Consistent truncations and G_2-invariant AdS_4 solutions of $D = 11$ supergravity

Bastien Duboeuf,[1,*] Michele Galli,[2,†] Emanuel Malek,[3,‡] and Henning Samtleben[1,4,§]

[1]ENS de Lyon, CNRS, LPENSL, UMR5672, 69342, Lyon cedex 07, France
[2]School of Mathematics and Physics, University of Queensland, St Lucia, Brisbane, Queensland 4072, Australia
[3]Department of Physics, Astronomy and Mathematics, University of Hertfordshire, College Lane, Hatfield, AL10 9AB, United Kingdom
[4]Institut Universitaire de France (IUF), Paris, France

(Received 12 January 2025; accepted 10 February 2025; published 5 March 2025)

Maximal supergravities in ten and eleven dimensions admit consistent truncations on particular spheres to maximal supergravities in lower dimensions. Concurrently, the truncation to singlets under any subgroup of the sphere isometry group leads to consistent truncations with less or no supersymmetry. We review the relation between these truncations in the framework of exceptional field theory. As an application, we derive three new G_2-invariant solutions of $D = 11$ supergravity. Their geometry is of the form $AdS_4 \times \Sigma_7$ where Σ_7 is a deformed seven-sphere, preserving SO(7) isometries.

DOI: 10.1103/PhysRevD.111.066007

I. INTRODUCTION

Generalized geometry and exceptional field theory (ExFT) have proven to be invaluable tools in the construction of consistent truncations of type II and 11-dimensional supergravity [1–3]. In particular, the language of generalized *G*-structures reduces the problem of constructing a truncation *Ansatz*, to that of understanding the generalized intrinsic torsion of a particular generalized *G*-structure [3]. An interesting example are maximally supersymmetric truncations on spheres or products thereof, which arise from the generalized Leibniz parallelisability of spheres S^n [1,2]. These truncations are powerful because they always contain a large number of scalar fields, which is a promising starting point when one is looking for new AdS_d solutions. Specifically, the scalars in maximal supergravity in $D = 11 - d$ dimensions parametrize the target space

$$\mathcal{M}_{\rm scalar} = \frac{E_{d(d)}}{K_d}, \tag{1.1}$$

where K_d denotes the maximal compact subgroup of the exceptional group $E_{d(d)}$. More recently, exceptional field theory has also been adopted as a universal tool to compute the full Kaluza-Klein (KK) spectra around any solution which fits within a maximally supersymmetric truncation [4–6].

The key advantage of using ExFT for the computation of KK masses, is that it allows one to express all fluctuations in terms of only the scalar harmonics on the internal manifold, without having to resort to any tensorial or spinorial harmonics. This is because all the nontrivial tensor structure is encoded in the generalized frame that defines the parallelization. This feature is retained whenever the internal space is generalized parallelisable, i.e. crucially the backgrounds that one considers do not necessarily have to fit within a maximal truncation. This observation was used in [7,8] to compute the spectra around the supersymmetric $AdS_4 \times S^7_{\rm squashed}$ background, which is not contained within maximal $\mathcal{N} = 8$ supergravity.

A complementary approach to consistent truncations which has been exploited since the early years of supergravity [9] makes use of the fact that the truncation of a higher-dimensional theory to all fields invariant under a subgroup K of the isometry group $SO(n+1)$ of the internal space $M_{\rm int} = S^n$ is automatically consistent. This is because the retained K-singlet fields cannot source the truncated nonsinglet fields. Consequently, consistency in general requires to retain all the K-singlet fields. In general, such a truncation will contain an infinite number of fields (including in particular an infinite number of massive spin-2 fluctuations), unless the group K acts transitively

*Contact author: bastien.duboeuf@ens-lyon.fr
†Contact author: m.galli@uq.edu.au
‡Contact author: emanuel.malek@gmail.com
§Contact author: henning.samtleben@ens-lyon.fr

on $M_{\rm int}$ [9]. A consistent truncation to a finite number of fields thus requires that the internal manifold can be represented as a coset space

$$M_{\rm int} = S^n = \frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n)} = \frac{\mathrm{K}}{\mathrm{L}}, \tag{1.2}$$

with L the isotropy subgroup of K. The field content of the associated consistent truncation is given by the L-singlets within the field content of the maximal supergravity. In particular, the scalar target space of the truncation is given by

$$\mathcal{M}_{\rm scalar} = \frac{\mathrm{Com}_{\mathrm{L}}(\mathrm{E}_{d(d)})}{\mathrm{Com}_{\mathrm{L}}(\mathrm{K}_d)}, \tag{1.3}$$

where $\mathrm{Com}_{\mathrm{L}}(G)$ denotes the commutant of L within G. In the notation of [3], L is the reduced structure group of the exceptional generalized geometry and the intrinsic torsion is a constant L-singlet, reflecting the consistency of the truncation. When the action of K is not transitive, the truncation to K-singlets contains an infinite number of fields and the internal space is not of the form (1.2), but rather becomes a foliation of K/L over another space X with the coordinates on X parametrizing the family of singlets kept in the truncation [10].

In general, the consistent truncation to K-singlets around a sphere is not a subtruncation of the maximal supergravity. From the perspective of the maximal supergravity, they contain higher KK modes. However, the coset structure of (1.2) can be combined with the twist matrix of the maximally supersymmetric truncation in order to construct the generalized frame of exceptional field theory. This leads to fairly compact formulas for the resulting Kaluza-Klein mass matrices. In particular, the mass spectrum can still be computed in a convenient basis of scalar harmonics organized under $\mathrm{SO}(n+1)$, even though the actual vacuum breaks this group to the smaller group K. This structure was exploited in [7,8] to compute the full KK spectrum around the squashed seven-sphere [11,12] represented as

$$M_{\rm int} = S^7_{\rm squashed} = \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)}. \tag{1.4}$$

Another well known example is the $\mathcal{N}=4$ truncation of type IIB that comes from viewing S^5 as a Sasaki-Einstein space [13–15]. The SU(2) structure group in this case, corresponds to the SU(2) denominator in the coset $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$.

In this paper, we review the relation between truncations to K-singlets and exceptional field theory and exploit the structure in order to construct new G_2-invariant AdS_4 solutions of $D=11$ supergravity compactified on squashed seven-spheres. Specifically, we consider the truncation of $D=11$ supergravity to singlets under the G_2 subgroup of the SO(8) isometry group of the round S^7. Since G_2 does not act transitively on S^7, the induced consistent truncation contains an infinite number of fields. For the description within exceptional field theory, we represent the seven sphere as a foliation of $S^6 = \mathrm{G}_2/\mathrm{SU}(3)$ over an interval $\mathcal{I}$

$$M_{\rm int} = \Sigma_7 = \mathcal{I} \times \frac{\mathrm{G}_2}{\mathrm{SU}(3)}. \tag{1.5}$$

The resulting truncation then takes the form of an $\mathcal{N}=2$ four-dimensional supergravity with all fields depending on an additional internal coordinate $w \in \mathcal{I}$, parametrizing the infinite families of KK states. In particular, the spin-2 and spin-1 towers are described by a metric $g_{\mu\nu}(x,w)$, and two vector fields $A^a_\mu(x,w)$, $a = 1, 2$, with x denoting the AdS_4 coordinates. The scalar fields parametrize the coset space

$$\mathcal{M}_{\rm scalar} = \frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)} \times \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}, \tag{1.6}$$

while still depending on the extra coordinate w. We show that this truncation is in fact a rewriting of $D=5$ minimal gauged supergravity coupled to one hypermultiplet. In turn, this is the theory obtained by consistent truncation of $D=11$ supergravity to the (finitely many) G_2-singlets on an internal S^6.

