British Review of Economic Issues, Volume 17, Number 43, October 1995

Chaotic Behaviour in a Bank Account

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ABSTRACT

This paper presents a very simple one variable model, apparently not previously studied, of a bank account with no random elements yet which displays chaos. Interest is periodically added to the account and when the balance exceeds a pre-set limit then a fixed amount is removed into another account. The owner of the account is not required to make any additional deposits or withdrawals, nor is the rate of interest required to change for chaotic behaviour to be observed in the balance in the account. In the process of investigating this model we aim to introduce ideas from chaos theory to a wider audience. No previous knowledge of chaos theory or dynamical systems is assumed and all technical terms used from these areas are explained, these include: strange attractor, Lyapunov exponent, ergodicity, mixing, dense set, invariant set, measure zero, countable and uncountable infinities. The three defining properties of a chaotic system are presented and are shown to be possessed by the model.

1 INTRODUCTION

Chaos theory is part of the broader field of dynamics. In dynamics one studies the behaviour of a system whose state changes over time; the system can be anything from a lifeless pendulum to the heartbeat of a person receiving their first kiss, it can be the Earth's climate or the motion of a comet , it can be the fluctuations on the stock exchange or the spread of disease amongst a population. Hence people working in dynamics have no shortage of material and can cross subject boundaries whenever it suits them. In the natural sciences there has been a tremendous growth in chaos-related research; one bibliography references 7000 papers and books on the subject (Zhang Shu-yu 1991). Robert May's review paper in Nature (1976) appears to have been very influential in drawing researchers into the subject. May is a population biologist and was able to show how a simple quadratic model for the size of a biological population could display very complicated behaviour.

Turning now to economics, here there have been two main approaches: set up a mathematical model and then show it is chaotic (see for example De Grauwe and Vansanten (1990)) or, study a set of observations taken over time (a time series) and then try to show that the data come from a chaotic system. The latter type of investigation was widely applied to financial and macroeconomic data after the stock market crash of October 1987. One attraction for economists is that irregular behaviour need not be represented by adding random shocks into a model, instead it may be inherent in a model which is deterministic i.e. does not include any random inputs. (There is now a rather technical textbook applying chaos theory to economics by Medio 1992; an extensive list of references can be found in Parker and Stacey (1994), see also the collection of papers in Anderson et al 1988, and Baumol and Benhabib 1989). The impact of chaos theory in the area of business does not appear to have been very great so far, this may be due to the lack of a relevant and sufficiently simple model for people to comprehend. Nevertheless there have been some applications, ranging from Stacey's (1991) broad framework view of business organizations as feedback systems, to the more technical such as the unpredictability of the value of the internal rate of return (of a series of cash flows) provided by numerical methods (Osborne 1990).

2 THE MODEL

We shall be looking at the balance in a bank account at regular intervals of time (t = 0,1,2,3,etc.). Let the initial amount placed in the account be x_0, the owner of the account will not deposit any more money into the account - it is not necessary to include such a complication to observe complicated behaviour. The balance in the account at time t is represented as x_t. In dynamics this is referred to as the state of the system at time t, and the set of all possible values of this variable is called the state space. The account gains interest periodically at a rate i per period (this is represented as a decimal e.g. 11% is represented as 0.11). The interest is allowed to stay in the account. Notice that the state of our system only changes once per period, hence we are dealing with a discrete (time) system and not a continuous one. The interest rate is kept fixed as we watch the system evolve over time; once again this keeps things simple and emphasises the fact that chaos does not need a complicated system to appear. We will of course investigate the behaviour for different rates of interest i.e. the interest rate is a parameter.

