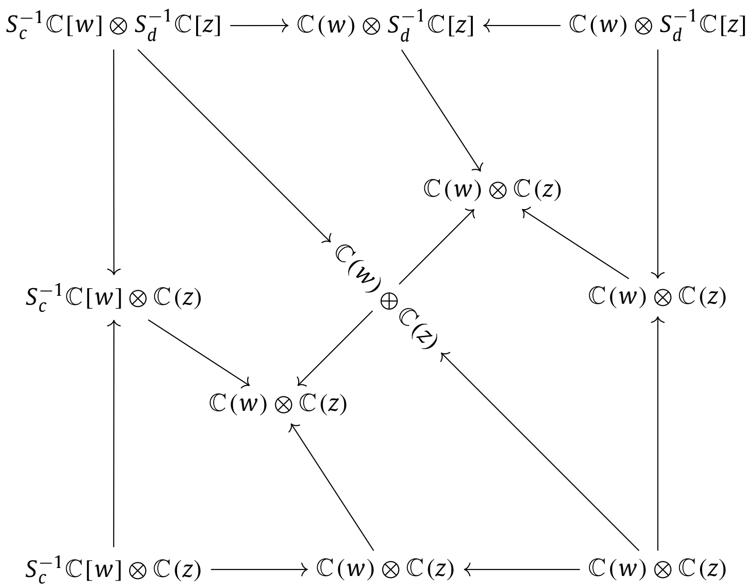
It is enough to show that $\mathcal{C}^\bullet(\Pi'\mathbf{A}_V)$ is homotopy equivalent to this complex.

To do that, we shall use the fact that the associated complex $\mathcal{C}^\bullet(\Pi'\mathbf{A}_V)$ and the Thom-Whitney complex $\mathrm{Th}^\bullet(\Pi'\mathbf{A}_V)$ are homotopy equivalent, as we recalled in §5.2. We shall show that $\mathcal{O}(V)$ is a deformation retract of $\mathrm{Th}^\bullet(\Pi'\mathbf{A}_V)$:

$$\big(0 \to \mathcal{O}(V) \to 0\big) \underset{p}{\overset{i}{\rightleftarrows}} \mathrm{Th}^\bullet(\Pi'\mathbf{A}_V) \circlearrowleft h \tag{46}$$

The argument below is similar to that in the proofs of Theorem 25 and Theorem 33. The only new ingredient conceptually is that the semisimplicial set $\mathrm{Flag}_\bullet(V)$ now consists of *uncountably many* copies of $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2)$, one for each closed point $(c, d) \in V$, appropriately sewn together along edges of the form $(\mathit{line}) \subset \mathrm{E}$. The restricted products Π' are needed to keep the sums finite in the definition of the homotopy h.

A cochain $\omega \in \mathrm{Th}^\bullet(\Pi'\mathbf{A}_V)$ is a polynomial differential form on $\mathrm{Flag}_\bullet(V)$, valued in $\mathbb{C}(w) \otimes \mathbb{C}(z)$ and subject to certain boundary conditions: namely for each $(c, d) \in V$, $\mathbf{A}_V$ restricts on the embedded copy of $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2(c, d))$ to the $\mathrm{Flag}_\bullet(\mathrm{PDisc}_2(c, d))$-algebra



(47)

Every element of $\mathbb{C}(w)\otimes\mathbb{C}(z)$ has a partial fraction decomposition, in w and then z, which is a sum of finitely many terms. Thus we have the direct sum decomposition of vector spaces

$$\mathbb{C}(w)\otimes\mathbb{C}(z)\cong_{\mathbb{C}}\mathcal{O}(V)\oplus\bigoplus_{c\notin\{a_1,\dots,a_m\}}(w-c)^{-1}\mathbb{C}[(w-c)^{-1}]\otimes\mathbb{C}(z)$$
$$\oplus\bigoplus_{d\notin\{b_1,\dots,b_n\}}\mathbb{C}(w)_{a_1,\dots,a_m}\otimes(z-d)^{-1}\mathbb{C}[(z-d)^{-1}]$$

