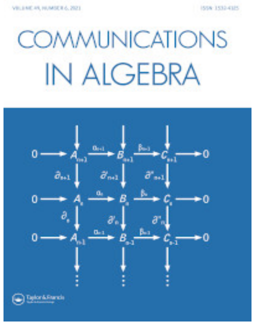

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Syzygy modules for dihedral groups

J. D. P. Evans

School of Computing, Electronics and Mathematics, Coventry University, Coventry, UK

ABSTRACT

Let p be an odd prime and $\Lambda = \mathbb{Z}[D_{2p}]$ the integral group ring of the dihedral group D_{2p} of order $2p$. The syzygies $\Omega_r(\mathbb{Z})$ are the stable classes of the intermediate modules in a free Λ-resolution of the trivial module. We will discuss explicitly the interaction of the stable syzygies under $-\otimes_{\mathbb{Z}}-$.

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Throughout, p will be an odd prime and C_p the cyclic group of order p generated by x. We then denote the dihedral group of order $2p$ by D_{2p} whose description can be written thus

$$D_{2p} = \langle x,\ y \mid x^p = y^2 = 1,\ yxy^{-1} = x^{p-1}\rangle.$$

The integral group ring of D_{2p} will then be denoted by $\Lambda = \mathbb{Z}[D_{2p}]$, unless stated otherwise.

Throughout, we work with finitely generated right modules and say that two Λ-modules $M,\ M'$ are *stably equivalent* (written $M \sim M'$) when $M \oplus \Lambda^a \cong M' \oplus \Lambda^b$ for some integers $a,\ b \geq 0$. We denote the set of isomorphism classes of modules M' such that $M' \sim M$ by $[M]$ and call this the stable module of M. Next consider the following free resolution,

$$\cdots \to F_{n+1} \xrightarrow{\partial_{n+1}} F_n \to \cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z},$$

where $\mathbb{Z}$ is the trivial Λ-module and each F_i is finitely generated and free. The syzygy modules $(J_r)_{r\geq 1}$ are defined to be the intermediate modules

$$J_r = Im(\partial_r) = Ker(\partial_{r-1}).$$

The stable syzygy $\Omega_r(\mathbb{Z})$ is then defined to be the stable class $[J_r]$ of any such J_r. It is a straightforward consequence of Schanuel's Lemma (cf. [5], p. 94) that $\Omega_r(\mathbb{Z})$ is independent of the choice of free resolution.

Our goal in what follows is to explicitly discuss the interaction of the above stable syzygies under the tensor product $-\otimes_{\mathbb{Z}}-$. In the hope of maintaining as succinct a notation as possible,

CONTACT J. D. P. Evans ad3290@coventry.ac.uk School of Computing, Electronics and Mathematics, Coventry University, Priory Street, Coventry, CV1 5FB, UK.
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we shall hereafter drop the subscript $\mathbb{Z}$ wherever it shall cause no confusion to do so. The key point is that the syzygies decompose into indecomposable modules representing what we may think of as the x-strand and the y-strand. For clarity, we denote the modules representing the x-strands of $\Omega_0(\mathbb{Z}),\ \Omega_1(\mathbb{Z}),\ \Omega_2(\mathbb{Z}),\ \Omega_3(\mathbb{Z})$ as $K,\ P,\ L,\ R,$ respectively.

Theorem A. *Let $\Lambda = \mathbb{Z}[D_{2p}]$ where $p = 2n+1$ is prime. For K as above, we will show that K acts as the identity within the stable class under the tensor product; that is, $K \otimes X \cong X \oplus \Lambda^a$ for some $a \geq 0$, and where X is either $K,\ P,\ L$ or R.*

Theorem B. *Let $\Lambda = \mathbb{Z}[D_{2p}]$ where $p = 2n+1$ is prime. For $K,\ P,\ L,\ R$ as above, the following identities hold:*

- $P \otimes P \cong L \oplus \Lambda^{n-1}$;
- $P \otimes L \cong R \oplus \Lambda^{n}$;
- $P \otimes R \cong K \oplus \Lambda^{n-1}$.

Using Theorems A and B it follows immediately that the x-strand of the stable syzygies $\Omega_r(\mathbb{Z})$ $(r = 0,\ 1,\ 2,\ 3)$ forms a cyclic group with identity $[K]$ and generated by $[P]$.

1. Preliminaries 1: the syzygies of cyclic groups

Throughout, we work with $\mathbb{Z}[G]$-lattices, i.e. $\mathbb{Z}[G]$-modules whose underlying abelian group is finitely generated and free. When $M,\ N$ are $\mathbb{Z}[G]$-lattices of ranks $m,\ n,$ respectively, the tensor product $M \otimes N$ is a $\mathbb{Z}[G]$-lattice of rank mn with G-action given by $(\nu \otimes \omega)g = \nu g \otimes \omega g$. Working with lattices confers several advantages, notably that the dual of a short exact sequence of $\mathbb{Z}[G]$-lattices is again a short exact sequence of $\mathbb{Z}[G]$-lattices (in which the arrows are reversed). This property extends to exact sequences of finite length (see [4], pp. 117-118). We denote the category of finitely generated $\mathbb{Z}[G]$-lattices by $\mathcal{F}(\mathbb{Z}[G])$.

We first discuss the stable syzygies of $\mathbb{Z}[C_p]$. There are only two such stable syzygies, reflecting the fact that C_p has period two. Indeed, it is the only such group to have free period two (see [9], Lemma 5.2, p.205). Whereas the result of this section is certainly well-known, the explicit construction of the isomorphism does not seem to appear in the literature.

As there is no benefit to working with prime numbers, we shall instead work with cyclic groups of order $n \geq 1$. To avoid confusion, it is stressed that for later sections, we will be working with $n = p$, prime. Furthermore, we shall frequently write $p = 2n+1$ in these later sections. The n in this description should not be confused with the n used in this section.

We describe the cyclic group of order n as $C_n = \langle x \mid x^n = 1 \rangle$ and set $\Lambda_0 = \mathbb{Z}[C_n]$. There is a free resolution of period two given by:

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda_0 \xrightarrow{x-1} \Lambda_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where ϵ is the augmentation map, and ϵ^* is its dual. We denote the augmentation ideal of Λ_0 by $I_C = ker(\epsilon)$. We can now read off the syzygies from the above as follows:

$$\Omega_r(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & r \equiv 0 (mod\ 2); \\ I_C, & r \equiv 1 (mod\ 2). \end{cases}$$

We will show the following:

$$I_C \otimes I_C \cong \mathbb{Z} \oplus \Lambda_0^{n-2}. \tag{1.1}$$

Recall that I_C can be written, $I_C = span_{\mathbb{Z}}\{x-1,\ x^2-1, \ldots,\ x^{n-1}-1\}$. In particular $rk_{\mathbb{Z}}(I_C) = n-1$. Consider the exact sequence, $0 \to I_C \xrightarrow{i} \Lambda_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$ and dualize,

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda_0 \xrightarrow{i^*} I_C^* \to 0$$

where $\epsilon^*(1) = \Sigma = \sum_{r=0}^{n-1} x^r$ is central. Therefore, $Im(\epsilon^*)$ is the two-sided ideal of Λ_0, generated by Σ. Consequently, we identify I_C^* with $\Lambda_0/(\Sigma)$, which is naturally a ring. Next, we put $\nu_r = i^*(x^r)$ where $\nu_0 = 1$, and observe that we can write $\nu_r = (\nu_1)^r = \nu^r$. If we think of I_C^* as a Λ_0-module, then I_C^* has a $\mathbb{Z}$-basis $\{1, \ \nu, \ \nu^2, \dots, \ \nu^{n-2}\}$, in which we have $\nu^{n-1} = -1 - \nu - \cdots - \nu^{n-2}$ and the action of x is to multiply by ν. It is well-known that $I_C \cong_{\Lambda_0} I_C^*$. However, once we introduce the y-action of D_{2p} it will be beneficial to distinguish between I_C and I_C^*.

Now, if $n=2$ then (1.1) is immediate. We, therefore, let $n \geq 3$ and define the following for $1 \leq r \leq n-2$:

$$V(r) = span_{\mathbb{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq n-1\} \subset I_C^* \otimes I_C^*.$$

Proposition 1.2. *For each $1 \leq r \leq n-2$, we have $V(r) \cong \Lambda_0$.*

Proof. Consider the map $f_r : \Lambda_0 \to V(r)$ which sends $x^k \mapsto \nu^{r+k} \otimes \nu^k$. This map is clearly surjective by the definition of $V(r)$. Thus, to show f is an isomorphism it suffices to show the defining set of $V(r)$ is linearly independent. This is straightforward and so the details are omitted. The reader is directed to [2] for a full proof. □

Proposition 1.3. *For any $1 \leq r \leq n-2$,*

$$V(r) \cap (V(1) + \cdots + V(r-1) + V(r+1) + \cdots + V(n-2)) = \{0\}.$$

Again, this is straightforward to check. The reader is directed to [2] for the details.