We only need a one-dimensional mathematical model for this system because

only one quantity is varying:

$$x_{t+1} = (1+i)\, x_t$$

If we know the balance at time t then we can use this equation to calculate it one period later and then keep re-applying ('iterating') the equation to find the balance for any time thereafter. This is an example of what is called a recurrence relation or difference equation. Ours is a linear model because if we plotted a graph of x_{t+1} against x_t we would get a straight line whose slope is $1+i$. Unfortunately linear models cannot produce chaos so we need to introduce one more feature. Suppose that the account owner does not wish his balance to rise above some threshold amount and that to prevent this happening funds are sometimes transferred out of the account. Bank customers often use such a 'sweep' facility to transfer money out of a low interest current account into a high interest savings account; it is also known as a (conditional) standing order. For convenience we shall measure the account balance in terms of this threshold or maximum amount to which we give a value of 1. This means that we are measuring x_t as a proportion of the maximum balance. For example if the maximum is £100 then a value of x_t = 0.7564 represents 0.7564*£100 = £75.64 in the account. Hence our state space is the set of all points or values from 0 to 1, because x_t can only lie in this range. We shall assume that the bank does not deliberately round off values e.g. to the nearest penny, at each time period, so that errors do not accumulate by this means. This apparently innocuous assumption will be discussed further in a later section.

The precise rule for our sweep is that if, when the interest is added in the balance exceeds 1, then a fixed amount s is swept away into some other account. We shall treat s as another parameter to see how it affects the evolution of our system over time. Note that s is measured in the same units as x_t.

Our model is therefore:

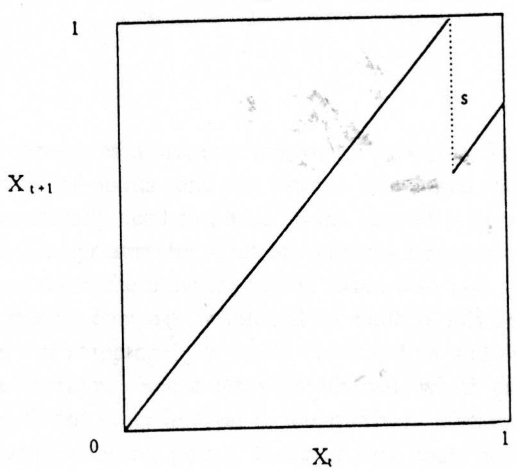
$$x_{t+1} = (1+i)\, x_t \qquad \text{if } (1+i)x_t \text{ does not exceed } 1$$
$$x_{t+1} = (1+i)\, x_t - s \qquad \text{if } (1+i)x_t \text{ exceeds } 1$$

When choosing the sweep quantity s we need to ensure that it is not exceeded by the amount accumulated as interest (ix_t) in any one period, otherwise it will not be fulfilling its function. The largest accumulation of interest occurs when the balance is at its maximum value of $x_t = 1$, so the condition we need is:

$$s \geq i$$

The graph is shown in figure 1. As it is made up of linear pieces it is an example of what is called a piecewise-linear mapping (or map). The map is said to be discontinuous because of the gap between the two segments. (The vertical dashed line is to aid the eye and does not form part of the mapping.) It is a model of a 'discrete time system', i.e. changes do not occur continuously over time but suddenly at specific points in time. Increasing the interest rate makes the slope (of both parallel segments) increase. Varying the sweep quantity affects the vertical drop between the segments. The condition $s \geq i$ forces the second segment to intersect the box (the unit square) on the right hand side rather than at the top of the square. For every value of x_t there is a unique value of x_{t+1} so the future will be uniquely determined. However, if we choose a relatively large value of x_{t+1} (greater than 1-s) the graph shows that there may be two values of x_t that could have given rise to it and we cannot tell which of the two actually did so. This implies that we cannot work out the previous history prior to any data we are given, even though we have a deterministic system.

Figure 1



Consider making an initial deposit which grows over time until it exceeds the threshold. After the first sweep the balance must be at least 1-s; this is clear

from the lowest value of x_{t+1} on the second graph segment. The balance can never again fall below this amount, so all future behaviour will occur in the range from 1-s to 1. This set of points is called the 'attractor' in dynamics because points outside this set get dragged into it and can never leave. The initial behaviour in the range 0 to 1-s is referred to as being 'transient'. In our case the transient behaviour is the well known geometric growth associated with compound interest. What is more interesting is what happens in the long term i.e. the post-transient behaviour or 'motion on the attractor'. Below we shall be considering the case when there is any form of repeating (periodic) pattern and then the case when there is no pattern (chaotic). First though we can make some very important deductions simply by considering the slope of our mapping.