Searching for AdS_4 solutions within this truncation, we set $g_{\mu\nu}(x,w) = g^{\mathrm{AdS}_4}_{\mu\nu}(x)$, $A^a_\mu = 0$, and restrict to scalar fields independent of the AdS coordinates x. We provide explicit uplift formulas for these fields to $D=11$ dimensions which produces the most general G_2-invariant AdS_4 *Ansatz* in $D=11$ supergravity.[1] The field equations result in a system of second order ordinary differential equations for the w-dependent scalar fields. The system is singular at the endpoints of the interval $\mathcal{I}$, and we find that imposing regularity reduces its solutions to a finite discrete set. Among them, we recover the known analytic solutions [17–20] which all live within the consistent truncation to $\mathcal{N}=8$ supergravity [21] and correspond to the four G_2-invariant extremal points of its scalar potential [22]. On top of these solutions, we identify three new regular numerical solutions of the system. Their uplift yields geometries of the form $\mathrm{AdS}_4 \times \Sigma_7$ where Σ_7 is a deformed seven-sphere, preserving SO(7) isometries, together with a nonvanishing three-form flux which preserves $\mathrm{G}_2 \subset \mathrm{SO}(7)$ symmetry. The analysis suggests that this is the complete set of G_2-invariant AdS_4 solutions of $D=11$ supergravity. The description of these solutions within ExFT paves the way for a future analysis of their stability, mass spectra, supersymmetry, etc., which we leave for future work.

[1] Earlier constructions [16,17] were restricted to solutions living within the consistent truncation to $\mathcal{N}=8$ supergravity.

The rest of the paper is organized as follows. In Sec. II we revisit the maximal consistent truncations and the truncations to K-singlets in the ExFT framework. In Sec. III, we apply the construction to the truncation of $D = 11$ supergravity to singlets under the G_2 subgroup of the SO(8) isometry group of the round S^7. We recover the four known G_2-invariant solutions and present three new numerical solutions. We close in Sec. IV with some concluding remarks and outlook.

II. CONSISTENT TRUNCATIONS TO K-SINGLETS IN EXFT

In this section, we review the construction of maximal consistent truncations and consistent truncations to K-singlets in the framework of ExFT.

A. ExFT and maximal consistent truncations

Exceptional field theory provides a reformulation of $D = 11$ and IIB supergravity in terms of new variables that mimic the field content of the lower-dimensional maximal supergravity. As such, it offers a natural description of the consistent truncation of $D = 11$ supergravity to the lower-dimensional maximal supergravity. For the purpose of this paper, and in particular the construction of AdS_4 solutions, we will focus on the $E_{7(7)}$ ExFT, constructed in [23,24] to which we refer for details. Its bosonic field content comprises a 4×4 metric $g_{\mu\nu}$, $\mu, \nu = 0, \ldots, 3$, a set of 56 vector fields $A_\mu{}^M$, $M = 1, \ldots, 56$, and scalar fields parametrizing a coset representative $\mathcal{V}$ of $E_{7(7)}/SU(8)$. The latter encodes the $D = 11$ fields according to

$$\mathcal{V} \equiv \exp\left[A_{klmnpq} t_{(+4)}^{klmnpq}\right] \exp\left[A_{kmn} t_{(+2)}^{kmn}\right] V_{GL(7)}, \quad (2.1)$$

where $V_{GL(7)} \in GL(7) \subset E_{7(7)}$ is proportional to the internal block of the 11D vielbein, and A_{kmn} and A_{klmnpq} are the internal components of the $D = 11$ three-form and dual six-form, respectively, with indices $k, l, m = 1, \ldots, 7$. The $t_{(+n)}$ are the $E_{7(7)}$ generators of positive grading $+n$ in the algebra decomposition

$$\mathfrak{e}_{7(7)} \longrightarrow 7'_{-4} \oplus 35_{-2} \oplus \mathfrak{gl}(7)_0 \oplus 35'_{+2} \oplus 7_{+4}. \quad (2.2)$$

The remaining fields of $D = 11$ supergravity parametrize the 56 vector fields $A_\mu{}^M$ and the external metric $g_{\mu\nu}$. For later use, we state the relevant parts of the ExFT Lagrangian

$$\begin{aligned} g^{-1/2}\mathcal{L}_{\text{kin}} &= \frac{1}{48} g^{\mu\nu} \mathcal{D}_\mu \mathcal{M}^{MN} \mathcal{D}_\nu \mathcal{M}_{MN}, \\ g^{-1/2}\mathcal{L}_{\text{pot}} &= \frac{1}{48} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL} \\ &\quad - \frac{1}{2} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_L \mathcal{M}_{NK} + \frac{1}{2} g^{-1} \partial_M g \partial_N \mathcal{M}^{MN} \\ &\quad + \frac{1}{4} \mathcal{M}^{MN} g^{-2} \partial_M g \partial_N g + \frac{1}{4} \mathcal{M}^{MN} \partial_M g^{\mu\nu} \partial_N g_{\mu\nu}, \end{aligned} \quad (2.3)$$

in terms of the external metric $g_{\mu\nu}$, its determinant $g = \det g_{\mu\nu}$, and the internal metric $\mathcal{M} = \mathcal{V}\mathcal{V}^T$.

The $E_{7(7)}$ ExFT formulation of $D = 11$ supergravity allows for a natural description of the consistent truncation on the round S^7 to $D = 4$ maximal gauged supergravity [21,25]. In particular, the embedding into $D = 11$ supergravity of the 70 scalar fields of $D = 4$, $\mathcal{N} = 8$ supergravity, parametrizing an $E_{7(7)}/SU(8)$ coset representative V is given by

$$\mathcal{V}(x, y) = \mathring{U}(y) V(x), \quad (2.4)$$

in terms of the variables (2.1). Here, x and y denotes the four-dimensional coordinates, and the coordinates of the seven-sphere, respectively. The $SL(8) \subset E_{7(7)}$ valued twist matrix $\mathring{U}(y)$ which encodes the embedding (2.4) is explicitly given by [1,2]

$$\mathring{U}_{\underline{m}}{}^a(\mathcal{Y}) = \begin{pmatrix} \mathring{\omega}^{3/4}(\mathcal{Y}^a - 6\zeta^n \partial_n \mathcal{Y}^a) \\ \mathring{\omega}^{-1/4} \partial_m \mathcal{Y}^a \end{pmatrix} \in SL(8), \qquad \underline{m} = \{0, m\}, \qquad a = \{1, \ldots, 8\}, \quad (2.5)$$

in terms of the geometric data of the round S^7, specifically the fundamental sphere harmonics $\mathcal{Y}^a \mathcal{Y}^a = 1$, the vector field ζ^k satisfying $\mathring{\nabla}_k \zeta^k = 1$, and $\mathring{\omega} = \sqrt{\det \mathring{g}_{mn}^{S^7}}$. After embedding this matrix $\mathring{U}$ into the 56-dimensional fundamental representation of $E_{7(7)}$, its algebra valued currents

$$\Gamma_{\underline{MN}}{}^{\underline{K}} = \rho^{-1} (U^{-1})_{\underline{M}}{}^P (U^{-1})_{\underline{N}}{}^L \partial_P U_L{}^{\underline{K}}, \quad \rho = \mathring{\omega}^{-1/2}, \quad (2.6)$$

define the constant intrinsic torsion

$$X_{\underline{MN}}{}^{\underline{K}} = -7 [\Gamma_{\underline{MN}}{}^{\underline{K}}]_{\mathbf{912}}, \quad (2.7)$$

after projection onto the irreducible **912** representation.

As an interesting consequence of the ExFT formulation, the embedding (2.4) of $D = 4$ maximal supergravity can be extended to the higher Kaluza-Klein scalar modes around the round S^7 as

$$\mathcal{V}(x, y) = \mathring{U}(y) V(x) \exp\left[\mathcal{P}_I j^{I,\Sigma}(x) \mathcal{Y}_\Sigma(y)\right], \quad (2.8)$$

where the index Σ labels the scalar harmonics $\mathcal{Y}_\Sigma(y)$ on S^7, and the 70 noncompact generators of $E_{7(7)}$ are denoted by $\mathcal{P}_I$. In terms of SO(8) representations, these correspond to

$$\Sigma: \bigoplus_n [n,0,0,0], \qquad I: [2,0,0,0] \oplus [0,0,2,0] \tag{2.9}$$

The scalar fluctuations (including the Goldstone modes) thus fill the tensor product of these representations. In particular, the fluctuation *Ansatz* (2.8) allows to straightforwardly derive universal and compact mass formulas for the full KK spectrum of scalar fluctuations and higher point couplings [4–6,26].