We now set $V = V(1) \oplus \cdots \oplus V(n-2)$ and observe that $rk_{\mathbb{Z}}(V) = n(n-2)$. Thus, $rk_{\mathbb{Z}}((I_C^* \otimes I_C^*)/V) = 1$. By considering the underlying abelian group of $(I_C^* \otimes I_C^*)/V$, we see that this is isomorphic to $\mathbb{Z} \oplus$ (finite abelian). However, in [2] it was shown that $(I_C^* \otimes I_C^*)/V$ is torsion free, and so is isomorphic to $\mathbb{Z}$. Consider the basis $\{\nu^i \otimes \nu^j \mid 0 \leq i, \ j \leq n-2\}$ of $I_C^* \otimes I_C^*$. By performing elementary basis transformations (see Proposition 4.11), this can be replaced by the following basis:

$$\{\nu^{r+k} \otimes \nu^k \mid 1 \leq r \leq n-2, \ 0 \leq k \leq n-1\} \cup \{T\},$$

where

$$\begin{aligned} T = \ & 1 \otimes 1 + 1 \otimes \nu + 1 \otimes \nu^2 + \cdots + 1 \otimes \nu^{n-2} \\ & + \nu \otimes \nu + \nu \otimes \nu^2 + \cdots + \nu \otimes \nu^{n-2} \\ & + \nu^2 \otimes \nu^2 + \cdots + \nu^2 \otimes \nu^{n-2} \\ & \ddots \qquad \vdots \\ & + \nu^{n-2} \otimes \nu^{n-2}. \end{aligned}$$

So $(I_C^* \otimes I_C^*)/V$ is generated by $\natural(T)$, where $\natural : I_C^* \otimes I_C^* \to (I_C^* \otimes I_C^*)/V$ is the natural surjection. It is clear that $T \in I_C^* \otimes I_C^*$ but $T \notin V$, and that $Tx = T$, thereby showing that x acts trivially on $(I_C^* \otimes I_C^*)/V$. Since x clearly acts trivially on $\mathbb{Z}$, our isomorphism extends to one over Λ_0, i.e. $(I_C^* \otimes I_C^*)/V \cong_{\Lambda_0} \mathbb{Z}$.

For notational convenience, we shall adopt the slight abuse of notation by writing T for the monogenic module of rank 1. The above arguments have shown the existence of the following short exact sequence, $0 \to V \to I_C^* \otimes I_C^* \to T \to 0$. By dualizing and using the self-duality of $V \cong \Lambda_0^{n-2}$ and $T \cong \mathbb{Z}$, we, therefore, arrive at the desired isomorphism:

Proposition 1.4. *$I_C \otimes I_C \cong T \oplus V \cong \mathbb{Z} \oplus \Lambda_0^{n-2}$.*

2. Preliminaries 2: useful results for dihedrals

We now summarize several results found in [6]. We find it useful to write $p = 2n+1$, however, the reader is once again reminded that this n is not the same n as the previous section. As we shall see, many of our calculations will not overtly require $2n+1$ to be prime. Rather, the necessity of this is to ensure our syzygies are periodic, and to ensure our modules of interest are indecomposable.

Hereafter, we write $\Lambda = \mathbb{Z}[D_{2p}]$ and $\Lambda_0 = \mathbb{Z}[C_p]$. Associated to Λ_0 is the canonical injection $i : \Lambda_0 \hookrightarrow \Lambda$. From this we can induce two maps on the categories of finitely generated lattices, $i^* : \mathcal{F}(\Lambda) \to \mathcal{F}(\Lambda_0)$ and $i_* : \mathcal{F}(\Lambda_0) \to \mathcal{F}(\Lambda)$, where i^* is given by restricting scalars to Λ_0, and i_* is given by extending scalars; that is, $i_*(M) = M \otimes_{\Lambda_0} \Lambda$. Similarly, we have the canonical injection $j : \mathbb{Z}[C_2] \hookrightarrow \Lambda$ and this too induces two maps on the categories of finitely generated lattices. Both the restriction and extension of scalars functors have easily verified properties; they are additive, exact, and take free modules to free modules. In addition, the restriction and extension of scalars functors arise in the context of the well-known Eckmann-Shapiro relations. An exposition of these relations in our context can be found in Appendix B of [5]. A related result is sometimes referred to as the 'projection formula for Frobenius reciprocity' in the literature (see [1]).

Proposition 2.1 **(Frobenius Reciprocity).** *Let $i : H \subset G$ be the inclusion map of the subgroup H into a finite group G. If M is a $\mathbb{Z}[H]$-module, and N is a $\mathbb{Z}[G]$-module, then there exists an isomorphism $\varphi : i_*(M) \otimes_{\mathbb{Z}} N \xrightarrow{\simeq} i_*(M \otimes_{\mathbb{Z}} i^*(N))$.*

The augmentation ideal of Λ will be denoted by I_G, and the augmentation ideals of Λ_0 and $\mathbb{Z}[C_2]$ will be denoted by I_C and I_2, respectively. By $[\alpha)$ we mean the right ideal generated by α; that is $[\alpha) = \{\alpha\lambda \mid \lambda \in \Lambda\}$. In particular, any ideal in Λ is a Λ-lattice.

In keeping with the notation of [6] we set,

$$\pi = (x^n - 1)(y - 1) \tag{2.2}$$

$$\tilde{\rho} = (y - 1)(x - 1). \tag{2.3}$$

We then define

$$P = [\pi), \quad R = [\tilde{\rho}). \tag{2.4}$$

The author stresses the importance of the expressions denoted by π and $\tilde{\rho}$. In addition, we write $\Sigma_x = 1 + x + \cdots + x^{2n}$, which we observe is central in Λ.

Proposition 2.5. *The ideal $[x-1)$ decomposes as a direct sum $[x-1) = P \oplus R$.*

We provide two alternative descriptions for P and R using the results of Section 1. We saw that I_C^* has a $\mathbb{Z}$-basis, $\{\nu^r \mid 0 \le r \le 2n-1\}$ where $1 + \nu + \cdots + \nu^{2n} = 0$. The action of C_p on I_C^* may be extended in one of two ways to an action of the dihedral group:

- Either: $\nu^r \cdot y = \nu^{-r} = \nu^{2n+1-r}$ for $0 \le r \le 2n-1$;
- or: $\nu^r \cdot y = -\nu^{-r} = -\nu^{2n+1-r}$ for $0 \le r \le 2n-1$.

Under the former, we denote $(I_C^*)_+$, and under the latter we denote $(I_C^*)_-$.

Proposition 2.6. *$P \cong (I_C^*)_-$ and $R \cong (I_C^*)_+$.*

Proof. We use the classification result in Section 5 of [6]. Take $\nu^0 \in (I_C^*)_-$ and note that $\nu^0 \cdot x^r = \nu^r$. Moreover, $\nu^0 \cdot y = -\nu^0$ by our choice of Galois action. Thus, Johnson's $\mathcal{M}(-)$ property is satisfied. It remains to show the $\mathcal{M}(\Sigma)$ property. Let $\alpha \in (I_C^*)_-$ be written as $\alpha = \Sigma_r a_r \nu^r$. Since $1 + \nu + \nu^2 + \cdots + \nu^{2n} = 0$,

$$\nu^r \Sigma_x = \nu^r(1 + x + x^2 + \cdots x^{2n}) = \nu^r + \nu^{r+1} + \nu^{r+2} + \cdots + \nu^{r+2n} = \nu^r(1 + \nu + \nu^2 + \cdots + \nu^{2n})$$
$$= 0.$$

From this we can clearly deduce $M(\Sigma)$ is satisfied and $P \cong (I_C^*)_-$, as required. A similar argument shows $R \cong (I_C^*)_+$. □

Note, P and R are *not* isomorphic as Λ-modules, nor even stably isomorphic. Nevertheless, $P^* \cong R$ and $R^* \cong P$. Next, we define the modules K and L to be

$$K = [\Sigma_x,\ y-1) \text{ and } L = [\Sigma_x,\ y+1). \tag{2.7}$$

In [6], it was shown that both K and L have $\mathbb{Z}$-rank $2n+2$, and that $\Lambda/K \cong R$ and $\Lambda/L \cong P$. Finally, both K and L are self-dual; that is $K^* \cong K$ and $L^* \cong L$.

3. The module *K* acts as the identity

With the preliminary legwork complete, we now show Theorem 1; that is, we show

$$K \otimes ? \cong ? \oplus \Lambda^r \tag{3.1}$$

for some $r \geq 0$ and where $? = K,\ P,\ L,\ R$. In [6] it was shown K has a $\mathbb{Z}$-basis given by $\{(y-1)x^i \mid 0 \leq i \leq 2n\} \cup \{\Sigma_x\}$. Define $K_0 = span_{\mathbb{Z}}\{(y-1),\ (y-1)x, \ldots,\ (y-1)x^{2n}\}$ so that K/K_0 is represented by the class of Σ_x. Observe $\Sigma_x \cdot x = \Sigma_x$ and $\Sigma_x \cdot y = (y-1)\Sigma_x + \Sigma_x = \Sigma_x$ in K/K_0. Thus, x and y act trivially on K/K_0 and it is, therefore, isomorphic to $\mathbb{Z}$. In particular, we have an exact sequence of the form $0 \to K_0 \to K \to \mathbb{Z} \to 0$. Recall $j : \mathbb{Z}[C_2] \hookrightarrow \Lambda$. It is straightforward to show

$$j_*(I_2) \cong [y-1). \tag{3.2}$$

Likewise, $[y+1)$, can be thought of as an induced module of $\mathbb{Z}$. Observe that both $[y-1)$ and $[y+1)$ are self-dual, but are *not* isomorphic, as Λ-modules.