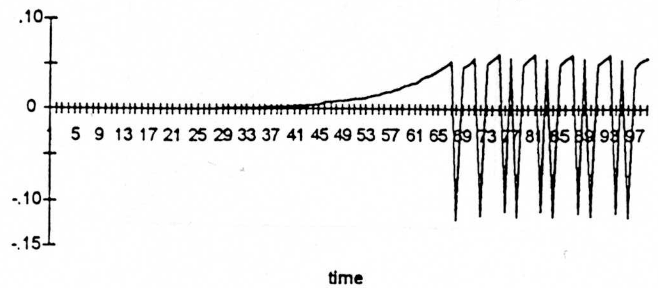
3 THE FUTURE IS SENSITIVELY DEPENDENT ON THE PRESENT

The slope or gradient of our mapping is everywhere the same, it is $1+i$, so for any positive interest rate the slope is greater than 1. Consider two very close initial points on the horizontal axis (i.e. two accounts with similar but not identical balances), suppose the distance between them is d, this is the difference in the accounts. By going up to the plotted graph and horizontally to the vertical axis we shall see that after one time step the distance between them has grown; it grows by a factor $(1+i)$ to become $(1+i)d$. After two steps it will be $(1+i)^2d$ and after t steps $(1+i)^td$. Thus nearby points diverge not by a constant or fixed amount at each step (this would correspond to simple interest), but geometrically - this is a much faster type of growth. We have thus demonstrated one of the hallmarks of chaotic systems: sensitive dependence on initial conditions. In dynamics the rate of divergence is expressed in terms of the exponential function so we write $(1+i)^t = e^{\lambda t}$. The exponent (or power) λ is called the Lyapunov exponent, and must be positive for sensitive dependence on initial conditions to be present. In our case $(1+i) = e^{\lambda}$ so $\lambda = \ln(1+i)$ which is always positive.

We have just demonstrated *local* exponential divergence of nearby points i.e. the divergence in the first few steps. We have yet to throw in the complication of the sweeping mechanism - this will make the time series for our two accounts even more dissimilar. Our two initially similar accounts may start by differing only by a penny, this difference will of course grow because that penny will accumulate interest, but there will eventually come a time when one account reaches the threshold and is swept down whilst the other does not, and this will make the two accounts very different. Such sensitivity is not uncommon; consider an old-fashioned pinball machine with pins nailed into an inclined board - as a ball rolls down and hits a pin it is forced to pass by on one side, but if it had approached at a slightly different angle then it would pass on the other side. Now think of an asteroid hurtling toward our planet, a slight difference in its angle of approach can make the difference between a near-miss and a collision. Human behaviour too is filled with small incidents which lead to greater things.

Figure 2

DIFFERENCE BETWEEN 2 INITIALLY CLOSE POINTS (initial difference less than 1 part in 8000)



(Parameter values: i = 0.10, s = 0.18)

Our system also possesses a property known as 'mixing'. This requires that, given two intervals of points, one can always find initial values in the first interval which eventually lead to points in the second interval. This can be shown to be true for our attractor - but obviously we cannot ever return from the attractor to points in the transient region (from 0 to 1-s). To demonstrate the presence of mixing consider an interval of width d, this gets magnified at each iteration of our mapping by a factor (1+i) and so will eventually cover the whole of the attractor. Hence from our interval, which can be arbitrarily small, we can reach any other interval in the attractor. This means that inside any range of possible starting points, however tiny, there exist points which will eventually explore the whole of the attractor in the sense that they can get arbitrarily close to any given point; such a sequence of points is called a dense orbit.