B. Consistent truncations to K-singlets

We have reviewed in the above subsection, how the maximal consistent truncations on round spheres are naturally described within the ExFT formulation of higher-dimensional supergravity. Their consistency is based on the underlying exceptional geometry together with the constant intrinsic torsion (2.7). As discussed in the introduction, there is a different class of consistent truncations to singlets under some subgroup K of the isometry group of the round sphere whose consistency is ensured by a simple symmetry argument [9]. Let us briefly review how these truncations fit into the above framework.

Within the scalar sector, a truncation to the K-singlets in the spectrum can be described by restricting the fluctuation *Ansatz* (2.8) according to

$$\mathcal{V}(x,y) = \mathring{U}(y)\Phi(x,y), \tag{2.10}$$

with $\Phi(x,y)$ given by

$$\Phi(x,y) = \exp\left[\sum_{\text{K-singlets}} \phi^\sigma(x) s_\sigma^{I,\Sigma}\mathcal{P}_I\mathcal{Y}_\Sigma(y)\right]. \tag{2.11}$$

The index σ here labels the K-singlets found in the tensor product of the SO(8) representations (2.9), thereby defining the constant tensor $s_\sigma^{I,\Sigma}$. The statement that $\phi^\sigma(x)s_\sigma^{I,\Sigma}$ be K-singlet fields corresponds to

$$\phi^\sigma(x)s_\sigma^{I,\Sigma}(k^{-1})_I{}^J = \phi^\sigma(x)s_\sigma^{J,\Lambda}k_\Lambda{}^\Sigma, \tag{2.12}$$

where $k \in \mathrm{K}$. Upon contracting with the harmonics $\mathcal{Y}_\Sigma$, (2.12) corresponds to the following equivariance condition for Φ

$$k\Phi(x,y)k^{-1} = \Phi(x,k\cdot y), \tag{2.13}$$

where "$k\cdot$" is the action of K on M_{int}. I.e., the relevant K-invariant fields correspond to K-equivariant functions on the internal space M_{int}.

In case the group K acts transitively on the internal manifold S^7, the number of singlets is finite, and the truncation (2.10) can be explicitly represented in terms of a coset representative $S(y)$ of the internal space (1.2)

$$M_{\text{int}} = \frac{\mathrm{K}}{\mathrm{L}}, \tag{2.14}$$

where L is the isotropy subgroup of K. This is best understood through the language of G-structures as one typically does in generalized geometry [3].

C. Consistent truncation via the L-structure

The truncation to K-singlets, as described in the previous section is consistent by the usual argument of singlets not sourcing nonsinglets in the equations of motion [9]. Despite its simplicity, this picture has some drawbacks. It is nontrivial to identify the field content of the truncated theory from the K-singlets point of view. This applies in particular to the scalar target space and the gauging of the lower-dimensional supergravity. On the other hand, both the gauging and the scalar coset space can be computed systematically using the G-structure and intrinsic torsion data of generalized geometry [3]. In this section, we will review how the truncation to K-singlets corresponds to an "ordinary" truncation arising from an appropriate L-structure in generalized geometry. Let us first consider the case of finite truncations, i.e. the case, when the K-action on the internal space $M_{\text{int}} = \mathrm{K}/\mathrm{L}$ is transitive.

Crucially, we always work on internal spaces which also admit maximal truncations, associated to a generalized frame denoted by

$$\mathring{\mathcal{U}}_{\bar{M}}{}^M = \rho^{-1}(\mathring{U}^{-1})_{\bar{M}}{}^M. \tag{2.15}$$

The truncation consists of all KK modes in the maximal theory which are invariant under K. Since (2.15) defines a generalized parallelization, the generalized tangent bundle E, its dual E^*, and all their tensor powers are trivial. So we can view the space of sections $\Gamma(E)$ as

$$\Gamma(E) \cong C^\infty(M) \times R_1, \tag{2.16}$$

where R_1 is the relevant $\mathrm{E}_{d(d)}$ representation. Analogous statements hold for the various tensor powers by replacing R_1 with other $\mathrm{E}_{d(d)}$ representations. In this sense, sections can be defined by simply prescribing a set of well-defined functions, valued in the relevant $\mathrm{E}_{d(d)}$ representation.

The K-invariant modes in the truncation *Ansatz* correspond to K-equivariant functions multiplying $\mathring{\mathcal{U}}$, as seen explicitly for the scalar sector in (2.10), (2.13). The remaining fields obey analogous equivariance conditions. As anticipated above, it is natural to introduce a local coset representative $S: \frac{\mathrm{K}}{\mathrm{L}} \to \mathrm{K}$, which obeys the fundamental property

TABLE I. Spheres as coset spaces and the corresponding consistent truncations.

AdS × sphere	Coset $\frac{K}{L}$	SUSY	Scalar target space	Truncation
$AdS_4 \times S^7$	$\frac{USp(4)}{SU(2)}$	$\mathcal{N}=4$	$\frac{SO(6,3)}{SO(6)\times SO(3)}\times\frac{SL(2)}{SO(2)}$	[28]
$AdS_4 \times S^7$	$\frac{SU(4)}{SU(3)}$	$\mathcal{N}=2$	$\frac{SL(2)}{SO(2)}\times\frac{SU(2,1)}{U(2)}$	[29]
$AdS_4 \times S^7$	$\frac{USp(4)\times SU(2)}{SU(2)\times SU(2)}$	$\mathcal{N}=1$	$\frac{SL(2)}{SO(2)}\times\frac{SL(2)}{SO(2)}$	[28,30]
$AdS_4 \times S^7$	$\frac{SO(7)}{G_2}$	$\mathcal{N}=1$	$\frac{SL(2)}{SO(2)}$	[29]
$AdS_4 \times S^6$	$\frac{G_2}{SU(3)}$	$\mathcal{N}=2$	$\frac{SL(2)}{SO(2)}\times\frac{SU(2,1)}{U(2)}$	[31,32]
$AdS_5 \times S^5$	$\frac{SU(3)}{SU(2)}$	$\mathcal{N}=2$	$\frac{SO(5,2)}{SO(5)\times SO(2)}\times\mathbb{R}^+$	[13–15]
$AdS_7 \times S^3$	SU(2)	$\mathcal{N}=1$	$\frac{SO(3,3)}{SO(3)\times SO(3)}\times\mathbb{R}^+$	[1]

$$S(k\cdot y) = kS(y)\ell(k,y), \tag{2.17}$$

where $k\in K$ and $\ell(k,y)\in L$. The local frame

$$\mathcal{U}_{\bar M}{}^M = (S^{-1})_{\bar M}{}^{\bar N}\mathring{\mathcal{U}}_{\bar N}{}^M, \tag{2.18}$$

then defines an L-structure on $M_{\rm int}$ and allows to read off the change of basis between the global frame, and the local L-frame. Namely, if V are the components of a section in the K-basis, the corresponding global function $\mathring{V}$ in the parallelization basis is

$$\mathring{V} = S\cdot V, \tag{2.19}$$

where "·" acts in the relevant $E_{d(d)}$ representation of V. We can then show that $\mathring{V}$ is K-equivariant if and only if V is a constant L-singlet.

First, notice that S is constructed by picking a fixed point $\hat n\in\frac{K}{L}$, and building a K-valued local function $S\colon\frac{K}{L}\to K$ such that

$$S(y)\cdot\hat n = y, \tag{2.20}$$

where $y\in\frac{K}{L}$.[2] Taking V to be a constant L-singlet,[3] a quick calculation shows that $\mathring{V}$ in (2.19) is K-equivariant. For the converse, assume $\mathring{V}$ to be K-equivariant, i.e.

$$k\cdot\mathring{V}(y) = \mathring{V}(k\cdot y). \tag{2.21}$$

Then by definition $V = S^{-1}\cdot\mathring{V}$, and it follows that V is constant,

$$\begin{aligned} V(y) &= S(y)^{-1}\cdot\mathring{V}(y) \\ &= \mathring{V}(S(y)^{-1}\cdot y) = \mathring{V}(\hat n). \end{aligned} \tag{2.22}$$

It is then straightforward to show that V is also an L-singlet. Let $\ell\in L$, then

$$\ell\cdot\mathring{V}(\hat n) = \mathring{V}(\ell\cdot\hat n) = \mathring{V}(\hat n). \tag{2.23}$$

We have thus shown the equivalence

$$\begin{aligned} &\{\text{K equivariant functions multiplying}\,\mathring{\mathcal{U}}\} \\ &\leftrightarrow \{\text{Constant L-singlets multiplying}\,\mathcal{U}\}. \end{aligned} \tag{2.24}$$

Evaluating (2.19) on the scalar sector straightforwardly reproduces the truncation *Ansatz* of [8]

$$\mathcal{V}(x,y) = \mathring{U}(y)S(y)W(x)S^{-1}(y), \tag{2.25}$$

where

$$W(x)\in\frac{\mathrm{Com}_L(E_{d(d)})}{\mathrm{Com}_L(K_d)}, \tag{2.26}$$

in agreement with (1.3). Analogous expressions hold for the *Ansätze* of all remaining ExFT fields.