We now show K acts as the identity within the stable class of our cyclic group of order 4. Observe that tensoring with any of the $P,\ R,\ K$ or L yields the exact sequence

$$0 \to K_0 \otimes ? \to K \otimes ? \to ? \to 0.$$

Proposition 3.3. $j^*(P) \cong j^*(R) \cong \mathbb{Z}[C_2]^n$.

Proof. Consider the exact sequence $0 \to I_C \to \Lambda_0 \to \mathbb{Z} \to 0$, and apply the exact functor $i_*(-)$ to yield $0 \to i_*(I_C) \to \Lambda \to \mathbb{Z}[C_2] \to 0$. Next, observe that the induced module $i_*(I_C)$ is simply another description of $[x-1)$. So, using the isomorphism $[x-1) \cong P \oplus R$ (see Proposition 2.5), we have the following exact sequence,

$$0 \to R \oplus P \to \Lambda \to \mathbb{Z}[C_2] \to 0.$$

Now apply the exact functor $j^*(-)$,

$$0 \to j^*(R \oplus P) \to \mathbb{Z}[C_2]^{2n+1} \to \mathbb{Z}[C_2] \to 0.$$

This sequence splits and we observe $j^*(R \oplus P)$ is stably free of rank $2n$. Furthermore, as C_2 satisfies the Eichler condition (see [5], pp. 175–176), $j^*(R \oplus P)$ is free by Swan-Jacobinski (see [3, 8, 10]). Hence, both $j^*(R)$ and $j^*(P)$ are projective $\mathbb{Z}[C_2]$-modules of equal $\mathbb{Z}$-rank. Since $\tilde{K}_0(\mathbb{Z}[C_2]) = 0$ (see, for example, [7]), any projective module is necessarily stably free. Using Swan-Jacobinski once more, $j^*(P)$ and $j^*(R)$ are both free, each of rank n. □

Proposition 3.4. *$K \otimes R(i) \cong R(i) \oplus \Lambda^n$, for $1 \leq i \leq 2$ where $R(1) \cong P$ and $R(2) \cong R$.*

Proof. Consider the following exact sequence,

$$0 \to K_0 \otimes R(i) \to K \otimes R(i) \to R(i) \to 0.$$

Note that K_0 is simply $[y-1)$. So, by using (3.2), two applications of Frobenius Reciprocity (Proposition 2.1), and Proposition 3.3, we have the following isomorphism:

$$j_*(I_2) \otimes R(i) \cong j_*(I_2 \otimes j^*(R(i))) \cong j_*(\mathbb{Z}[C_2]^n) \cong \Lambda^n.$$

Replacing this in the above exact sequence we, therefore, get

$$0 \to \Lambda^n \to K \otimes R(i) \to R(i) \to 0$$

which splits, yielding $K \otimes R(i) \cong R(i) \oplus \Lambda^n$, as required. □

Proposition 3.5. $j^*(K) \cong \mathbb{Z}[C_2]^{n+1}$.

Proof. Start with the following exact sequence, $0 \to K \to \Lambda \to R \to 0$ and apply $j^*(-)$,

$$0 \to j^*(K) \to \mathbb{Z}[C_2]^{2n+1} \to j^*(R) \to 0.$$

By Proposition 3.3, we know $j^*(R) \cong \mathbb{Z}[C_2]^n$ and so the above exact sequence splits, yielding $j^*(K) \oplus \mathbb{Z}[C_2]^n \cong \mathbb{Z}[C_n]^{2n+1}$, i.e. $j^*(K)$ is stably free of rank $n+1$. Since stably free modules are free over $\mathbb{Z}[C_2]$ (Swan-Jacobinski), $j^*(K) \cong \mathbb{Z}[C_2]^{n+1}$, as required. □

A similar argument shows the following:

Proposition 3.6. $j^*(L) \cong \mathbb{Z}[C_2]^{n+1}$.

Proposition 3.7. $K \otimes K_i \cong K_i \oplus \Lambda^{n+1}$, *for* $1 \leq i \leq 2$ *where* $K_1 = L$ *and* $K_2 = K$.

Proof. Consider the exact sequence,

$$0 \to K_0 \otimes K_i \to K \otimes K_i \to K_i \to 0.$$

Using Frobenius Reciprocity and Propositions 3.5 and 3.6, we have

$$j_*(I_2) \otimes K_i \cong j_*(I_2 \otimes j^*(K_i)) \cong j_*(\mathbb{Z}[C_2]^{n+1}) \cong \Lambda^{n+1}.$$

For each $1 \leq i \leq 2$, the above exact sequence now splits, yielding $K \otimes K_i \cong K_i \oplus \Lambda^{n+1}$. □

Evidently, Theorem 1 follows directly from Propositions 3.4 and 3.7.

4. $\mathbf{P \otimes P \sim L}$

In this section, we show

$$P \otimes P \cong L \oplus \Lambda^{n-1}. \tag{4.1}$$

Recall Section 1 in which we showed $I_C^* \otimes I_C^* \cong_{\Lambda_0} T \oplus V$, where $V = V(1) \oplus \cdots \oplus V(2n-1)$ and $V(r) = span_{\mathbb{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq 2n\}$. Moreover, in Section 2 we introduced the following action of y, $\nu^r \cdot y = -\nu^{2n+1-r}$. As before, $\{\nu^r \mid 0 \leq r \leq 2n-1\}$ is a $\mathbb{Z}$-basis for I_C^*, and under this action $(I_C^*)_- \cong P$. We first construct the free part by showing that for $r \geq 2$,

$$V_r = V(r) + V(2n+1-r) \text{ is a } \Lambda - \text{module, and } V_r \cong \Lambda.$$

Proposition 4.2. *Define* Ψ *to be the* $(2n+1) \times (2n+1)$ *matrix where*

$$\Psi_{ij} = \begin{cases} 1, & i = 1, \; j = 2n+1; \\ 1, & j = i-1, \; 2 \leq i \leq 2n+1; \\ 0, & o/w. \end{cases}$$

Then $\rho_{V_r}(x^{-1}) = \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix}$.

Proof. Label

$$e_i = \nu^{r+i-1} \otimes \nu^{i-1}, \quad 1 \le i \le 2n+1$$
$$e_{(2n+1)+i} = \nu^{2n-r+i} \otimes \nu^{i-1}, \quad 1 \le i \le 2n+1.$$

Then $e_{2n+1} \cdot x = \nu^r \otimes 1 = e_1$ and for $2 \le i \le 2n+1, e_i \cdot x = \nu^{r+i} \otimes \nu^i = e_{i+1}$. Likewise, we have $e_{4n+2} \cdot x = \nu^{2n+1-r} \otimes 1 = e_{2n+2}$. In general, for $2 \le i \le 2n+1$, x acts on $e_{(2n+1)+i}$ by $e_{(2n+1)+i} \cdot x = \nu^{2n+1-r+i} \otimes \nu^i = e_{(2n+1)+i+1}$. The result now follows. □

Proposition 4.3. *Define the matrix Φ by*

$$\Phi_{ij} = \begin{cases} 1, & i=j=1; \\ 1, & j = 2n+3-i, \ 2 \le i \le 2n+1; \\ 0, & o/w. \end{cases}$$

Then $\rho_{V_r}(y) = \begin{pmatrix} 0 & \Phi \\ \Phi & 0 \end{pmatrix}$.

Proof. With the e_i, $e_{(2n+1)+i}$ as defined above, first observe $e_1 \cdot y = \nu^{2n+1-r} \otimes 1 = e_{2n+2}$. Now consider y acting on a general basis element of $V(r)$ for $2 \le i \le 2n+1$,

$$e_i \cdot y = (\nu^{r+i-1} \otimes \nu^{i-1})y = (\nu^{2n+2-r-i} \otimes \nu^{2n+2-i}) = (\nu^{2n-r+(2-i)} \otimes \nu^{2n+2-i}) = e_{2n+1+(2n+3-i)}.$$

Next, let y act on the basis elements of $V(2n+1-r)$. As before, we have $e_{2n+2} \cdot y = \nu^r \otimes 1 = e_1$. For a general basis element $2 \le i \le 2n+1$,

$$e_{2n+1+i} \cdot y = (\nu^{2n-r+i} \otimes \nu^{i-1})y = \nu^{r+(2n+2-i)} \otimes \nu^{2n+2-i} = e_{2n+3-i}.$$

□

Proposition 4.4. *With Ψ as defined above* $\rho_{reg}(x^{-1}) = \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix}$.