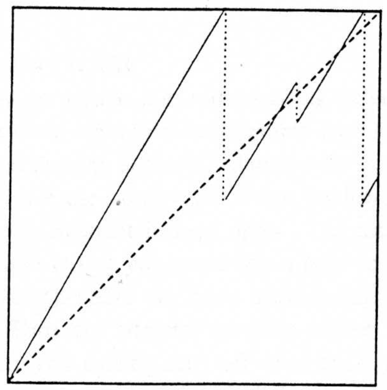
4 PERIODIC BEHAVIOUR

In what follows we will make use of the 45 degree (diagonal) line which joins the origin to the opposite corner of the unit square. If we choose our parameters (i and s) so that this line crosses the plotted lines on the graph then the point of intersection would correspond to what is called a fixed point or equilibrium point. At such a point we have that $x_{t+1} = x_t$ so that the balance in the account stays the same from one time to the next and so never changes. This occurs when the amount gained as interest (ix_t) is exactly equal to the amount swept away (s). Setting these equal to each other shows that the starting balance for this to occur is $xt = s/i$, however since we have ruled that $s \geq i$, the only equilibrium point we have which does not exceed 1 is $x = 1$ (which occurs when the interest gained is precisely equal to the amount swept away).

Let us now plot the relationship which shows us where each starting value will lead to in two time steps i.e. we wish to plot x_{t+2} versus x_t. On a computer spreadsheet this can be done by taking a large number of equally spaced starting values between 0 and 1 and calculating where they will be in one time step and then using these latter values in the mapping again to find where they will be in one further step. The graph now shows three or four segments, each with slope $(1+i)^2$. What do these segments correspond to? Moving from left to right, the first corresponds to those initial points (balances) which do not grow sufficiently to experience a sweep in two steps; then come the points which grow in the first step and then get swept down in the second step; thirdly we have values which are large enough to experience a sweep in the first step but not in the second; and finally there are the points which are swept down in both steps (this latter segment will not be present if s is much greater than i because if the drop in value is very great the balance will not then grow sufficiently to have to be reduced again).

We can choose our parameter values so that there are intersections with the diagonal; there can only be two such points because the first and last segments can never cross the diagonal. The intersection points satisfy the condition $x_{t+2} = x_t$, this means that we have a cycle of period 2 i.e. it repeats every two time steps. The account balance will oscillate regularly between the two values corresponding to the points of intersection. For example if $i = 0.1$ and $x_t = 0.9$ then $x_{t+1} = 0.99$, multiplying this by (1+i) gives 1.089, so if we choose $s = 0.189$ we will be returned back to 0.9 to repeat the cycle. There are points that lead in to this repeating sequence and get stuck there (e.g. 0.67618, 0.7438, 0.8181...). These are 'pre-periodic' points in the transient region leading into a limit-cycle. The other possibility is that there are no period-2 cycles, this occurs if $s >> i$, so that after a large first drop the balance does not have a chance to rise up above the diagonal.

Figure 3



If next we proceeded to construct the graph relating values at time t to values three time steps later we would find that the maximum number of segments doubles again, so that we may have up to eight of them. Thus the number of intersections with the diagonal may exceed three. This means that several different period-3 cycles may be present.

Is there any limit to the length of cycles? Can you have cycles which repeat every 100 or 1000 steps? The answer is that you can have cycles of any length whatsoever! To see how this can be, imagine a low starting balance (near $x = 0$) and a very low interest rate so that it takes a long time to approach the threshold ($x = 1$), then choose the sweep amount so that it returns you to precisely the point you started at; you will then of course find that all points that were visited before will be revisited in the same order ad infinitum. Now

if there are cycles of any integer length then there must be an infinite number of them because the number of integers is infinite. Hence there must be an infinity of periodic points in the range 0 to 1. Obviously, longer cycles involve more points to be visited and so increasing the cycle length will cause the points to be more tightly squeezed together in the range 0 to 1. For any arbitrarily small sub-interval we can always find a periodic point within it (imagine doubling the cycle length repeatedly until this occurs); the technical expression for this property is that the periodic points form a 'dense set'. Does this mean that the proportion or 'measure' of all possible starting points that are periodic is 1? The surprise is that not only is the answer 'no', but that, horror of horrors, the proportion is actually zero! This means that the probability of randomly choosing a starting point which is periodic is zero. How can this be so? The answer lies in the fact that some infinities are larger than others. Let's leave aside our bank account and allow ourselves to think abstractly for a while to demonstrate this.