As discussed, the consistent truncation to finitely many K-singlets in the spectrum of a round sphere $S^n = SO(n+1)/SO(n)$, requires an alternative representation of the sphere as a coset space K/L [9]. Such representations exist for a number of spheres [27] and Table I lists the examples relevant for supergravity. In the general framework of [3], L is the reduced structure group of the exceptional generalized geometry. In all cases, the scalar coset space is given by (2.26), and likewise the remaining supersymmetry of the truncation is given by the number of L-singlets among the gravitini of the maximal theory. The embedding of the scalar target space into the ExFT formulation of the higher-dimensional supergravity is

[2] Note that this definition automatically implies (2.17), where L is the subgroup of K that fixes $\hat n$.

[3] In this context, constant means independent of $M_{\rm int}$. V can still depend on the external space.

given by (2.25). For $\mathrm{USp}(4)/\mathrm{SU}(2)$ and $(\mathrm{USp}(4)\times \mathrm{SU}(2))/(\mathrm{SU}(2)\times\mathrm{SU}(2))$ this was exploited in [7,8] to compute the full KK spectrum around the squashed seven-sphere [11,12]. The $\mathrm{SU}(4)/\mathrm{SU}(3)$ example has been discussed in detail in [10].

Finally, let us consider a nontransitively acting K. In this case $M_{\rm int}$ is foliated into K-orbits, and a transverse space T [10]. We denote coordinates on the orbits by y, and the transverse ones by w. It is straightforward to see that an argument identical to the transitive case still holds. However, the L-singlets in (2.25) will no longer be constant, but w-dependent, namely

$$W(x)\longrightarrow W(x,w). \tag{2.27}$$

D. Intrinsic torsion

The gauging associated to the lower-dimensional truncated theory is encoded in the embedding tensor, which contains all the information about couplings between scalars and vectors. For truncations that arise from an L-structure, the embedding tensor is contained in the intrinsic torsion, as discussed in [3]. The above construction ensures that the intrinsic torsion of the L-structure is a constant L-singlet as we shall now sketch. Let us begin with the case of finite trunctions. We will ignore trombone contributions, thus restricting to $\mathrm{E}_{d(d)}$-valued generalized connections, as opposed to $\mathrm{E}_{d(d)}\times\mathbb{R}^+$. It is useful to start with some general remarks.

A generalized connection $A_{M\bar{N}}{}^{\bar{P}}$ is L-compatible whenever the A_M are valued in the Lie algebra $\mathfrak{l}$ of L. Note that barred indices are flattened with U, not with $\mathring{U}$. A covariant derivative D_M is defined by

$$D_M=\partial_M+(A_M\cdot), \tag{2.28}$$

where $(A_M\cdot)$ acts in the appropriate $\mathfrak{l}$ representation. One can compute the torsion $\tau(D)$ of D [33][4]

$$\tau(D)=\rho^{-1}(\mathbb{P}_{R_3})_{\bar{M}}{}^{\bar{\alpha}}{}_{\bar{\beta}}{}^{\bar{N}}(A_{\bar{N}}{}^{\bar{\beta}}-\Gamma_{\bar{N}}{}^{\bar{\beta}}), \tag{2.29}$$

where R_3 is the $\mathrm{E}_{d(d)}$ representation of the embedding tensor. The greek indices in (2.29) span the adjoint of $\mathrm{E}_{d(d)}$. We denote by Γ the standard "current" (2.6) associated to U. One can further expand Γ in terms of $\mathring{U}$ and S

$$\begin{aligned}\Gamma_{\overline{MN}}{}^{\bar{P}}=&(S^{-1})_{\bar{M}}{}^{\bar{R}}(S^{-1})_{\bar{N}}{}^{\bar{S}}\mathring{\Gamma}_{\overline{RS}}{}^{\bar{T}}S_{\bar{T}}{}^{\bar{P}}\\&+(S^{-1})_{\bar{M}}{}^{\bar{Q}}(S^{-1})_{\bar{N}}{}^{\bar{S}}\mathring{\partial}_{\bar{Q}}S_{\bar{S}}{}^{\bar{P}},\end{aligned} \tag{2.30}$$

where $\mathring{\partial}_{\bar{Q}}=(\mathring{U}^{-1})_{\bar{Q}}{}^{M}\partial_M$, and $\mathring{\Gamma}$ is the $\mathring{U}$ current.

[4]Note that again we are ignoring trombone contributions.

In order to compute the intrinsic torsion, it is convenient to pick an origin in the affine space of generalized connections. One can choose

$$S^{-1}\partial S|_{\mathfrak{l}}, \tag{2.31}$$

where the $\mathfrak{l}$ projection is taken in order to get an L-connection. Let us denote the intrinsic torsion by $\tau_{\rm int}$. We then deduce that the R_3 component of $\tau_{\rm int}$ is

$$\tau_{\rm int}=\rho^{-1}(\mathbb{P}_{R_3})(-\mathring{\Gamma}-S^{-1}\partial S|_{\mathfrak{k}\ominus\mathfrak{l}}), \tag{2.32}$$

where all indices are suppressed. Let us make a couple of observations about (2.32)

(1) The current $\mathring{\Gamma}$ is contracted with S as in (2.30).

(2) The derivative in $S^{-1}\partial S|_{\mathfrak{k}\ominus\mathfrak{l}}$ is really $(S^{-1})_{\bar{M}}{}^{\bar{N}}\mathring{\partial}_{\bar{N}}$.

The $\mathring{\Gamma}$ term in (2.32) gives the constant embedding tensor $\mathring{X}$ of the maximal truncation associated to the global frame $\mathring{\mathcal{U}}$. It is crucial to note that $S\in\mathrm{K}$ belongs to some subgroup of the gauging, thus $\mathring{X}$ is also a K-singlet. Hence, the S dressing leaves $\mathring{X}$ invariant. We can now focus on the $S^{-1}\partial S$ term.

1. Finite case

We will specialise to the situation of interest, i.e., when the internal space is S^n, and $\mathring{U}$ is its usual parallelization of [1,2]. S should be viewed as a local function on the sphere. Furthermore, derivatives obey the section condition. Hence, we can write the action of $\mathring{\partial}$ as

$$\rho^{-1}\mathring{\partial}_{\bar{M}}=K_{\bar{M}}{}^{i}\partial_i, \tag{2.33}$$

where $K_{\bar{M}}{}^{i}$ are the $\mathrm{SO}(n+1)$ Killing vectors of the round n-sphere. From now on, let us introduce $\mathrm{SO}(n+1)$ fundamental indices $\bar{a}=1,\ldots,n+1$, and denote the nontrivial Killing vectors by $K_{\overline{ab}}$. It is also convenient to introduce $\mathrm{SO}(n+1)$ generators

$$(T_{\overline{ab}})_{\bar{c}}{}^{\bar{d}}=2\delta_{\bar{c}[\bar{a}}\delta_{\bar{b}]}{}^{\bar{d}}. \tag{2.34}$$

The action of $K_{\overline{ab}}$ on a scalar function f reduces to

$$K_{\overline{ab}}{}^{i}\partial_i f(y)=\frac{d}{dt}|_{t=0}f(e^{tT_{\overline{ab}}}y). \tag{2.35}$$