In this way, our polynomial differential form ω decomposes uniquely as a sum

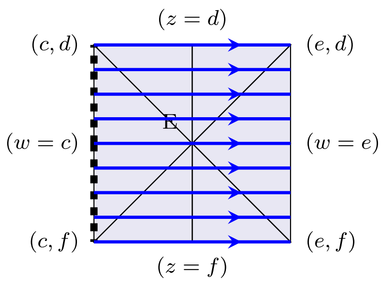
$$\omega=\omega_{\mathcal{O}(V)}+\sum_{c\notin\{a_1,\dots,a_m\}}\omega_c+\sum_{d\notin\{b_1,\dots,b_n\}}\omega'_d.$$

Here, it must be the case that after pulling ω back to any individual simplex S of $\mathrm{Flag}_\bullet(V)$ only finitely many summands are nonzero, so that $\omega|_S$ correctly takes values in $\mathbb{C}(w)\otimes\mathbb{C}(z)$. *Which* summands contribute, though, can depend on which of the uncountably many simplices one considers, so it need not be true that only finitely many summands are nonzero.

We define the homotopy h to act diagonally with respect to this decomposition,

$$h(\omega):=h_{\mathcal{O}(V)}(\omega_{O(V)})+\sum_{c\notin\{a_1,\dots,a_m\}}h_c(\omega_c)+\sum_{d\notin\{b_1,\dots,b_n\}}h'_d(\omega'_d),$$

as follows. The summand ω_c must vanish when pulled back to the boundaries (pt., $(w=c)$), and, in the same notation we used in the main text above, we define $h_c(\omega_c)$ to be a retract back to this boundary:

$h_c(\omega_c) :=$ 

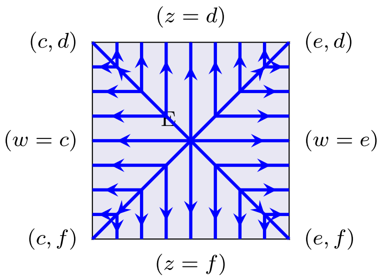
for every $d,f\notin\{b_1,\dots,b_n\}$ and $e\notin\{a_1,\dots,a_m,c\}$. What has to be checked is that for any simplex S, the resulting sum $\sum_{c\notin\{a_1,\dots,a_m\}}h_c(\omega_c)|_S$ has only finitely many non-zero terms. This is what the conditions in the definition (13) of Π' ensure. For example, consider a flag $((w=e),\mathrm{E})$. By our definition $h(\omega)|_{((w=e),\mathrm{E})}$ receives contributions from *all* flags $((w=c),\mathrm{E})$ in this sum. But for all but finitely many such flags, the pull-back $\omega|_{((w=c),\mathrm{E})}$ actually takes values not just in $\mathbf{A}((w=c),\mathrm{E})=\mathbb{C}(w)\otimes\mathbb{C}(z)$ but in $\mathbb{A}((w=c))=S_c^{-1}\mathbb{C}[w]\otimes\mathbb{C}(z)$, which means that the pullback of the summand ω_c to this flag actually vanishes. Similarly, consider a flag $((e,d),(w=e),\mathrm{E})$. By definition $h(\omega)|_{(e,d),(w=e),\mathrm{E}}$ receives a contribution from all flags

$((c,d),(w=c),\mathrm{E})$ and $((c,d),(z=d),\mathrm{E})$ for $c \notin \{b_1,\dots,b_m,e\}$, but at most finitely many of these contributions are actually nonzero.

We define $h'_d(\omega'_d)$ similarly to be given by retracting to the boundary of $\mathrm{Flag}_\bullet(V)$ defined by $(z=d)$.