Consider the set of all numbers in the interval from 0 to n, and compare this with the set of all the integers from 0 to n, obviously the first set is larger because it includes whole *intervals* of points between any two adjacent integers, therefore it is infinitely larger and the integers are of measure (proportion) zero. (The strict definition says that a set of points has 'measure zero' if it can be covered by a set of intervals whose total length is arbitrarily small. Clearly this cannot be done for the continuous intervals between integers because they have a measurable width.) This argument remains true for any value of n, so we can let n be infinite. We then conclude that the set of all positive integers , although infinite, is as nothing compared to the set of all positive numbers. To distinguish between these two types of infinity, mathematicians refer to the smaller one as being 'countable' and the larger one as being 'uncountable'. The integers are countable in the sense that one can at least embark on the process of counting them up even though it would take forever. (Any infinite set where there is a 1-to-1 association with all the integers is said to be a 'countable' infinity.) But when we move from isolated points to intervals or continuous ranges of numbers then we do not even know how we would start counting all the numbers they contain. Any countable set will have measure zero; to show this cover the first point by an interval of arbitrarily small width w, the second by one of width w/2, the third by w/4 (we are halving the width each time). The total width is 2w and so is arbitrarily small and so the set has measure zero. One further result we shall use is that any set which can be split into a countable number of subsets , each of which is countable, must itself be a countable set. We now apply these results from number theory to our cycles: we have cycles of length n for every integer n, hence we have a countable number of cycle lengths. For a given length of cycle there are a given number (countable) of points which can form part of such a cycle (these were the points of intersection with the diagonal). Therefore the total number of all periodic points is countable and so is of measure zero.

5 CHAOTIC BEHAVIOUR

It follows from the last paragraph that most starting values do not lead to any form of periodic behaviour, therefore most points lead to irregular or chaotic behaviour. This means that the account balance will, in theory, forever jump about and never ever have the same value twice for however many centuries or millennia or aeons the account is kept open. The reason you cannot ever revisit any previous point is that you would then have to repeat the behaviour subsequent to that point (because we have a deterministic system), and this would form a cycle. Thus our attractor consists of two sets of points which always remain distinct: you cannot start off on a cycle (also called periodic orbit/trajectory/motion) and move to chaotic behaviour, neither can you move from chaos and end up being periodic. Sets with this property are said to be invariant, once you are in such a set you remain in it for all time. Our attractor differs from classical steady states (fixed point equilibria, limit cycles) in that it contains cycles of all periods as well as an aperiodic regime, moreover it displays sensitive dependence on initial conditions - for these reasons it is called a strange attractor.

A system is said to be chaotic if it possesses the following three properties:

1. Sensitive dependence on initial conditions

2. Dense set of periodic points

3. Mixing

(Instead of mixing, Devaney (1987) and Barnsley (1993) use the equivalent property of transitivity in their definitions of chaotic systems.)

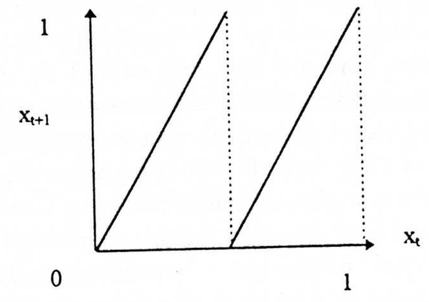
We have demonstrated all three of these properties for our bank model.

6 THE SAWTOOTH MAP

There is a special case of our model that has been previously studied, not as a model of anything real but as an abstract dynamical system used for gaining insight. This is for the parameter values of $i = 1$ and $s = 1$. In our context an interest rate of 100% is unrealistic (except perhaps over a long period of time or in a country experiencing hyper-inflation) and sweeping away the full threshold amount is also unlikely - note however that this would not bring the balance to zero since a sweep only occurs if the threshold is strictly *exceeded*. The graph of this so-called 'sawtooth map' shows two linear segments of equal size. We have:

$$x_{t+1} = 2x_t \quad \text{if } 2x_t \text{ does not exceed } 1,$$
$$x_{t+1} = 2x_t - 1 \quad \text{if } 2x_t \text{ does exceed } 1.$$