More specifically, in order to match (2.30), $\mathring{\partial}$ must be contracted with S^{-1}. Hence, (2.35) becomes

$$(S^{-1})_{\overline{ab}}{}^{\overline{cd}}K_{\overline{cd}}{}^{i}\partial_i f(y)=\frac{d}{dt}\bigg|_{t=0}f(S(y)e^{tT_{\overline{ab}}}S^{-1}(y)y), \tag{2.36}$$

where we used that $T_{\overline{ab}}$ are K-singlets. Applying (2.36) to our situation, leaves us with the following expression

$$S^{-1}(y)\frac{d}{dt}\bigg|_{t=0} S(S(y)e^{tT_{\overline{ab}}}S^{-1}(y)y). \tag{2.37}$$

By definition the coset representative satisfies $S^{-1}(y)y = \hat{n}$. Hence, (2.37) vanishes whenever $T_{\overline{ab}}$ is an element of the Lie subalgebra $\mathfrak{so}(n)$. Therefore, the only nontrivial contribution comes when $T_{\overline{ab}} \in \mathfrak{so}(n+1) \ominus \mathfrak{so}(n) = \mathfrak{k} \ominus \mathfrak{l}$.[5] One can then take $T_{\overline{ab}}$ to lie in $\mathfrak{k} \ominus \mathfrak{l}$, without loss of generality. Let us label generators of $\mathfrak{k} \ominus \mathfrak{l}$ by indices $\bar{I}, \bar{J}, \ldots$. Crucially, $T_{\bar{I}} \in \mathfrak{k}$, so we can use the fundamental property of S to pull out the exponential

$$S(S(y)e^{tT_{\bar{I}}}S^{-1}(y)y) = S(y)e^{tT_{\bar{I}}}e^{\chi(t,y)}, \tag{2.38}$$

where $\chi(t,y)$ is a curve in $\mathfrak{l}$ with $\chi(0,y) = 0$. It is then straightforward to see that (2.37) reduces to

$$\begin{aligned} S^{-1}(y)\frac{d}{dt}\bigg|_{t=0} S(S(y)e^{tT_{\bar{a}b}}S^{-1}(y)y) &= \frac{d}{dt}\bigg|_{t=0} e^{tT_{\bar{I}}}e^{\chi(t,y)} \\ &= T_{\bar{I}} + \dot{\chi}(0,y), \end{aligned} \tag{2.39}$$

where $\dot{\chi}$ is the t derivative of χ. To conclude, substituting (2.39) into (2.32), shows that the remaining component of τ_{int} corresponds to the $(\mathfrak{k}\ominus\mathfrak{l}) \otimes (\mathfrak{k}\ominus\mathfrak{l})$ block of the Cartan-Killing form κ of $\mathrm{SO}(n+1)$, appropriately embedded in $\mathrm{E}_{d(d)}$, and projected onto R_3. Crucially, $\kappa_{\overline{IJ}}$ is a constant L-singlet.

2. *Infinite case*

The infinite case is more involved. In this paper, we will restrict to the case where the transverse space T is one-dimensional, such that the fields of the truncation (2.27) depend on one transverse coordinate w only. Specifically, we view the internal space $M_{\text{int}} = S^n$ as a fibration of S^{n-1} over the interval $\mathcal{I} = [-1,1]$. We take $S^{n-1} \cong \mathrm{K/L}$, where K acts transitively on S^{n-1}. More concretely, we write the S^n embedding coordinates as

$$\mathcal{Y} = \left(\sqrt{1-w^2}\,y, w\right), \tag{2.40}$$

where $y = (y^1, \ldots, y^n)$ are embedding coordinates of S^{n-1}, and ω parametrizes $\mathcal{I}$. The coset representative S is then only a function of y, while $\hat{n}$ is replaced with a copy of $\mathcal{I}$. More specifically, we introduce

$$\hat{n}(w) = \left(\sqrt{1-w^2}, 0, \ldots, 0, w\right), \tag{2.41}$$

if we then view S and $\hat{n}$ as (local) functions on S^n, the defining property (2.20) becomes[6]

$$S(\mathcal{Y})\hat{n}(\mathcal{Y}) = \mathcal{Y}. \tag{2.42}$$

Crucially, the isotropy group L does not stabilize a single point anymore, but a copy of $\mathcal{I}$. Namely, it is defined by

$$\ell\hat{n}(\mathcal{Y}) = \hat{n}(\mathcal{Y}), \quad \text{with} \quad \hat{n}(\mathcal{Y}) = \left(\sqrt{1-w^2}, 0, \ldots, 0, w\right). \tag{2.43}$$

Since K acts on S^{n-1}, the modification (2.42) still implies the universal property (2.17). Much of the finite case analysis still applies. In particular, the nontrivial term we should compute is still (2.37). However, there are two significant differences

(1) $S^{-1}(y)y = \hat{n}$ is not a constant: it corresponds to $\hat{n}(w)$ of (2.41).
(2) In the infinite case, (2.37) is nontrivial *also* for some generators outside of $\mathfrak{k}\ominus\mathfrak{l}$. For example, in our case, the T's transforming in the vector representation of $\mathrm{SO}(n)$ give nonvanishing contributions.

Because of point 2 above, we cannot use the fundamental property of S to pull out $Se^{tT}S^{-1}$, as we did for the finite case. We instead write

$$S(S(z)e^{tT_{\bar{a}b}}S^{-1}(z)z) = S(z)S(e^{tT_{\overline{ab}}}\hat{n}(w))\ell(t,z). \tag{2.44}$$

Note that here we denote the n local coordinates on S^n by z, which in turn correspond to w along with the S^{n-1} local coordinates. From now on, let us assume that $S(\hat{n}(w)) = 1$.[7] One can easily see that $\ell(0,z) = 1$. Thus, for sufficiently small t, we can again assume

$$h(t,z) = e^{\chi(t,z)}, \quad \text{with } \chi \text{ some curve in } \mathfrak{l}. \tag{2.45}$$

We now show that in the infinite case, the $\mathfrak{k}\ominus\mathfrak{l}$ projection of $S^{-1}\partial S$ *only* depends on w, and is an L-singlet.

The w-dependence follows from (2.44). Namely

$$\begin{aligned} S^{-1}(z)\frac{d}{dt}\bigg|_{t=0} & S(S(z)e^{tT_{\bar{a}b}}S^{-1}(z)z) \\ &= \frac{d}{dt}\bigg|_{t=0} S(e^{tT_{\bar{a}b}}\hat{n}(w))e^{\chi(t,z)} \\ &= \frac{d}{dt}\bigg|_{t=0} S(e^{tT_{\bar{a}b}}\hat{n}(w)) + \dot{\chi}(0,z), \end{aligned} \tag{2.46}$$

[5]This equality holds up to $\mathfrak{so}(n)$ shifts.

[6]We slightly abuse notation here, one has to keep in mind that S only depends on the S^{n-1} coordinates. Similarly $\hat{n}$ is a function of w only.

[7]This can be done without loss of generality.

it is clear that the $\mathfrak{k}\ominus\mathfrak{l}$ component of (2.46), can only depend on w. We will now show that (2.37) is an L-singlet.

Let $\Lambda\in\mathrm{L}$, one can act directly on (2.46)

$$\Lambda\frac{d}{dt}\bigg|_{t=0}S(\Lambda^{-1}e^{tT_{\overline{ab}}}\hat{n}(w))e^{\chi(t,z)}\Lambda^{-1} = \frac{d}{dt}\bigg|_{t=0}S(e^{tT_{\overline{ab}}}\hat{n}(w))\lambda(t,w)e^{\chi(t,z)}\Lambda^{-1}, \quad (2.47)$$

where we again use the fundamental property of S. The extra factor $\lambda(t,w)\in\mathrm{L}$ "compensates" for Λ^{-1} being pulled out of S on the left-hand side. Let us now consider $t=0$,

$$\begin{aligned}1 = S(\hat{n}(w)) &= S(\Lambda^{-1}\hat{n}(w))\\ &= \Lambda^{-1}S(\hat{n}(w))\lambda(0,w)\\ &= \Lambda^{-1}\lambda(0,w),\end{aligned} \quad (2.48)$$

so that $\lambda(0,w)=\Lambda$. Hence, again for small t, we can take

$$\lambda(t,w)=\Lambda\, e^{\xi(t,w)},\quad \text{with } \xi \text{ a curve in } \mathfrak{l} \text{ such that } \xi(0,w)=0. \quad (2.49)$$

Substituting (2.49) into (2.47) gives

$$\Lambda\frac{d}{dt}\bigg|_{t=0}S(\Lambda^{-1}e^{tT_{\overline{ab}}}\hat{n}(w))e^{\chi(t,z)}\Lambda^{-1} = \frac{d}{dt}\bigg|_{t=0}S(e^{tT_{\overline{ab}}}\hat{n}(w)) + \text{some } \mathfrak{l} \text{ contribution}, \quad (2.50)$$

Thus we conclude that the $\mathfrak{k}\ominus\mathfrak{l}$ projection is an L-singlet.