Proof. Set

$$f_i = x^{i-1}, \quad 1 \le i \le 2n+1$$
$$f_{2n+1+i} = yx^{i-1}, \quad 1 \le i \le 2n+1.$$

Clearly, $f_{2n+1} \cdot x = 1 = f_1$ *and* $f_i \cdot x = x^i = f_{i+1}$ *for* $1 \le i \le 2n$. Similarly, $f_{4n+2} \cdot x = y = f_{2n+2}$ *and* $f_{2n+1+i} \cdot x = yx^i = f_{(2n+1)+i+1}$. The result now follows. □

Proposition 4.5. *With Φ as defined above,* $\rho_{reg}(y) = \begin{pmatrix} 0 & \Phi \\ \Phi & 0 \end{pmatrix}$.

Proof. Let f_i, f_{2n+1+i} be as above. First observe $f_1 \cdot y = y = f_{2n+2}$. For a more general element where $2 \le i \le 2n+1$, we have $f_i \cdot y = x^{i-1}y = yx^{2n+2-i} = f_{(2n+1)+2n+3-i}$. Similarly $f_{2n+2} \cdot y = 1 = f_1$. For a more general element, $f_{2n+1+i} \cdot y = yx^{i-1}y = x^{2n+2-i} = f_{2n+3-i}$. □

Proposition 4.6. *For $r \ge 2$, $V_r = V(r) + V(2n+1-r) \cong \Lambda$.*

Proof. Immediate since $\rho_{V_r}(g) = \rho_{reg}(g)$ for all $g \in D_{4n+2}$, by Propositions 4.2 - 4.5. □

We are, therefore, left with $T + V(1)$ which can be used to give us L. Consider the following map $\psi : I_C^* \otimes I_C^* \to \mathbb{Z}$ defined by,

$$\nu^r \otimes \nu^s \mapsto \begin{cases} 1, & \text{if } r = s+1; \\ -1, & \text{if } s = r+1; \\ 0, & \text{if } |r-s| \ne 1 \end{cases}$$

where $\mathbb{Z}$ is taken to mean the trivial Λ_0-module and $0 \le r, \ s \le 2n-1$.

Proposition 4.7. *The map ψ, as defined above, is a Λ_0-homomorphism.*

Proof. It is straightforward to check ψ is well defined and a $\mathbb{Z}$-homomorphism. By applying the x-action, note that $\psi(\nu^r x \otimes \nu^s x) = \psi(\nu^{r+1} \otimes \nu^{s+1}) = \psi(\nu^r \otimes \nu^s) = \psi(\nu^r \otimes \nu^s)\ x$ since x acts trivially. We, therefore, conclude that ψ is a Λ_0-homomorphism. □

Proposition 4.8. *The map $\psi : (I_C)_- \otimes (I_C)_- \to \mathbb{Z}_-$ induced from ψ above, is a* Λ-homomorphism.

Proof. Consider how ψ behaves on $\nu^r \otimes \nu^s$. By Proposition 4.7, we need only consider the y-action. There are three cases to consider:

$$(a)\ \ r = s + 1, \qquad (b)\ \ s = r + 1, \qquad (c)\ \ |r - s| \neq 1.$$

For (a) we have the following:

$$\nu^{s+1}y \otimes \nu^s y = \begin{cases} -\sum_{i=0}^{2n-1} \nu^i \otimes 1, & s = 0; \\ -\nu^{2n-1} \otimes \sum_{i=0}^{2n-1} \nu^i, & s = 1; \\ \nu^{2n-s} \otimes \nu^{2n+1-s}, & 2 \leq s \leq 2n-2. \end{cases}$$

As such,

$$\psi\left(-\sum_{i=0}^{2n-1} \nu^i \otimes 1\right) = \psi(-1 \otimes 1 - \nu \otimes 1 - \cdots - \nu^{2n-1} \otimes 1) = -1,$$
$$\psi\left(-\nu^{2n-1} \otimes \sum_{i=0}^{2n-1} \nu^i\right) = \psi(-\nu^{2n-1} \otimes 1 - \cdots - \nu^{2n-1} \otimes \nu^{2n-2} - \nu^{2n-1} \otimes \nu^{2n-1}) = -1,$$

and finally $\psi(\nu^{2n-s} \otimes \nu^{2n+1-s}) = -1$. In each case, y acts by -1 when $r = s + 1$. A similar argument applies to $s = r + 1$.

When $|r - s| \neq 1$ there are the following possibilities:

$$\nu^r y \otimes \nu^s y = \begin{cases} 1 \otimes 1, & r = s = 0; \\ 1 \otimes \nu^{2n+1-s}, & r = 0,\ 2 \leq s \leq 2n-1; \\ \nu^{2n} \otimes \nu^{2n}, & r = s = 1; \\ \nu^{2n} \otimes \nu^{2n+1-s}, & r = 1,\ 3 \leq s \leq 2n-1; \\ \nu^{2n+1-r} \otimes 1, & 2 \leq r \leq 2n-1,\ \ s = 0; \\ \nu^{2n+1-r} \otimes \nu^{2n}, & 3 \leq r \leq 2n-1,\ \ s = 1; \\ \nu^{2n+1-r} \otimes \nu^{2n+1-s}, & 2 \leq r,\ \ s \leq 2n-1. \end{cases}$$

The only cases that do not obviously go to zero are those involving ν^{2n}. For $\nu^{2n} \otimes \nu^{2n+1-s}$ observe $2 \leq 2n + 1 - s \leq 2n - 2$. It is clear that both $r - (2n + 1 - s) = 1$ and $(2n + 1 - s) - r = 1$ occurs as r varies between 0 and $2n - 1$. Hence, the $+\ 1$ and -1 cancel each other out and the effect is that $\nu^{2n} \otimes \nu^{2n+1-s}$ is sent to zero. Likewise, $\nu^{2n+1-r} \otimes \nu^{2n}$ is sent to zero.

For the case $\nu^{2n} \otimes \nu^{2n}$, we sum over all $\nu^r \otimes \nu^s$ where both r, s vary as above. By the above remark, we are left with elements of the form $1 \otimes \nu^s$ and $\nu^{2n-1} \otimes \nu^s$ where $0 \leq s \leq 2n - 1$. The only elements sent to something other than zero in the former is of the form $1 \otimes \nu$, and in the latter $\nu^{2n-1} \otimes \nu^{2n-2}$. These are clearly sent to -1 and 1, respectively, and, therefore, cancel. It follows that ψ will map each of the above elements to zero when $|r - s| \neq 1$. Consequently, y acts by -1 and $\psi : (I_C^*)_- \otimes (I_C^*)_- \to \mathbb{Z}_-$ is a Λ-homomorphism. □

Let $V' = V(2) + \cdots + V(2n-1)$ and observe that V' is free and $V' \subset Ker(\psi)$. Let

$$\natural : \left[(I_C^*)_- \otimes (I_C^*)_-\right] \to \left[(I_C^*)_- \otimes (I_C^*)_-\right]/V'$$

be the natural map and restrict ψ to $[(I_C^*)_- \otimes (I_C^*)_-]/V'$. Then the following is exact,

$$0 \to Ker(\psi) \to \left[(I_C^*)_- \otimes (I_C^*)_-\right]/V' \xrightarrow{\psi} \mathbb{Z}_- \to 0. \tag{4.9}$$

To emphasize which part of $(I_C^*)_- \otimes (I_C^*)_-$ remains when we quotient by V', we write this as $\natural(T + V(1))$. Observe that the sum $T + V(1)$ will not be direct over Λ. We now prove:

Proposition 4.10. *$Ker(\psi) = span_{\mathbb{Z}}\{\natural(\nu^i \otimes \nu^i) \mid 0 \leq i \leq 2n\}$.*

To do so, we first prove several preliminary results.

Proposition 4.11. *For each $j \in \{0, 1, \ldots, p-1\}$, $\natural(\nu^j \otimes \nu^j) = \natural(T) + \sum_{i=j+1}^{j+p-2} \natural(\nu^{i+1} \otimes \nu^i)$.*

Proof. Start with the case $j = 0$ in which we successively subtract the 'columns' from T.

$$\begin{aligned} 1 \otimes 1 &= T + \nu^{p-1} \otimes \nu^{p-2} - (1 + \nu + \cdots + \nu^{p-3}) \otimes \nu^{p-3} - \cdots \\ &\quad \cdots - (1 + \nu + \cdots + \nu^i) \otimes \nu^i - \cdots - (1 + \nu) \otimes \nu \\ &= T + \nu^{p-1} \otimes \nu^{p-2} + (\nu^{p-1} + \nu^{p-2}) \otimes \nu^{p-3} + \cdots \\ &\quad \cdots + (\nu^{p-1} + \cdots + \nu^{i+1}) \otimes \nu^i + \cdots + (\nu^{p-1} + \cdots + \nu^2) \otimes \nu \\ &= T + \sum_{i=1}^{p-2} (\nu^{p-1} + \cdots + \nu^{i+1}) \otimes \nu^i. \end{aligned}$$

However, $\nu^{i+k} \otimes \nu^i \in V(k)$ for some $1 \leq k \leq p-2$. Thus, $\natural(1 \otimes 1) = \natural(T) + \sum_{i=1}^{p-2} \natural(\nu^{i+1} \otimes \nu^i)$.