Figure 4



Thus we are doubling the value at each iteration and keeping only the fractional part of the answer i.e. any integer part is discarded; mathematicians express this as $x_{t+1} = 2x_t$ modulo 1. The effect of one application of this map is that the interval of points 0 to 0.5 (half the horizontal axis on our graph) is stretched to cover the full interval 0 to 1, and the other half of the interval is treated likewise. So the original interval has been stretched to double its length, cut in the middle and the second piece placed on top of the first. It is because this is akin to rolling and cutting pastry that this is sometimes called a kneading transformation. Interesting properties of this map become apparent if we work in binary. The initial value is then a string of noughts and ones preceded by a decimal point, moreover any such string, finite or infinite, will correspond to some initial value in the range 0 to 1. The application of the map merely shifts the decimal point one place to the right and any whole number part is discarded. This implies that at each time step the most significant digit in the data is being thrown away. On a digital computer a shift of the decimal point can be executed with no computational round-off error. The other name for this map is the Bernoulli shift; a Bernoulli process being one where there are two possible outcomes - in our case corresponding to landing in the left or right half of the interval, and this is determined by whether the next binary digit is a zero or one respectively. In a sense we have the entire future behaviour displayed before us in the sequence of ones and noughts, there is no calculation to be done apart from shifting the decimal point to the right. One thing that this map makes abundantly clear is that any error in specifying the initial value will gradually creep up to haunt you in a big way, e.g. an initial error in the tenth binary digit will, after nine time steps, appear as the most significant digit, placing you in the wrong half of the interval. The effect of finite precision in stating the initial value means that the binary representation will end with a series of zeros and so the motion will inevitably terminate at a fixed point at $x = 0$.

If the initial binary value is made up of a repeating pattern of noughts and ones then we have periodic behaviour e.g. if $x_0 = 2/3$ (= 0.1010101...in binary), then $x_1 = 1/3$ (= 0.0101010...in binary), $x_2 = 2/3$, etc., the repeating sequence of length two in the binary representation manifests itself a cycle of period-2. Of course perfectly repeating patterns are rare and so, as we saw earlier we would not expect to hit on a cycle or periodic sequence by chance. It is fascinating to observe here that the numbers which have perfectly repeating sequences are precisely those which, in everyday parlance, are called fractions (but are technically referred to as rationals) i.e. can be written m/n where m and n are whole numbers. By contrast, irrational numbers have infinitely many digits with no pattern (think of π for example) and they correspond to chaotic or aperiodic evolution. It follows from our earlier argument that irrationals must greatly exceed the rationals, they are uncountable, whereas the rationals are countable; almost all numbers are irrational. In fact we would expect the digits 0 and 1 in our binary representation to appear equally and

randomly. This implies that the series of points representing the evolution of this special form of the system will uniformly, but irregularly, explore the whole of the 0 to 1 range i.e. if we split this range into a number of equal-sized bins then on average the system will spend equal amounts of time in each bin. This is called ergodic behaviour.
Surprisingly, despite the chaos, the simplicity of this mapping allows us to write down a general solution:

$$x_t = 2^t x_0 \quad \text{modulo } 1$$

This gives the balance at any future time t, for any initial value x_0, simply double the value t times and throw away the integer part. It now becomes clear that the chaos actually comes from the data - in the irregular string of digits that make up x_0.