To conclude, in the infinite case, the intrinsic torsion is still an L-singlet. However, unlike the finite case, it is not constant. Instead, it depends on the transverse coordinate, w. It would be interesting to explicitly evaluate τ_{int} for the infinite case.

III. G_2-INVARIANT SOLUTIONS OF $D=11$ SUPERGRAVITY

In this section, we will use the ExFT structures in order to revisit and construct new G_2-invariant $AdS_4\times\Sigma_7$ solutions of $D=11$ supergravity. Solutions of this type have been constructed in the past directly in $D=11$ dimensions [18–20], and most systematically in [17]. However, all previous constructions have been restricted to solutions that live within the consistent truncation to $\mathcal{N}=8$ supergravity [21]. In terms of the $D=4$ theory, they correspond to the G_2-invariant extremal points of the scalar potential [22].

Instead, here we will allow for a deformation of the round S^7 by the most general combination of the infinitely many G_2-invariant scalar modes in the KK spectrum. In the ExFT framework, this corresponds to analysing a consistent truncation to infinitely many fields [10]. Since we are interested in AdS_4 solutions, we focus on the scalar sector of the four-dimensional theory.

A. Consistent truncation to G_2-singlets

We start by representing the seven-sphere S^7 as a foliation of S^6 over an interval $\mathcal{I}$. Specifically, we use its embedding coordinates $\mathcal{Y}^I$ inside $\mathbb{R}^8$, satisfying $\mathcal{Y}^I\mathcal{Y}^I=1$, and represent them as

$$\mathcal{Y}^i=\sqrt{1-w^2}\,y^i,\qquad \mathcal{Y}^8=w\in[-1,1],\quad i=1,\ldots,7, \quad (3.1)$$

with the embedding coordinates y^i of the round S^6 satisfying $y^iy^i=1$. We will also introduce the angle coordinate $\theta\in[0,\pi]$ by

$$w=-\cos\theta. \quad (3.2)$$

According to the general discussion, within ExFT the consistent truncation to G_2-singlets in the scalar sector is described by a parametrization of the generalized vielbein as

$$\mathcal{V}(x,y,\theta)=\mathring{U}(y,\theta)S(y)W(x,\theta)S^{-1}(y). \quad (3.3)$$

Here, $S(y)$ is a coset representative for

$$S^6=\frac{G_2}{SU(3)}, \quad (3.4)$$

and the W is a coset representative of the coset (1.3)

$$\frac{\mathrm{Com}_{SU(3)}(E_{7(7)})}{\mathrm{Com}_{SU(3)}(SU(8))}=\frac{SU(2,1)}{U(2)}\times\frac{SU(1,1)}{U(1)}, \quad (3.5)$$

still depending on the additional coordinate $\theta\in[0,\pi]$, parametrizing the infinite families of KK states. We denote by $\{\phi,\chi\}$ the coordinates of the second factor of (3.5), and parametrize the quaternionic manifold $SU(2,1)/U(2)$ by coordinates

$$\{\phi_1,\chi^m\}=\{\phi_1,\chi_{1a},\chi_{1b},\chi_2\}. \quad (3.6)$$

By virtue of their θ-dependence, each of these six fields represents an infinite family of four-dimensional scalars, which, however, still include both physical scalars together with the Goldstone modes. As for the remaining bosonic fields, the truncation carries two infinite families of vector fields, parametrized by θ as $A_\mu^\alpha(\theta)$, $\alpha=1$, 2, as well as the spin-2 tower described by $g_{\mu\nu}(\theta)$. After Higgsing (for spin-1 and spin-2 fields), the theory then describes the massive spin-2 tower, together with one massive spin-1 tower and four infinite towers of massive scalar fields. In the fermionic sector, the four-dimensional theory after Higgsing carries two infinite towers of massive gravitino fields $\psi_\mu^u(\theta)$, $u=1$, 2, together with two towers of massive spin 1/2 fermions. Indeed, this matches the counting of G_2-singlets within the KK spectrum around the round sphere S^7 [34–37].

When searching for AdS_4 solutions, we will impose vanishing fermions, and set

$$A^a_\mu(\theta) = 0, \qquad \partial_\theta g_{\mu\nu}(\theta) = 0. \tag{3.7}$$

The external part $\mathcal{L}_{\rm kin}$ of the ExFT Lagrangian (2.3) in this truncation is computed by evaluating the Lagrangian with the *Ansatz* (3.3), leading to

$$g^{-1/2}\mathcal{L}_{\rm kin} = -2\rho^{-2}\bigg(\partial_\mu\phi_1\partial^\mu\phi_1 + e^{-4\phi_1}M_{mn}\partial_\mu\chi^m\partial^\mu\chi^n + \frac{3}{4}(\partial_\mu\phi\partial^\mu\phi + e^{-2\phi}\partial_\mu\chi\partial^\mu\chi)\bigg), \tag{3.8}$$

with the weight factor

$$\rho = (\sin\theta)^{-3}, \tag{3.9}$$

and the scalar matrix M_{mn} given by

$$M_{mn} = \begin{pmatrix} e^{2\phi_1} + \chi_{1b}^2 & -\chi_{1a}\chi_{1b} & -\chi_{1b} \\ -\chi_{1a}\chi_{1b} & e^{2\phi_1} + \chi_{1a}^2 & \chi_{1a} \\ -\chi_{1b} & \chi_{1a} & 1 \end{pmatrix}. \tag{3.10}$$

This confirms that the scalar kinetic term (3.8) is given by a four-dimensional sigma-model on the six-dimensional target space (3.5) with all fields carrying an additional dependence on the coordinate θ.

In the search for AdS_4 solutions, we will further restrict to scalar fields that are constant in AdS_4, i.e. reduce to functions of only the additional coordinate θ. For such solutions, the kinetic Lagrangian (3.8) vanishes and does not contribute to the field equations. The relevant Lagrangian is thus obtained from the internal part $\mathcal{L}_{\rm pot}$ of the ExFT Lagrangian (2.3). After some lengthy but straightforward computation this yields the truncated Lagrangian

$$\mathcal{L}_{\rm pot} = \rho^{-2}g^{-1/2}L_{\rm pot}, \tag{3.11}$$

with

$$\begin{aligned} L_{\rm pot} = {}& \frac{3}{2}e^{-3\phi}D_\theta\phi D_\theta\phi - e^{-3\phi}D_\theta\phi_1 D_\theta\phi_1 - e^{-3\phi-4\phi_1}M_{mn}D_\theta\chi^m D_\theta\chi^n \\ & + 3e^{-\phi}\{2\cot^2\theta(5\chi_{1a}^2 + 3\chi_{1b}^2) - 4e^{-4\phi_1}(\chi_{1a} + \cot\theta(\chi_{1a}^2\chi_{1b} + 2\chi_{1a}\chi_2 + \chi_{1b}^3))^2 \\ & + e^{-2\phi_1}((5 + \cot\theta(10\chi_2 - 8\chi_{1a}\chi_{1b}))(1 + 2\cot\theta\chi_2) + \cot^2\theta(5\chi_{1a}^2 - 3\chi_{1b}^2)(\chi_{1a}^2 + \chi_{1b}^2))\} \\ & + 15e^{-\phi+2\phi_1}\cot^2\theta - 3e^{-3\phi}(1 + 3\cos(2\theta))\csc^2\theta, \end{aligned} \tag{3.12}$$