Now suppose we have shown the result for $j \in \{0, \ldots, p-2\}$, i.e. we have shown $\natural(\nu^{j-1} \otimes \nu^{j-1}) = \natural(T) + \sum_{i=j}^{j+p-3} \natural(\nu^{i+1} \otimes \nu^i)$. We need to show the result holds for $\natural(\nu^j \otimes \nu^j)$. The result will then follow from a straightforward inductive argument.

$$\begin{aligned} \natural(T) + \sum_{i=j+1}^{j+p-2} \natural(\nu^{i+1} \otimes \nu^i) &= \natural\left(\nu^{j-1} \otimes \nu^{j-1} - \sum_{i=j}^{j+p-3} \nu^{i+1} \otimes \nu^i + \sum_{i=j+1}^{j+p-2} \nu^{i+1} \otimes \nu^i\right) \\ &= \natural(\nu^{j-1} \otimes \nu^{j-1} - \nu^{j+1} \otimes \nu^j + \nu^{j-1} \otimes \nu^{j-2}) \\ &= \natural(\nu^{j-1} \otimes (\nu^{j-1} + \nu^{j-2}) - \nu^{j+1} \otimes \nu^j) \\ &= \natural(-\nu^{j-1} \otimes (1 + \nu + \cdots + \nu^{j-3} + \nu^j + \cdots + \nu^{p-1}) - \nu^{j+1} \otimes \nu^j) \\ &= \natural(-(\nu^{j-1} + \nu^{j+1}) \otimes \nu^j - \nu^{j-1} \otimes (1 + \nu + \cdots + \nu^{j-3} + \nu^{j+1} + \cdots + \nu^{p-1})) \\ &= \natural((1 + \nu + \cdots + \nu^{j-2} + \nu^j + \nu^{j+2} + \cdots + \nu^{p-1}) \otimes \nu^j - \\ &\quad - \nu^{j-1} \otimes (1 + \nu + \cdots + \nu^{j-3} + \nu^{j+1} + \cdots + \nu^{p-1})) \\ &= \natural(\nu^j \otimes \nu^j + (1 + \nu + \cdots + \nu^{j-2} + \nu^{j+2} + \cdots + \nu^{p-1}) \otimes \nu^j - \\ &\quad - \nu^{j-1} \otimes (1 + \nu + \cdots + \nu^{j-3} + \nu^{j+1} + \cdots + \nu^{p-1})). \end{aligned} \tag{*}$$

It remains to show $(1 + \nu + \cdots + \nu^{j-2} + \nu^{j+2} + \cdots + \nu^{p-1}) \otimes \nu^j$ and $-\nu^{j-1} \otimes (1 + \nu + \cdots + \nu^{j-3} + \nu^{j+1} + \cdots + \nu^{p-1})$ both belong in V'. This is left to the reader. □

Corollary 4.12.

- $\natural(\nu^2 \otimes \nu^2) - \natural(1 \otimes 1) = \natural(1 \otimes \nu^{p-1}) + \natural(\nu \otimes 1) - \natural(\nu^2 \otimes \nu) - \natural(\nu^3 \otimes \nu^2)$;
- For $3 \leq j \leq p-4$, $\natural(\nu^{j+1} \otimes \nu^{j+1}) - \natural(\nu^j \otimes \nu^j) = \natural(\nu^j \otimes \nu^{j-1}) - \natural(\nu^{j+2} \otimes \nu^{j+1})$;
- $\natural(\nu^{p-1} \otimes \nu^{p-1}) - \natural(\nu^{p-2} \otimes \nu^{p-2}) = \natural(\nu^{p-2} \otimes \nu^{p-3}) - \natural(1 \otimes \nu^{p-1})$.

Proof of Proposition 4.10. Clearly $\Omega = span_{\mathbb{Z}}\{\natural(\nu^i \otimes \nu^i) \mid 0 \le i \le p-1\} \subset Ker(\psi)$. For the converse, suppose $\nu \in Ker(\psi)$ and write $\nu = \sum_{r=0}^{p-1} \alpha_r \natural(\nu^{r+1} \otimes \nu^r) + \alpha_T \natural(T)$, where α_r, $\alpha_T \in \mathbb{Z}$. By the results of Proposition 4.11, we can rewrite $\natural(T)$ as $\natural(T) = \natural(1 \otimes 1) - \sum_{i=1}^{p-2} \natural(\nu^{i+1} \otimes \nu^i)$. We can, therefore, rewrite ν as

$$\nu = \sum_{r=0}^{p-1} \beta_r \natural(\nu^{r+1} \otimes \nu^r) + \alpha_T \natural(1 \otimes 1),$$

where $\beta_0 = \alpha_0, \beta_r = \alpha_r - \alpha_T$ for $1 \le r \le p-2$ and $\beta_{p-1} = a_{p-1}$. Thus, $\nu \in \Omega$ if and only if $\nu' = \sum_{r=0}^{p-1} \beta_r \natural(\nu^{r+1} \otimes \nu^r) \in \Omega$.

Next, since $\psi(\natural(\nu^{r+1} \otimes \nu^r)) = 1$ and $\psi(\nu') = 0$, it follows that $\sum_{r=0}^{p-1} \beta_r = 0$. In particular:

$$\begin{aligned}
\nu' &= \sum_{r=0}^{p-1} \beta_r \natural(\nu^{r+1} \otimes \nu^r) - \sum_{r=0}^{p-1} \beta_r \natural(\nu \otimes 1) \\
&= \sum_{r=0}^{p-1} \beta_r \natural(\nu \otimes 1) x^r - \sum_{r=0}^{p-1} \beta_r \natural(\nu \otimes 1) \\
&= \natural(\nu \otimes 1) \left(\sum_{r=0}^{p-1} \beta_r x^r - \sum_{r=0}^{p-1} \beta_r \right) \\
&= \natural(\nu \otimes 1) \left(\sum_{r=0}^{p-1} \beta_r (x^r - 1) \right) \\
&= \natural(\nu \otimes 1)(x-1) \left(\sum_{r=0}^{p-1} \beta_r (x^{r-1} + \cdots + 1) \right) \\
&= \left[\natural(\nu^2 \otimes \nu) - \natural(\nu \otimes 1) \right] \left(\sum_{r=0}^{p-1} \beta_r (x^{r-1} + \cdots + 1) \right).
\end{aligned}$$

It is, therefore, sufficient to show $\natural(\nu^2 \otimes \nu) - \natural(\nu \otimes 1) \in \Omega$. To do so, first note that for $3 \le j \le p-4$, we can rewrite $\natural(\nu^{j+3} \otimes \nu^{j+3}) - \natural(\nu^{j+2} \otimes \nu^{j+2}) + \natural(\nu^{j+1} \otimes \nu^{j+1}) - \natural(\nu^j \otimes \nu^j)$ as

$$\natural(\nu^{j+2} \otimes \nu^{j+1}) - \natural(\nu^{j+4} \otimes \nu^{j+3}) + \natural(\nu^j \otimes \nu^{j-1}) - \natural(\nu^{j+2} \otimes \nu^{j+1}) = \natural(\nu^j \otimes \nu^{j-1}) - \natural(\nu^{j+4} \otimes \nu^{j+3}).$$

We now have two cases to consider:

- $p - 3 \equiv 0 \mod 4$: In this case, it is a straightforward use of Corollary 4.12 to show $\sum_{j=3}^{p-1} (-1)^j \natural(\nu^j \otimes \nu^j) = \natural(\nu^3 \otimes \nu^2) - \natural(1 \otimes \nu^{p-1})$ and so

$$\begin{aligned}
\sum_{j=2}^{p-1} (-1)^j \natural(\nu^j \otimes \nu^j) - \natural(1 \otimes 1) &= \natural(\nu^3 \otimes \nu^2) - \natural(1 \otimes \nu^{p-1}) + \natural(1 \otimes \nu^{p-1}) + \\
&\quad + \natural(\nu \otimes 1) - \natural(\nu^2 \otimes \nu) - \natural(\nu^3 \otimes \nu^2) \\
&= \natural(\nu \otimes 1) - \natural(\nu^2 \otimes \nu).
\end{aligned}$$

- $p - 5 \equiv 0 \mod 4$: This time, $\sum_{j=5}^{p-1} (-1)^j \natural(\nu^j \otimes \nu^j) = \natural(\nu^5 \otimes \nu^4) - \natural(1 \otimes \nu^{p-1})$. The proof follows in the same manner as the previous case. We once again reach the desired conclusion, $\sum_{j=2}^{p-1} (-1)^j \natural(\nu^j \otimes \nu^j) - \natural(1 \otimes 1) = \natural(\nu \otimes 1) - \natural(\nu^2 \otimes \nu)$. □

With $\natural$ as above, we use Proposition 4.6 to construct the following split exact sequence,

$$0 \to \Lambda^{n-1} \to (I_C^*)_- \otimes (I_C^*)_- \to \natural(T + V(1)) \to 0. \tag{4.13}$$