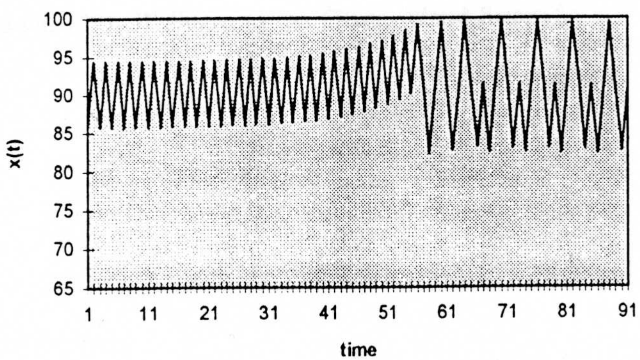
7 THE EFFECT OF ROUND-OFF ERROR

Computers cannot store most numbers exactly, some rounding is incurred because of the limited amount of memory that is set aside for the storage of any number fed into them. For instance the number 0.1 when converted to binary becomes a never-ending sequence of noughts and ones, and so the bank's computer will approximate this by storing only a limited number of these binary digits. It follows from this that only certain numbers can be stored exactly on the computer, these are called the machine numbers. In our case we can think of these as a set of stepping stones between 0 and 1 upon which our bank balance jumps about. Because there is a limited number of these stones we shall eventually jump on one a second time and end up repeating a cycle; therefore we cannot expect to sit at our computer and watch a truly non-repeating sequence. For similar reasons random number generators on computers eventually cycle back to repeat numbers they gave in the past. So by using a digital computer we are forced to observe a type of behaviour which for our model we have demonstrated to be so exceedingly rare that we should not expect to observe it at all! It should be pointed out however that the cycle times may be very long because there are so many machine numbers. In other words when we look at plots of time series we will not immediately be able to distinguish an aperiodic series from one with an extremely long cycle time.

Suppose we do some algebra and deliberately calculate our starting balance so that it is a periodic point. Can the computer be relied upon to keep it in its intended cycle? Once again the surprising answer is 'no'. Any error in specifying the initial value, or any unanticipated round-off errors arising in computing future values will get magnified (compounded). In the jargon of dynamics the periodic points would therefore be classed as unstable - the slightest disturbance and you are thrown off the cycle you thought you were on.

Figure 5

Effect of error in a period-2 cycle



(The balance x_t is expressed as a percentage of the maximum level.)

The fact that the bank is likely to record balances to the nearest penny will have the same effect as having a finite quantity of values which the balance can take on. This too will restrict us to eventual cycling - and much sooner than any cycling implied by the much larger set of machine numbers. For instance suppose the maximum amount permitted in the account is £100. This means that there are only 10 000 different values (to the nearest penny) that the balance can show.

8 CONCLUSION

This paper has explored the behaviour of a model of a simple bank account incorporating a sweep facility. Despite the fact that only one deposit is ever made into the account and that no cheques are drawn and no interest rate changes are assumed, a wealth of possible behaviours are exhibited by the balance in the account over time. Depending on the initial conditions one can demonstrate the existence of cyclic patterns of any given period, however it was shown that periodic behaviour is far from typical (akin to a thrown coin landing on its edge) and that one would almost always expect future changes in the account to show no repeating cycles i.e. irregular or chaotic evolution. The initial conditions which give rise to periodic and irregular behaviour do not fall into nicely separated ranges, instead the periodic points are densely scattered among the infinitely more numerous chaotic points.

It was pointed out that the limited memory of any computer places a restriction on how many different numbers can be represented in any given range. It then follows that the computed balance will eventually fall on a previously visited value and will then be required to repeat from there on. Nevertheless, the quantity of numbers representable on a modern computer is so great that one is unlikely to notice such cycles, because they would be very long indeed. More serious would be rounding to the nearest penny, this would reduce the richness of the dynamics that theory predicts: aperiodic behaviour and very long cycles would not be observed.

REFERENCES

Anderson P., Arrow K., and Pines D., 1988, The Economy as an Evolving Complex System, Addison Wesley.

Barnsley M.F., 1993, Fractals Everywhere, Academic Press.

Baumol W.J., Benhabib J., 1989, Chaos:Significance, Mechanism, and Economic Applications, Journal of Economic Perspectives, 3, 77-105.

De Grauwe P. and Vansanten K., 1990, Deterministic Chaos in the Foreign Exchange Market. CEPR Discussion Paper no. 370.

Devaney R.L., 1987, An Introduction to Chaotic Dynamical Systems, Addison Wesley.

Lorenz H.W., 1989, Non-linear Dynamical Economics and Chaotic Motion, Springer.

May R., 1976, Simple Mathematical Models with very Complicated Dynamics, Nature, 261, 459-467.

Medio A., 1992, Chaotic Dynamics: Theory and Applications to Economics, Cambridge Univ. Press.

Osborne M.J., 1990, Financial Chaos, Management Accounting, Nov. 1990.

Parker D., Stacey R., 1994, Chaos, Management and Economics: The Implications of Nonlinear Thinking, Institute of Economic Affairs, London.

Zhang Shu-yu, 1991, Bibliography on Chaos, World Scientific.