We define $h_{\mathcal{O}(V)}$ to be the retraction to the vertex E: for every $c,d \notin \{a_1,\dots,a_m\}$ and $e,f \notin \{b_1,\dots,b_n\}$,

$h_{\mathcal{O}(V)}(\omega_{\mathcal{O}(V)}) :=$



In this case there are no potentially infinite sums: for example, $h_{\mathcal{O}(V)}(\omega_{\mathcal{O}(V)})|_{((c,d),(w=c),\mathrm{E})}$ receives contributions only from $\omega_{\mathcal{O}(V)}|_{((c,d),(w=c),\mathrm{E})}$ and $\omega_{\mathcal{O}(V)}|_{((c,d),\mathrm{E})}$.

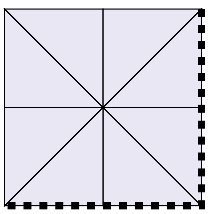
We let i be the map embedding an element of $\mathcal{O}(V)$ as a constant 0-form on $\mathrm{Flag}(V)$. (Observe that it obeys all the boundary conditions). We let p be the map which picks out the component in $\mathcal{O}(V)$ of the pull-back of a form to the vertex E:

$$p(\omega) := \omega_{\mathcal{O}(V)}|_{\mathrm{E}}$$

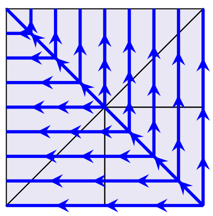
With these definitions, one checks that (46) is a deformation retract of dg vector spaces, as we wanted to show. This completes the proof of Theorem 6.

Appendix B. An example computation in detail

Suppose that Φ is a $\mathbb{C}$-valued polynomial differential form on the semisimplicial set shown on the left below. Assume that the boundary conditions on Φ are that it must vanish on the dotted edges shown. We define $h(\Phi)$ to be given as shown on the right.



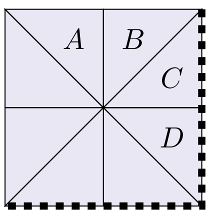
$h(\Phi) :=$



We now describe in detail what we mean by this pictorial definition of $h(\Phi)$, and show that it implies that

$$[\mathrm{d},h](\Phi) = \Phi.$$

Let us label individual 2-simplices by letters A, B, C, D



and write Φ^A for the form Φ on the simplex labelled A, etc, and likewise $h(\Phi)^A$ etc for $h(\Phi)$. On each individual 2-simplex we use the coordinates from §5.5; as in the main text, the vertex E is the one in the centre, and the vertices corresponding to closed points are the outer corners. We have

$$\Phi^A(s,t) = f^A(s,t) + f^A_s(s,t)\mathrm{d}s + f^A_t(s,t)\mathrm{d}t + f^A_{st}(s,t)\mathrm{d}s \wedge \mathrm{d}t$$

for some coefficient functions $f^A, f^A_s, f^A_t, f^A_{st}$, and similarly on the other 2-simplices.

Consider first the simplex labelled D. We define

$$h(\Phi)^D := a(\Phi^D) + b(\Phi^D)$$

with a and b as in the proof of Proposition 21. Exactly as in that proof, one then has

$$[\mathrm{d}, h(\Phi)]^D(s,t) = \Phi^D(s,t) - f^D(0,0),$$

and in our present case $f^D(0,0) = 0$ by the boundary conditions.

Now consider the simplex labelled A. We define

$$h(\Phi)^A := \widetilde{a}(\Phi^A) + \widetilde{b}(\Phi^A) + \int_0^1 \left(f_s^D(s',s') + f_t^D(s',s') \right) \mathrm{d}s' \tag{48}$$

with

$$\widetilde{a}(\Phi)(s,t) := \left(\int_t^s f_s(s',t)\mathrm{d}s' \right) + \left(\int_t^s f_{st}(s',t)\mathrm{d}s' \right) \mathrm{d}t$$

and

$$\widetilde{b}(\Phi)(s,t) := \left(\int_1^t \left(f_s(s',s') + f_t(s',s') \right) \mathrm{d}s' \right)$$