Proposition 4.14. *L, $\natural(T + V(1)) \in Ext^1(\mathbb{Z}_-, L_0)$.*

Proof. Using [6] and a similar argument to that of K, we have the short exact sequence $0 \to L_0 \to L \to \mathbb{Z}_- \to 0$. Thus, by (4.9) and Proposition 4.10 it remains to show

$$\Omega = span_{\mathbb{Z}}\{\natural(\nu^i \otimes \nu^i) \mid 0 \le i \le 2n\} \cong L_0.$$

To do so, denote the bases of Ω and L_0 by $\{e_i \mid 1 \le i \le 2n+1\}$, $\{f_j \mid 1 \le j \le 2n+1\}$, respectively, where $e_i = \natural(\nu^{i-1} \otimes \nu^{i-1})$ and $f_j = (y+1)x^{i-1}$. Clearly, we have an isomorphism as abelian groups. The result now follows as both sets of basis elements are easily shown to be equivariant under the actions of x and y. □

Corollary 4.12. *$L \cong \natural(T + V(1))$.*

Proof. It is sufficient to show L and $\natural(T + V(1))$ belong to the same class of $Ext^1(\mathbb{Z}_-, L_0)$. First, recall $j_*(\mathbb{Z}) = [y+1)$ and observe $j_*(\mathbb{Z}) \cong L_0$. Using Eckmann-Shapiro, $Ext^1(\mathbb{Z}_-, j_*(\mathbb{Z})) \cong Ext^1(\mathbb{Z}_-, \mathbb{Z}) \cong \mathbb{Z}/2$. Since L is indecomposable, it clearly does not belong in the trivial class.

For $\natural(T + V(1))$, we observe this is free as a $\mathbb{Z}[C_2]$-module. Start with the exact sequence $0 \to \Lambda^{n-1} \to (I_C^*)_- \otimes (I_C^*)_- \to \natural(T + V(1)) \to 0$ and apply the exact functor $j^*(-)$,

$$0 \to \mathbb{Z}[C_2]^{(2n+1)(n-1)} \to j^*((I_C^*)_- \otimes (I_C^*)_-) \to j^*(\natural(T + V(1))) \to 0.$$

As $(I_C^*)_- \cong P$, and since $j^*(P)$ is free of rank n (Proposition 3.3), $j^*(\natural(T + V(1)))$ is stably free of rank $n+1$. By the Swan-Jacobinski Theorem, $j^*(\natural(T + V(1)))$ is, therefore, free of rank $n+1$, i.e. $j^*(\natural(T + V(1))) \cong \mathbb{Z}[C_2]^{n+1}$.

Next, we show $j^*(L_0) \cong \mathbb{Z} \oplus \mathbb{Z}[C_2]^n$. Consider the exact sequence $0 \to L_0 \to L \to \mathbb{Z}_- \to 0$ and apply $j^*(-)$. Then $0 \to j^*(L_0) \to \mathbb{Z}[C_2]^{n+1} \to \mathbb{Z}_- \to 0$ is also exact. By comparing this with $0 \to \mathbb{Z} \to \mathbb{Z}[C_2] \to \mathbb{Z}_- \to 0$ we use Schanuel's Lemma to deduce $j^*(L_0) \oplus \mathbb{Z}[C_2] \cong \mathbb{Z} \oplus \mathbb{Z}[C_2]^{n+1}$. The identity now follows from Proposition 29.5 of [4] (p. 122).

We now suppose $\natural(T + V(1))$ is in the trivial class of $Ext^1(\mathbb{Z}_-, L_0)$, i.e. the exact sequence containing $\natural(T + V(1))$ splits. In particular, so too does the restriction to $\mathbb{Z}[C_2]$:

$$0 \to \mathbb{Z} \oplus \mathbb{Z}[C_2]^n \xrightarrow{\iota} \mathbb{Z}[C_2]^{n+1} \to \mathbb{Z}_- \to 0.$$

This exact sequence can be altered so that

$$0 \to \mathbb{Z} \to \mathbb{Z}[C_2]^{n+1}/\iota(\mathbb{Z}[C_2]^n) \to \mathbb{Z}_- \to 0 \tag{4.16}$$

is also exact. By Johnson's 'destabilization theorem' (see [5], p.97), $\mathbb{Z}[C_2]^{n+1}/\iota(\mathbb{Z}[C_2]^n)$ is projective. We can, therefore, construct the following split short exact sequence

$$0 \to \mathbb{Z}[C_2]^n \xrightarrow{\iota} \mathbb{Z}[C_2]^{n+1} \to \mathbb{Z}[C_2]^{n+1}/\iota(\mathbb{Z}[C_2]^n) \to 0.$$

Thus, $\mathbb{Z}[C_2]^{n+1}/\iota(\mathbb{Z}[C_2]^n)$ is stably free, and hence free of rank 1. Replacing this in (4.16), we have $0 \to \mathbb{Z} \to \mathbb{Z}[C_2] \to \mathbb{Z}_- \to 0$. However, this clearly does not split and so $\natural(T + V(1))$ cannot belong in the trivial class of $Ext^1(\mathbb{Z}_-, L_0)$. Hence, both L and $\natural(T + V(1))$ belong to the non-trivial class. It, therefore, follows that they are isomorphic, as required. □

Using Corollary 4.15 we can replace $\natural(T + V(1))$ with L in (4.13). We, therefore, have the following split the short exact sequence, $0 \to \Lambda^{n-1} \to (I_C^*)_- \otimes (I_C^*)_- \to L \to 0$. By recalling $(I_C^*)_- \cong P$, we have, therefore, shown:

Proposition 4.17. *$P \otimes P \cong L \oplus \Lambda^{n-1}$.*

Furthermore, note the following dual statement:

Proposition 4.18. $R \otimes R \cong L \oplus \Lambda^{n-1}$

5. $\mathbf{P \otimes L \sim R}$

The aim of this section will be to construct the following isomorphism:

$$L \otimes P \cong R \oplus \Lambda^n. \tag{5.1}$$

First, in [6] it was shown that:

- P has $\mathbb{Z}$-basis $\{\pi x,\ \pi x^2, \ldots,\ \pi x^{2n}\}, \pi = (x^n - 1)(y-1)$;
- L has $\mathbb{Z}$-basis $\{(y+1),\ (y+1)x, \ldots,\ (y+1)x^{2n},\ \Sigma_x\}, \Sigma_x = 1 + x + \cdots + x^{2n}$;
- R has $\mathbb{Z}$-basis $\{(y-1)(x-1),\ (y-1)(x^2-1), \ldots,\ (y-1)(x^{2n}-1)\}$.

We define L_0 to be the Λ-submodule of L with $\mathbb{Z}$-basis $\{(y+1),\ (y+1)x, \ldots,\ (y+1)x^{2n}\}$. Observe that L/L_0 is of rank 1, generated by the image of Σ_x, upon which x acts trivially and y acts by multiplication by -1. To reflect this, write $L/L_0 \cong \mathbb{Z}_-$ and construct the short exact sequence $0 \to L_0 \to L \to \mathbb{Z}_- \to 0$. Tensoring with P yields,

$$0 \to L_0 \otimes P \to L \otimes P \to \mathbb{Z}_- \otimes P \to 0. \tag{5.2}$$

Proposition 5.3. *$\rho_{\mathbb{Z}_-}(x^{-1}) = 1$, and $\rho_{\mathbb{Z}_-}(y) = -1$.*

Corollary 5.4. *For any module M, $\rho_{\mathbb{Z}_- \otimes M}(x^{-1}) = \rho_M(x^{-1})$ and $\rho_{\mathbb{Z}_- \otimes M}(y) = -\rho_M(y)$.*

Proposition 5.5. *The representation of the x-action on $\mathbb{Z}_- \otimes P$ is given by*

$$(\rho_{\mathbb{Z}_- \otimes P}(x^{-1}))_{ij} = \begin{cases} 1, & j = i-1,\ 2 \le i \le 2n; \\ -1, & j = 2n,\ 1 \le i \le 2n; \\ 0, & o/w. \end{cases}$$

Proof. Write $e_i = \pi x^i$, where $1 \le i \le 2n$. Then $e_i \cdot x = e_{i+1}$ for $1 \le i \le 2n-1$, and $e_{2n} \cdot x = \pi$. Since Σ_x is central we note $\pi \Sigma_x = 0$. It follows that $\pi = -\pi x - \pi x^2 - \cdots - \pi x^{2n} = -\sum_{i=1}^{2n} e_i$. As x acts trivially on $\mathbb{Z}_-$, the result follows. □

It will be convenient to use the following form for $\rho_{\mathbb{Z}_- \otimes P}(x^{-1})$. A proof is left to the reader.