with the matrix M_{mn} from (3.10) above and the "covariant" derivatives D_θ defined as

$$\begin{aligned} D_\theta\phi &= \partial_\theta\phi + 2\cot\theta, \\ D_\theta\phi_1 &= \partial_\theta\phi_1 + 2\chi_2 - 3\cot\theta - 6\chi\chi_{1a}\cot\theta, \\ D_\theta\chi_{1a} &= \partial_\theta\chi_{1a} - 3\chi_{1a}\cot\theta - \chi_{1b}(e^{2\phi_1} + \chi_{1a}^2 + \chi_{1b}^2) + 2\chi_{1a}\chi_2 + 3\chi\cot\theta(e^{2\phi_1} - \chi_{1a}^2 + 3\chi_{1b}^2), \\ D_\theta\chi_{1b} &= \partial_\theta\chi_{1b} - 3\chi_{1b}\cot\theta + \chi_{1a}(e^{2\phi_1} + \chi_{1a}^2 + \chi_{1b}^2) + 2\chi_{1b}\chi_2 - 3\chi(1 + 2\cot\theta(2\chi_{1a}\chi_{1b} + \chi_2)), \\ D_\theta\chi_2 &= \partial_\theta\chi_2 - \frac{1}{2}e^{4\phi_1} - \frac{5}{2} - 6\chi_2\cot\theta - \frac{1}{2}(\chi_{1a}^2 + \chi_{1b}^2)(2e^{2\phi_1} + \chi_{1a}^2 + \chi_{1b}^2) + 2\chi_2^2 \\ & \quad + 3\chi(\cot\theta(\chi_{1b}(e^{2\phi_1} + \chi_{1a}^2 + \chi_{1b}^2) - 2\chi_{1a}\chi_2) - \chi_{1a}). \end{aligned} \tag{3.13}$$

All fields depend on the coordinate θ only, variation of (3.12) thus implies a set of ordinary differential equations for the scalar fields. Furthermore, the Lagrangian (3.12) explicitly depends on the coordinate θ induced by the θ-dependence of the generalized frame of the round sphere $\mathring{U}$ in the truncation *Ansatz* (3.3).

It is straightforward to check that the combination

$$V = -L_{\rm pot} - \frac{3}{4}\rho^2\partial_\theta(\rho^{-2}e^{-3\phi}\partial_\theta\phi), \tag{3.14}$$

is conserved on-shell, i.e. $\partial_\theta V = 0$ as a result of the field equations implied by (3.12). This charge shows up in the four-dimensional Einstein field equations in this truncation, and encodes the AdS_4 radius ℓ_4 as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -Vg_{\mu\nu} = \frac{3}{\ell_4^2}g_{\mu\nu}. \tag{3.15}$$

The uplift of the consistent truncation can be computed by extracting the $D = 11$ fields upon combining (2.1) with the *Ansatz* (3.3). For instance, the $D = 11$ metric is expressed in terms of the scalar fields (3.6) as

$$ds^2 = \Delta^{-1} ds^2_{(4)} + e^{3\phi}\Delta^{-1} d\theta^2 + e^{-\frac{\phi}{2}} \sin^2\theta \Delta^{1/2} ds^2_{S^6}. \quad (3.16)$$

Here, $ds^2_{S^6}$ denotes the metric of the round S^6, and the warp factor Δ is given by

$$\Delta = e^{\phi + 4\phi_1/3}(\cos\theta)^{-4/3}((e^{2\phi_1} + \chi_{1a}^2 + \chi_{1b}^2)^2 + (\tan\theta + 2\chi_2)^2)^{-2/3}. \quad (3.17)$$

All fields are functions of θ. Similarly, one may extract the $D = 11$ three-form. We give explicit formulas below, cf. (3.52), after further simplification of the system.

By construction, every solution to the equations of motion of (3.12) locally describes a solution of $D = 11$ supergravity. However, searching for a solution with compact internal space given by a seven-sphere, the form of the metric (3.16) shows that we need to require all fields to remain regular at the endpoints of the interval $\theta \in [0, \pi]$. As we will make explicit below, the field equations derived from (3.12) are singular at these endpoints, such that the proof of existence and the construction of regular solutions becomes a rather nontrivial task. In particular, we will see that only a discrete and finite set of such solutions exists.

Before proceeding with the analysis of solutions, let us note that the Lagrangian (3.12) can be further simplified. First, we observe that it is invariant under the gauge transformations

$$\begin{aligned}
\delta\chi &= -\partial_\theta\Lambda - 2\Lambda\cot\theta, \\
\delta_\Lambda\phi_1 &= -6\Lambda\chi_{1a}\cot\theta, \\
\delta_\Lambda\chi_{1a} &= 3\Lambda\cot\theta(e^{2\phi_1} - \chi_{1a}^2 + 3\chi_{1b}^2), \\
\delta_\Lambda\chi_{1b} &= -3\Lambda(1 + 2\cot\theta(2\chi_{1a}\chi_{1b} + \chi_2)), \\
\delta_\Lambda\chi_2 &= 3\Lambda(\cot\theta(\chi_{1b}(e^{2\phi_1} + \chi_{1a}^2 + \chi_{1b}^2) - 2\chi_{1a}\chi_2) - \chi_{1a}),
\end{aligned} \quad (3.18)$$

with arbitrary $\Lambda = \Lambda(\theta)$, which can be used to eliminate one of the scalar fields. Moreover, the Lagrangian (3.12) only depends algebraically on the field χ which can thus be integrated out by virtue of its field equations, reducing the system to only four scalar fields. This is a remnant of the Higgs mechanism of the full theory.

As it turns out, the system can further be drastically simplified by going to different variables which are closer to the higher-dimensional origin of the fields. We will show in the following that upon change of coordinates and fields, the Lagrangians (3.8), (3.12) embed into a simple five-dimensional Lagrangian upon merging the AdS_4 coordinates x and the extra coordinate θ into a five-dimensional space-time. In turn, this significantly simplifies the equations of motion such that they can be treated by numerical methods. In order to illustrate this simplification of the system, we will first discuss the further subtruncation to the (still infinitely many) SO(7)-singlets in the spectrum of the round sphere.

B. SO(7) truncation

As an illustration, let us first discuss the truncation of the system to the SO(7)-singlets in the S^7 spectrum. This sector has been discussed in [10]. Within the above discussion, this corresponds to the further (consistent) truncation

$$\chi = \chi_{1a} = \chi_{1b} = 0, \quad (3.19)$$

such that we are left with three θ-dependent scalar fields $\{\phi, \phi_1, \chi_2\}$ parametrizing the target space

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \mathbb{R}, \quad (3.20)$$

with ϕ denoting the $\mathbb{R}$ coordinate. The ExFT action (3.8), (3.12) in this truncation reduces to

$$S = \int d^4x d\theta \rho^{-2} g^{-1/2}(L_{\mathrm{kin}} + L_{\mathrm{pot}}), \quad (3.21)$$

with

$$L_{\mathrm{kin}} = -\frac{3}{2}\partial_\mu\phi\partial^\mu\phi - 2\partial_\mu\phi_1\partial^\mu\phi_1 - 2e^{-4\phi_1}\partial_\mu\chi_2\partial^\mu\chi_2, \quad (3.22)$$

$$\begin{aligned}
L_{\mathrm{pot}} = &\frac{3}{2}e^{-3\phi}(\partial_\theta\phi + 2\cot\theta)^2 - e^{-3\phi}(\partial_\theta\phi_1 + 2\chi_2 - 3\cot\theta)^2 \\
&- e^{-3\phi-4\phi_1}\left(\partial_\theta\chi_2 + 2\chi_2^2 - \frac{1}{2}e^{4\phi_1} - 6\chi_2\cot\theta - \frac{5}{2}\right)^2 \\
&+ 15e^{-\phi-2\phi_1}(1 + 2\chi_2\cot\theta)^2 + 15\cot^2\theta e^{-\phi+2\phi_1} \\
&- 2e^{-3\phi}(1 + 3\cos(2\theta))\csc^2\theta).
\end{aligned} \quad (3.23)$$