The computation is then similar to that in the proof of Proposition 21. Indeed, we have

$$(\mathrm{d} \circ \widetilde{a}(\Phi))(s,t) = f_s(s,t)\mathrm{d}s - f_s(t,t)\mathrm{d}t + \left(\int_t^s f_{s,2}(s',t)\mathrm{d}s' \right) \mathrm{d}t + f_{st}(s,t)\mathrm{d}s \wedge \mathrm{d}t$$

and

$$\begin{aligned}
(\widetilde{a} \circ \mathrm{d}(\Phi))(s,t) &= \widetilde{a}\Big(f_{,1}(s,t)\mathrm{d}s + f_{,2}(s,t)\mathrm{d}t + \big(f_{t,1}(s,t) - f_{s,2}(s,t) \big) \mathrm{d}s \wedge \mathrm{d}t \Big) \\
&= \left(\int_t^s f_{,1}(s',t)\mathrm{d}s' \right) + \left(\int_t^s (f_{t,1}(s',t) - f_{s,2}(s',t))\mathrm{d}s' \right) \mathrm{d}t \\
&= f(s,t) - f(t,t) + f_t(s,t)\mathrm{d}t - f_t(t,t)\mathrm{d}t - \left(\int_t^s f_{s,2}(s',t)\mathrm{d}s' \right) \mathrm{d}t,
\end{aligned}$$

and therefore

$$([\mathrm{d}, \widetilde{a}](\Phi))(s,t) = \Phi(s,t) - f(t,t) - f_s(t,t)\mathrm{d}t - f_t(t,t)\mathrm{d}t. \tag{49}$$

At the same time, we have

$$(\mathrm{d} \circ \widetilde{b}(\Phi))(s,t) = f_s(t,t)\mathrm{d}t + f_t(t,t)\mathrm{d}t$$

and

$$\begin{aligned}
(\widetilde{b} \circ \mathrm{d}(\Phi))(s,t) &= \widetilde{b}\Big(f_{,1}(s,t)\mathrm{d}s + f_{,2}(s,t)\mathrm{d}t + \big(f_{t,1}(s,t) - f_{s,2}(s,t) \big) \mathrm{d}s \wedge \mathrm{d}t \Big) \\
&= \int_0^t \left(f_{,1}(s',s') + f_{,2}(s',s') \right) \mathrm{d}s' = \int_1^t \partial_{s'} f(s',s')\mathrm{d}s' \\
&= f(t,t) - f(1,1).
\end{aligned} \tag{50}$$

The final term in our definition (48) above gives an additional contribution of $f^D(1,1) - f^D(0,0)$ to $[\mathrm{d}, h](\Phi)^A$. In total, we see that

$$([\mathrm{d}, h](\Phi))^A(s,t) = \Phi^A(s,t) - f^A(1,1) + f^D(1,1) - f^D(0,0).$$

But $f^A(1,1) = f^D(1,1)$ by continuity of the form Φ at the central vertex, and, once more, $f^D(0,0) = 0$ by the boundary conditions.

Now consider the simplex labelled C. We define

$$\begin{aligned} h(\Phi)^C(s,t) &:= c(\Phi^C)(s,t) + h(\Phi)^D(s,1) \\ &= c(\Phi^C)(s,t) + a(\Phi^D)(s,1) + b(\Phi^D)(s,1) \end{aligned} \tag{51}$$

where a, b remain as in the proof of Proposition 21, and where we introduce

$$c(\Phi)(s,t) := \left(\int_1^t f_t(s,t')\mathrm{d}t'\right) - \left(\int_1^t f_{st}(s,t')\mathrm{d}t'\right)\mathrm{d}s.$$