Proposition 5.6.

$$\rho_{\mathbb{Z}_- \otimes P}(x^{-1}) = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$$

where $P_1,\ P_2,\ P_3,\ P_4$ are each $n \times n$ blocks such that

$$(P_1)_{ij} = \begin{cases} 1, & j = i-1,\ 2 \le i \le n; \\ 0, & o/w; \end{cases} \qquad (P_2)_{ij} = \begin{cases} -1, & j = n,\ 1 \le i \le n; \\ 0, & o/w; \end{cases}$$

$$(P_3)_{ij} = \begin{cases} 1, & i = 1,\ j = n; \\ 0, & o/w; \end{cases} \qquad (P_4)_{ij} = \begin{cases} 1, & j = i-1,\ 2 \le i \le n; \\ -1, & j = n,\ 1 \le i \le n; \\ 0, & o/w. \end{cases}$$

Proposition 5.7. *The representation of the y-action of* $\mathbb{Z}_- \otimes P$ *is given by,*

$$(\rho_{\mathbb{Z}_-\otimes P}(y))_{ij} = \begin{cases} 1, & j = 2n+1-i, \ 1 \le i \le 2n; \\ 0, & o/w. \end{cases}$$

Proof. Consider the y-action on a general basis element of P,

$$e_i y = \pi x^i y = (x^n - 1)(y-1)yx^{2n+1-i} = -\pi x^{2n+1-i} = -e_{2n+1-i}$$

By Corollary 5.4 we, therefore, have the desired representation. □

Now repeat the process for the $x, \ y$-actions on R.

Proposition 5.8. *The representation of the x-action on R is given by,*

$$(\rho_R(x^{-1}))_{ij} = \begin{cases} 1, & j = i-1, \ 2 \le i \le 2n; \\ -1, & i = 1, \ 1 \le j \le 2n; \\ 0, & o/w. \end{cases}$$

Proof. Set $f_i = (y-1)(x^i - 1)$, where $1 \le i \le 2n$. Then,

$$f_i \cdot x = (y-1)(x^i - 1)x = (y-1)(x^{i+1} - 1) - (y-1)(x-1) = f_{i+1} - f_1$$

for $1 \le i \le 2n-1$, and $f_{2n} \cdot x = -f_1$. The result now follows. □

As with Proposition 5.6, we can rewrite $\rho_R(x^{-1})$ as follows:

Proposition 5.9.

$$\rho_R(x^{-1}) = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$$

where $R_1, \ R_2, \ R_3, \ R_4$ *are each* $n \times n$ *blocks such that*

$$(R_1)_{ij} = \begin{cases} 1, & j = i-1, \ 2 \le i \le n; \\ -1, & i = 1, \ 1 \le j \le n; \\ 0, & o/w; \end{cases} \qquad (R_2)_{ij} = \begin{cases} -1, & i = 1, \ 1 \le j \le n; \\ 0, & o/w; \end{cases}$$

$$(R_3)_{ij} = \begin{cases} 1, & i = 1, \ j = n; \\ 0, & o/w; \end{cases} \qquad (R_4)_{ij} = \begin{cases} 1, & j = i-1, \ 2 \le i \le n; \\ 0, & o/w. \end{cases}$$

Proposition 5.10. *The representation of the y-action on R is given by,*

$$\rho_R(y) = \begin{cases} -1, & j = 2n+1-i; \\ 0, & o/w. \end{cases}$$

In particular, $\rho_R(y) = -\rho_{\mathbb{Z}_-\otimes P}(y)$.

Proof. As before, apply y to a general f_i to yield,

$$f_i y = (y-1)(x^i - 1)y = (y-1)y(x^{2n+1-i} - 1) = -f_{2n+1-i}.$$

□

To show the equivalence of $\mathbb{Z}_- \otimes P$ and R, we define the following $(2n) \times (2n)$ matrix X,

$$X = \begin{pmatrix} 0 & -\alpha^T \\ \alpha & 0 \end{pmatrix}, \qquad \text{where} \qquad \alpha_{ij} = \begin{cases} 1, & j \geq i; \\ 0, & j < i. \end{cases} \tag{5.11}$$

Proposition 5.12. *With X as defined in (5.11),* $\rho_{\mathbb{Z}_- \otimes P}(x^{-1})X = X\rho_R(x^{-1})$.

Proof. Using Propositions 5.6 and 5.9,

$$\rho_{\mathbb{Z}_- \otimes P}(x^{-1})X = \begin{pmatrix} P_2\alpha & -P_1\alpha^T \\ P_4\alpha & -P_3\alpha^T \end{pmatrix}, \text{ and } X\rho_R(x^{-1}) = \begin{pmatrix} -\alpha^T R_3 & -\alpha^T R_4 \\ \alpha R_1 & \alpha R_2 \end{pmatrix}.$$

We, therefore, have four calculations to check. First we show $P_2\alpha = -\alpha^T R_3$. Now, $(P_2\alpha)_{ik} = \sum_{j=1}^n (P_2)_{ij}\alpha_{jk} \neq 0$ only if $k \geq j$ and $j = n, 1 \leq i \leq n$. In other words, when $k = n$ and $1 \leq i \leq n$ then $(P_2\alpha)_{i,n} = -1$. For any other entry we get 0. Now, $(\alpha^T R_3)_{ik} = \sum_{j=1}^n \alpha_{ji}(R_3)_{jk} \neq 0$ when $i \geq j$ and $j = 1$, $k = n$. So, when $k = n$ and $i \geq 1$ $(\alpha^T R_3)_{i,n} = 1$ and zero otherwise. Thus, $P_2\alpha = -\alpha^T R_3$ as required.

Next, we show $P_1\alpha^T = \alpha^T R_4$ by first observing $(P_1\alpha^T)_{ik} = \sum_{j=1}^n (P_1)_{ij}\alpha_{kj} \neq 0$ when $j \geq k, j = i-1$ and $2 \leq i \leq n$. Putting this together, we see that for $i > k$ where $2 \leq i \leq n$ and $1 \leq k \leq n-1$ we have $(P_1\alpha^T)_{ik} = 1$, and zero otherwise. Likewise, $(\alpha^T R_4)_{ik} = \sum_{j=1}^n \alpha_{ji}(R_4)_{jk} \neq 0$ when $i \geq j, k = j-1$ and $2 \leq j \leq n$. So when $i > k$, $1 \leq k \leq n-1$ and $2 \leq i \leq n$ we have $(\alpha^T R_4)_{ik} = 1$, and zero otherwise.

For $P_4\alpha = \alpha R_1$ note $(P_4\alpha)_{ik} = \sum_{j=1}^n (P_4)_{ij}\alpha_{jk} \neq 0$ when either:

- $k \geq j, j = i-1$ and $2 \leq i \leq n$, in which case we have $+1$; or when
- $k \geq j, j = n$ and $1 \leq i \leq n$ when we have -1.

Putting the above two cases together we find that when $n-1 \geq k \geq i-1$ and $2 \leq i \leq n$, then $(P_4\alpha)_{ik} = 1$. However, if $k = n$, then $(P_4\alpha)_{i,n} \neq 0$ only when $i = 1$. This is due to a cancelation occurring from the $+1$ and -1 whenever $i \geq 2$. Putting this together, we have

$$(P_4\alpha)_{ik} = \begin{cases} 1, & i-1 \leq k \leq n-1, \; 2 \leq i \leq n; \\ -1, & i = 1, \; k = n; \\ 0, & o/w. \end{cases}$$

Adopting a similar approach shows $(\alpha R_1)_{ik} = \sum_{j=1}^n \alpha_{ij}(R_1)_{jk} \neq 0$ when either:

- $j \geq i, k = j-1$ and $2 \leq j \leq n$, in which case we have $+1$; or when
- $j \geq i, j = 1$ and $1 \leq k \leq n$ when we have -1.

By putting this together we once more get a cancelation between the $+1$, -1 when $i = 1$ and $k < n$, and so for $i = 1$, $k = n$ we get a value of -1. When $2 \leq i \leq n$ and $i-1 \leq k \leq n-1$, we get a value of 1. Thus, we conclude $P_4\alpha = \alpha R_1$.

Finally, we show $-P_3\alpha^T = \alpha R_2$. First, $(-P_3\alpha^T)_{ik} = \sum_{j=1}^n (-P_3)_{ij}\alpha_{kj} \neq 0$ only if $j \geq k, i = 1$ and $j = n$. In other words, we get a value of -1 when $i = 1$, and zero otherwise. Likewise, $(\alpha R_2)_{ik} = \sum_{j=1}^n \alpha_{ij}(R_2)_{jk} \neq 0$ only when $j \geq i, j = 1$ and $1 \leq k \leq n$, i.e. when $i = 1$. In this case we get -1, and zero otherwise. □

Proposition 5.13. *With X as defined in (5.11),* $\rho_{\mathbb{Z}_- \otimes P}(y)X = X\rho_R(y)$.