Then we see that

$$(\mathrm{d} \circ c(\Phi))(s,t) = \left(\int_1^t f_{t,1}(s,t')\mathrm{d}t'\right)\mathrm{d}s + f_t(s,t)\mathrm{d}t + f_{st}(s,t)\mathrm{d}s \wedge \mathrm{d}t$$

and

$$\begin{aligned} (c \circ \mathrm{d}(\Phi))(s,t) &= c\Big(f_{,1}(s,t)\mathrm{d}s + f_{,2}(s,t)\mathrm{d}t + \big(f_{t,1}(s,t) - f_{s,2}(s,t)\big)\,\mathrm{d}s \wedge \mathrm{d}t\Big) \\ &= \left(\int_1^t f_{,2}(s,t')\mathrm{d}t'\right) - \left(\int_1^t (f_{t,1}(s,t') - f_{s,2}(s,t'))\mathrm{d}t'\right)\mathrm{d}s \\ &= f(s,t) - f(s,1) - \left(\int_1^t f_{t,1}(s,t')\mathrm{d}t'\right)\mathrm{d}s + f_s(s,t)\mathrm{d}s - f_s(s,1)\mathrm{d}s, \end{aligned}$$

and therefore

$$([\mathrm{d}, c](\Phi))(s,t) = \Phi(s,t) - f(s,1) - f_s(s,1)\mathrm{d}s.$$

From our previous calculations (in the proof of Proposition 21) we know that

$$([\mathrm{d}, a+b](\Phi))(s,1) = f(s,1) + f_s(s,1)\mathrm{d}s - f(0,0).$$

Continuity of the form Φ on the edge between the 2-simplices C and D is the statement that

$$f(s,1)^C + f_s(s,1)^C\mathrm{d}s = f(s,1)^D + f_s(s,1)^D\mathrm{d}s.$$

Thus our definition (51) of $h(\Phi)^C$ implies that

$$[\mathrm{d}, h](\Phi)^C(s,t) = \Phi(s,t)^C - f^D(0,0) = \Phi(s,t)^C. \tag{52}$$

Finally, we turn to the vertex labelled B. We define

$$\begin{aligned} h(\Phi)^B(s,t) &:= \widetilde{a}(\Phi^B)(s,t) + h(\Phi)^C(t,t) \\ &= \widetilde{a}(\Phi^B)(s,t) + c(\Phi^C)(t,t) + a(\Phi^D)(t,1) + b(\Phi^D)(t,1) \end{aligned}$$

for the same functions $\widetilde{a}$, c, a, b as above. According to (49), we have

$$[\mathrm{d}, \widetilde{a}](\Phi^B)(s,t) = \Phi^B(s,t) - \Phi^B(t,t),$$

and here

$$\begin{aligned} \Phi^B(t,t) &= f^B(t,t) + f_s^B(t,t)\mathrm{d}t + f_t^B(t,t)\mathrm{d}t \\ &= f^C(t,t) + f_s^C(t,t)\mathrm{d}t + f_t^C(t,t)\mathrm{d}t = \Phi^C(t,t) \end{aligned}$$

by the continuity condition on the edge between the 2-simplices B and C. We have also, from (52), that

$$[\mathrm{d}, h](\Phi^C)(t,t) = \Phi^C(t,t).$$

We conclude that

$$[\mathrm{d},h](\Phi)^B(s,t)=\Phi^B(s,t).$$

This establishes that $[\mathrm{d},h](\Phi)=\Phi$ on each of the labelled simplices. The same is true on the remaining 4 simplices, by symmetry.

Appendix C. Homotopy Manin triples in L_∞ algebras

In the main text of this paper we choose to work with dg Lie algebras. Our definition of homotopy Manin triples in Section 6 extends straightforwardly to the larger category of L_∞ algebras. In this appendix we give this generalization, and relate it to notion of Manin L_∞ triples due to Kravchenko [37].

Indeed, Definition 9 goes over to L_∞ algebras unmodified except that one should generalize the invariance condition, as follows.