Proof. It is clear that $\rho_{\mathbb{Z}_-\otimes P}(y)$ may be written in the form

$$\rho_{\mathbb{Z}_-\otimes P}(y) = \begin{pmatrix} 0 & P_0 \\ P_0 & 0 \end{pmatrix}$$

where P_0 is an $n \times n$ block such that

$$(P_0)_{ij} = \begin{cases} 1, & j = n+1-i, \ 1 \le i \le n; \\ 0, & o/w. \end{cases}$$

It is then also clear that

$$\rho_R(y) = \begin{pmatrix} 0 & -P_0 \\ -P_0 & 0 \end{pmatrix}.$$

As such,

$$\rho_{\mathbb{Z}_-\otimes P}(y)X = \begin{pmatrix} P_0\alpha & 0 \\ 0 & -P_0\alpha^T \end{pmatrix}, \text{ and } \ X\rho_R(y) = \begin{pmatrix} \alpha^T P_0 & 0 \\ 0 & -\alpha P_0 \end{pmatrix}.$$

It remains to show $\alpha P_0 = P_0\alpha^T$ and $\alpha^T P_0 = P_0\alpha$. Observe $(\alpha P_0)_{ik} = \sum_{j=1}^n \alpha_{ij}(P_0)_{jk}$ which is non-zero when $j \ge i$ and $k = n+1-j$; that is, when $k \le n+1-i$. In this case we get a value of $+1$, and zero otherwise. Likewise, $(P_0\alpha^T)_{ik} = \sum_{j=1}^n (P_0)_{ij}\alpha_{kj} \ne 0$ when $j = n+1-i$ and $j \ge k$, i.e. when $n+1-i \ge k$. Thus, $\alpha P_0 = P_0\alpha^T$. A similar proof shows $\alpha^T P_0 = P_0\alpha$. □

Proposition 5.14. $\mathbb{Z}_- \otimes P \cong R$.

Proof. By Propositions 5.12 and 5.13, it remains to show X is invertible over $\mathbb{Z}$. Define the $n \times n$ matrix β by

$$\beta_{ij} = \begin{cases} 1, & i = j; \\ -1, & j = i+1; \\ 0, & o/w. \end{cases}$$

Observe $(\alpha\beta)_{ik} = \sum_{j=1}^n \alpha_{ij}\beta_{jk} = \sum_{j\ge i}\beta_{jk}$. Suppose now that $i = k$, then $(\alpha\beta)_{ii} = 1$ since $\beta_{ji} = 0$ for any $j \ne i$ in the range $i \le j \le n$. If $i \ne k$ then there are two cases to consider. If $k < i$ then $(\alpha\beta)_{ik} = 0$ since $\beta_{jk} = 0$ for any $k < i \le j \le n$. If $k > i$, then $(\alpha\beta)_{ik} = \beta_{k-1,k} + \beta_{kk} = -1+1 = 0$. Thus, $\alpha\beta = I_n$. A similar argument now shows $\beta\alpha = I_n$, i.e. $\alpha^{-1} = \beta$. Finally, we define the matrix

$$Y = \begin{pmatrix} 0 & \beta \\ -\beta^T & 0 \end{pmatrix},$$

and a straightforward calculation shows $XY = I_{2n} = YX$, i.e. X is invertible over $\mathbb{Z}$. □

Proposition 5.15. $L_0 \otimes P \cong \Lambda^n$.

Proof. Consider $j_*(\mathbb{Z}) \cong L_0$. By Proposition 3.3, $j^*(P) \cong \mathbb{Z}[C_2]^n$. So, by using Frobenius reciprocity we have $L_0 \otimes P \cong j_*(\mathbb{Z}) \otimes P \cong j_*(\mathbb{Z} \otimes \mathbb{Z}[C_2]^n) \cong j_*(\mathbb{Z}[C_2]^n) \cong \Lambda^n$. □

From Propositions 5.14 and 5.15 we can now rewrite (5.2) as, $0 \to \Lambda^n \to L \otimes P \to R \to 0$. Dualizing yields a split short exact sequence and so $(L \otimes P)^* \cong P \oplus \Lambda^n$. By dualizing once again, and since $M^{**} \cong M$ for any Λ-lattice M, we reach the desired conclusion; that is,

Proposition 5.16. $L \otimes P \cong R \oplus \Lambda^n$.

6. $P \otimes R \sim K$

To conclude the proof of Theorem 2, we construct the isomorphism,

$$R \otimes P \cong K \oplus \Lambda^{n-1}. \tag{6.1}$$

Proposition 6.2. $R \otimes P \cong (\mathbb{Z}_- \otimes L) \oplus \Lambda^{n-1}$.

Proof. In the previous section we showed $R \cong \mathbb{Z}_- \otimes P$. Using (4.1), we can show $R \otimes P \cong (\mathbb{Z}_- \otimes P) \otimes P \cong \mathbb{Z}_- \otimes (L \oplus \Lambda^{n-1}) \cong (\mathbb{Z}_- \otimes L) \oplus \Lambda^{n-1}$. □

Proposition 6.3. *Define the* $(2n+1) \times (2n+1)$ *matrix* Ψ *as in Proposition 4.2. Then* $\rho_{\mathbb{Z}_- \otimes L}(x^{-1}) = \begin{pmatrix} \Psi & 0_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & 1 \end{pmatrix}$.

Proof. Set $E_i = (y+1)x^{i-1}$ *for* $1 \leq i \leq 2n+1$, and $E_{2n+2} = \Sigma_x$. Clearly, $E_i \cdot x = E_{i+1}$ *for* $1 \leq i \leq 2n$and $E_{2n+1} \cdot x = E_1$. Finally, $E_{2n+2} \cdot x = E_{2n+2}$. The result now follows since x acts trivially on $\mathbb{Z}_-$, i.e. $\rho_{\mathbb{Z}_-}(x^{-1}) = 1$. □

Proposition 6.4. *Define the* $(2n+1) \times (2n+1)$ *matrix* Φ *as in Proposition 4.3. Then* $\rho_{\mathbb{Z}_- \otimes L}(y) = \begin{pmatrix} -\Phi & -1_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & 1 \end{pmatrix}$.

Proof. With E_i as above, consider how y acts on the basis elements of L. First, we observe that $E_1 + E_2 + \cdots + E_{2n+1} = \Sigma_x y + E_{2n+2}$. Now, $E_1 y = E_1$ and for a general E_i where $2 \leq i \leq 2n+1$,

$$E_i y = (y+1)x^{i-1}y = (y+1)yx^{2n+2-i} = E_{2n+3-i}$$
$$E_{2n+2}y = y + xy + \cdots + x^{2n}y = E_1 + E_2 + \cdots + E_{2n+1} - E_{2n+2}.$$

Finally, apply Corollary 5.4. □

Next, we do the same for K. Similar arguments show the following:

Proposition 6.5. *With* Ψ *as above,* $\rho_K(x^{-1}) = \begin{pmatrix} \Psi & 0_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & 1 \end{pmatrix}$.

Proposition 6.6. *With* Φ *as above,* $\rho_K(y) = \begin{pmatrix} -\Phi & 1_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & 1 \end{pmatrix}$.

Proposition 6.7. $\mathbb{Z}_- \otimes L \cong K$.

Proof. Define

$$X_{ij} = \begin{cases} 1, & i = j,\ 1 \leq i,\ j \leq 2n+1; \\ -1, & i = j = 2n+2; \\ 0, & o/w. \end{cases}$$

We claim $X\rho_K(g) = \rho_{\mathbb{Z}_- \otimes L}(g)X$ for all $g \in D_{4n+2}$. It is straightforward check that

$$\rho_{\mathbb{Z}_- \otimes L}(x^{-1})X = \begin{pmatrix} \Psi & 0_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & -1 \end{pmatrix} = X\rho_K(x^{-1}).$$

Likewise,

$$\rho_{\mathbb{Z}_-\otimes L}(y)X = \begin{pmatrix} -\Phi & 1_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & -1 \end{pmatrix} = X\rho_K(y).$$

Since X is clearly invertible, this completes the proof. □

Propositions 6.2 and 6.7, therefore, imply the following:

Proposition 6.8. $R \otimes P \cong K \oplus \Lambda^{n-1}$.

Evidently, by combining the main results of Sections 4–6, we have proven Theorem 2. We hope to explore the extent to which this result may be generalized to other metacyclic groups of order pq $(q|p-1)$ in an upcoming paper.

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References

[1] Carlson, J. (1996). *Modules and Group Algebras*. Basel: Birkhauser Verlag.
[2] Evans, J. D. P. (2018). Group algebras of metacyclic type. Ph.D. dissertation. London: University College London.
[3] Jacobinski, H. (1968). Genera and decomposition of lattices over orders. *Acta Math*. 121:1–29. DOI: 10.1007/BF02391907.
[4] Johnson, F. E. A. (2003). *Stable Modules and the D(2)-Problem*. Cambridge: Cambridge University Press.
[5] Johnson, F. E. A. (2011). *Syzygies and Homotopy Theory*, Vol. 17. London: Springer-Verlag.
[6] Johnson, F. E. A. (2016). Syzygies and diagonal resolutions for dihedral groups. *Commun. Algebra* 44(5): 2034–2047. DOI: 10.1080/00927872.2015.1027372.
[7] Rim, D. S. (1959). Modules over finite groups. *Ann. Math*. 69(3):700–712. DOI: 10.2307/1970033.
[8] Swan, R. G. (1962). Projective modules over group rings and maximal orders. *Ann. Math*. 76(1):55–61. DOI: 10.2307/1970264.
[9] Swan, R. G. (1965). Minimal resolutions for finite groups. *Topology*. 4(2):193–208. DOI: 10.1016/0040-9383(65)90064-9.
[10] Swan, R. G. (1983). Projective modules over binary polyhedral groups. *J. Reine Angew. Math*. 342:66–172.