Definition 35. A *homotopy Manin triple (of $L\infty$ algebras)* $(\mathfrak{a},\mathfrak{a}_\pm,\iota_\pm,\langle -\mid -\rangle,n)$ is the data of

(1) L_∞ algebras $\mathfrak{a}$, $\mathfrak{a}_+$ and $\mathfrak{a}_-$
(2) L_∞ algebra maps $\mathfrak{a}_+\xrightarrow{\iota_+}\mathfrak{a}\xleftarrow{\iota_-}\mathfrak{a}_-$, and
(3) a (degree zero) map of dg vector spaces $\langle -\mid -\rangle:\mathfrak{a}\otimes\mathfrak{a}\to \mathsf{s}^{-n}\mathbb{C}$

subject to the following conditions:

(i) the map of dg vector spaces $(\iota_+,\iota_-):\mathfrak{a}_+\oplus\mathfrak{a}_-\to\mathfrak{a}$ is a homotopy equivalence
(ii) the map $\langle -\mid -\rangle:\mathfrak{a}\otimes\mathfrak{a}\to \mathsf{s}^{-n}\mathbb{C}$ is
- (graded) symmetric: $\langle x\mid y\rangle=(-1)^{\mathrm{gr}\,x\,\mathrm{gr}\,y}\langle y\mid x\rangle$ for all $x\in\mathfrak{a}^{\mathrm{gr}\,x}$, $y\in\mathfrak{a}^{\mathrm{gr}\,y}$.
- cyclic: for each of the brackets $\ell_k(-,\dots,-):\mathfrak{a}^{\otimes k}\to \mathsf{s}^{2-k}\mathfrak{a}$,

$$\left\langle \ell_k(x_1,\dots,x_{k-1},x_k)\mid x_0\right\rangle+(-1)^{\mathrm{gr}\,x_0\,\mathrm{gr}\,x_k}\left\langle \ell_k(x_1,\dots,x_{k-1},x_0)\mid x_k\right\rangle=0.$$

(iii) the map $\langle -\mid -\rangle:\mathfrak{a}\otimes\mathfrak{a}\to \mathsf{s}^{-n}\mathbb{C}$ is non-degenerate up to homotopy: If $\langle x\mid -\rangle\simeq 0$ then $x\simeq 0$ (i.e. x is exact).
(iv) both $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are isotropic, i.e. the maps

$$\langle \iota_\pm(-)\mid \iota_\pm(-)\rangle:\mathfrak{a}_\pm\otimes\mathfrak{a}_\pm\to \mathsf{s}^{-n}\mathbb{C}$$

are homotopic to zero.

Kravchenko gives a definition of a Manin L_∞ triple, or strongly homotopy Manin triple, in [37]. Here we relax the definition given there slightly, in two ways: we do not insist the L_∞ algebras be finite dimensional, and we allow the bilinear form to be degree-shifted.

Definition 36. A *Manin L_∞ − triple* $(\mathfrak{g},\mathfrak{g}_\pm,\langle -\mid -\rangle,n)$ is a triple of L_∞ algebras $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ equipped with a nondegenerate bilinear form $\langle -\mid -\rangle:\mathfrak{g}\otimes\mathfrak{g}\to \mathsf{s}^{-n}\mathbb{C}$, such that

(i) $\mathfrak{g}_+$, $\mathfrak{g}_-$ are L_∞ subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-$ as (dg) vector spaces
(ii) $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to $\langle -\mid -\rangle$
(iii) the bilinear form is cyclic; that is, for each of the brackets $\ell_k(-,\dots,-):\mathfrak{g}^{\otimes k}\to \mathsf{s}^{2-k}\mathfrak{g}$ of $\mathfrak{g}$,

$$\left\langle \ell_k(x_1,\dots,x_{k-1},x_k)\mid x_0\right\rangle+(-1)^{\mathrm{gr}\,x_0\,\mathrm{gr}\,x_k}\left\langle \ell_k(x_1,\dots,x_{k-1},x_0)\mid x_k\right\rangle=0.$$

It is immediate that

Proposition 37. *Every Manin L_∞ triple in the sense Definition 36 is a homotopy Manin triple of L_∞ algebras in the sense of Definition 35.* □

To make a statement in the reverse direction, we need the notion of homotopy transfer of algebraic structures. Recall – from e.g. [19, §3,§6] and references therein, and cf. [41, §9.4.3–9.4.5] – that one has a homotopy transfer theorem for L_∞ algebras: if $\mathfrak{a}$ is an L_∞ algebra then any homotopy equivalence of dg vector spaces $\mathfrak{a}\simeq\mathfrak{b}$ induces the structure of an L_∞ algebra on the dg vector space $\mathfrak{b}$, in such a way that $\mathfrak{a}\simeq\mathfrak{b}$ becomes a quasi-isomorphism of L_∞ algebras. In particular the cohomology $H(\mathfrak{a})$ is a deformation retract of $\mathfrak{a}$ in dg vector spaces, so it gets an L_∞ algebra structure. This structure is compatible with the usual graded Lie algebra structure on $H(\mathfrak{a})$ – in particular, it has vanishing differential – but typically has non-vanishing higher brackets. In this way, every L_∞ algebra $\mathfrak{a}$ is quasi-isomorphic to a *minimal model*, $H(\mathfrak{a})$. Since minimal models have vanishing differential, two minimal models are quasi-isomorphic precisely if they are isomorphic.

Under certain conditions on the retract of $\mathfrak{a}$ onto $H(\mathfrak{a})$, it is also possible to transfer the cyclic structure: see [7, Appendix B] and cf. [8].

Proposition 38. *Let* $(\mathfrak{a}, \mathfrak{a}_\pm, \iota_\pm, \langle - | - \rangle, n)$ *be a homotopy Manin triple of* L_∞ *algebras in the sense of Definition 35. Assume the retract of* $\mathfrak{a}$ *onto* $H(\mathfrak{a})$ *allows the cyclic structure to be transferred from* $\mathfrak{a}$ *to* $H(\mathfrak{a})$. *Then* $(H(\mathfrak{a}), H(\mathfrak{a}_+), H(\mathfrak{a}_-))$ *gets the structure of a Manin* L_∞*-triple (with shift n) in the sense of Definition 36.*

Proof. As we noted in Remark 10, we certainly have that

(i') the map (ι_+, ι_-) induces an isomorphism of graded vector spaces

$$H(\mathfrak{a}_+) \oplus H(\mathfrak{a}_-) \cong_{\mathbf{grVect}} H(\mathfrak{a})$$

(iii') the map of graded vector spaces

$$H(\mathfrak{a}) \otimes H(\mathfrak{a}) \to \mathrm{s}^{-n}\mathbb{C}$$

induced by $\langle - | - \rangle$ is non-degenerate.

(iv') both $H(\mathfrak{a}_+)$ and $H(\mathfrak{a}_-)$ are isotropic as subspaces of $H(\mathfrak{a})$.

By the homotopy transfer theorem $H(\mathfrak{a}_\pm)$ and $H(\mathfrak{a})$ are minimal L_∞ algebras. Moreover the maps $H(\iota_\pm): H(\mathfrak{a}_\pm) \to H(\mathfrak{a})$ are maps of L_∞ algebras. (Indeed, by the decomposition theorem for L_∞ algebras – see e.g. [32], especially equation (2.55) – one has quasi-isomorphisms of L_∞ algebras $I_\pm : H(\mathfrak{a}_\pm) \to \mathfrak{a}_\pm$ and $P : \mathfrak{a} \to H(\mathfrak{a})$; and $H(\iota_\pm) = P \circ \iota_\pm \circ I_\pm$ is then the composition of these L_∞ algebra maps.)

By assumption, the bilinear form on the cohomology $H(\mathfrak{a})$ induced by the bilinear form $\langle - | - \rangle$ on $\mathfrak{a}$ remains cyclic for the L_∞ structure on $H(\mathfrak{a})$. □

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