Again, one finds that the combination V from (3.14) is conserved on-shell, i.e. as a result of the field equations implied by (3.23), and encodes the AdS_4 radius ℓ_4 according to (3.15).[8] Furthermore, one finds that the field equations obtained from (3.23) imply that

$$\Delta^{-3}\rho^2\partial_\theta(\rho^{-2}e^{-3\phi}\Delta^2 C_\theta) = \mathrm{const}, \quad (3.25)$$

[8]After change of coordinates

$$\phi_1 \to -\frac{1}{2}\varphi, \qquad \chi_2 \to \chi, \qquad \theta \to \pi - \theta_7, \quad (3.24)$$

one may further check that V in the SO(7) truncation precisely reproduces the effective potential derived in [10], where it is directly obtained via the embedding (3.3) with (3.4) replaced by SO(7)/SO(6) and given in their Eq. (3.31).

where C_θ is defined by

$$C_\theta = \tan\theta e^{3\phi}\Delta^{-2}(1-f(\theta)) - e^{\frac{3\phi}{2}-2\phi_1}\Delta^{-1/2}(\tan\theta + 2\chi_2), \tag{3.26}$$

with the function f satisfying the differential equation

$$f'(\theta) = (6-7f(\theta))\cot\theta - f(\theta)\tan\theta. \tag{3.27}$$

Equation (3.25) is inherited from the Bianchi identity in $D=11$ supergravity, with C_θ describing the $D=11$ internal 6-form as $\star_7 C_{(6)} = C_\theta d\theta$.

The existence of two conserved charges (3.14) and (3.25) suggests that the model (3.21) can be further simplified exploiting the associated global symmetries. Indeed this is the case and leads to a compact reformulation in terms of redefined coordinates and fields that are naturally associated with a $D=5$ uplift of the field equations as we shall show now.

1. Redefined coordinates and fields

Exploiting the global symmetry derived from the conserved charge (3.14) reveals a redefinition of the θ-coordinate, which together with a θ-dependent $\mathbb{R}\times\mathrm{SL}(2)$ transformation on the scalar fields, leads to a Lagrangian which no longer shows any explicit coordinate-dependence. Explicitly, this is achieved, by going to the coordinate u defined as

$$\begin{aligned} u &= -\frac{1}{96}(96 + 45\sin(2\theta) - 9\sin(4\theta) \\ &\quad + \sin(6\theta) - 60\theta) \in [-1,1], \\ &\Rightarrow \partial_\theta u = 2\rho^{-2} = 2\sin^6\theta. \end{aligned} \tag{3.28}$$

Simultaneously, we define the scalar fields as

$$\begin{aligned} &\{\phi,\phi_1,\chi_2\} \longrightarrow \{\Phi,\Phi_1,\mathcal{X}_2\}, \\ &\text{with} \quad e^\Phi = \rho^{4/3}e^\phi, \\ &\quad e^{\Phi_1} = \rho^{-1}e^{-3\phi/4}\Delta^{3/4}, \\ &\quad \mathcal{X}_2 = \frac{1}{2}\rho^{-2}(e^{-3\phi/2-2\phi_1}\Delta^{3/2}(\tan\theta + 2\chi_2) \\ &\qquad - (\tan\theta - 3\rho^2 u)), \end{aligned} \tag{3.29}$$

with ρ, Δ, and u from (3.9), (3.17), and (3.28), respectively. Although not manifest, one may verify that this transformation, corresponds to a nonlinear θ-dependent $\mathbb{R}\times\mathrm{SU}(1,1)$ transformation on the fields. As a result, the kinetic term (3.22) remains unchanged.

The Lagrangian (3.23) after redefinition (3.28), (3.29) takes the remarkably compact form

$$\begin{aligned} \mathcal{L}_{\rm pot} &= e^{-3\Phi}(3\partial_u\Phi\partial_u\Phi - 2\partial_u\Phi_1\partial_u\Phi_1 - 2e^{-4\Phi_1}\partial_u\mathcal{X}_2\partial_u\mathcal{X}_2) \\ &\quad + \frac{15}{2}e^{-\Phi-2\Phi_1}. \end{aligned} \tag{3.30}$$

As a result, the action given by (3.22) and (3.30) [upon temporarily relaxing the constraints (3.7)] can be written in manifestly five-dimensional form as

$$\begin{aligned} S = \int d^4x\,du\sqrt{|G_{(5)}|}\Big(&R^{(5)} - 2\partial_{\hat\mu}\Phi_1\partial^{\hat\mu}\Phi_1 \\ &- 2e^{-4\Phi_1}\partial_{\hat\mu}\mathcal{X}_2\partial^{\hat\mu}\mathcal{X}_2 + \frac{15}{2}e^{-2\Phi_1}\Big), \end{aligned} \tag{3.31}$$

with $\{x^{\hat\mu}\} = \{x^\mu, u\}$, $\hat\mu = 0,\dots,4$. This is the action of $D=5$ gravity coupled to an $\mathrm{SL}(2)/\mathrm{SO}(2)$ sigma model. It results from the consistent truncation of $D=11$ supergravity to the SO(6)-singlet modes around the round six-sphere $S^6 = \mathrm{SO}(7)/\mathrm{SO}(6)$. Splitting the 5D metric in the standard Kaluza-Klein fashion

$$G_{\hat\mu\hat\nu} = \begin{pmatrix} e^{-\Phi}g_{\mu\nu} + e^{2\Phi}A_\mu A_\nu & e^{2\Phi}A_\mu \\ e^{2\Phi}A_\mu & e^{2\Phi} \end{pmatrix}, \tag{3.32}$$

reinstating the truncation (3.7), together with inverting the change of coordinates (3.28) and fields (3.29), the Lagrangian (3.31) then yields back the ExFT Lagrangian (3.22), (3.23).

As seen above, solutions of type $\mathrm{AdS}_4\times\Sigma_7$ correspond to regular boundary behavior (at the endpoints of the interval $\theta=0$, $\theta=\pi$) for the fields $\{\phi,\phi_1,\chi_2\}$, which in turn will correspond to divergent boundary behavior of $\{\Phi,\Phi_1,\mathcal{X}_2\}$ at the endpoints of the interval $u\in[-1,1]$. In other words, in the $D=5$ reformulation (3.31) of this truncation, we need to identify particular singular solutions in order to describe a regular $\mathrm{AdS}_4\times\Sigma_7$ geometry.

2. Field equations

The reformulation (3.31) of the SO(7)-singlet sector of $D=11$ supergravity allows to quickly derive and further simplify the equations of motion. Let us first note that the global symmetry associated with the conserved charge V from (3.14) is nothing but the invariance of (3.30) under translations in u, i.e. corresponds to the conserved “energy” of this one-dimensional Lagrangian. Moreover, the field equations following from the Lagrangian (3.30) imply that

$$e^{-3\Phi-4\Phi_1}\partial_u\mathcal{X}_2 = F, \tag{3.33}$$

with some constant F, corresponding to equation (3.25) in the previous variables. In consequence, this equation can be used to eliminate $\mathcal{X}_2$ from the Lagrangian and arrive at

$$\mathcal{L}_{\text{pot,red}} = e^{-3\Phi}(3(\partial_u\Phi)^2 - 2(\partial_u\Phi_1)^2) + 2e^{3\Phi+4\Phi_1}F^2 + \frac{15}{2}e^{-\Phi-2\Phi_1}. \tag{3.34}$$

The conserved charge V from (3.14) then is simply given by

$$V = e^{-3\Phi}(3(\partial_u\Phi)^2 - 2(\partial_u\Phi_1)^2) - 2e^{3\Phi+4\Phi}{}_1F^2 - \frac{15}{2}e^{-\Phi-2\Phi_1}. \tag{3.35}$$

Let us spell out the field equations obtained from (3.34), however reexpressed in terms of the original fields ϕ and Δ, in order to better illustrate the boundary asymptotics

$$\begin{aligned}
0 &= 720\Delta^{-3/2}e^{\frac{7\phi}{2}} - 720\cos^2\theta - 36\sin(2\theta)(\Delta^{-1}\partial_\theta\Delta - 5\partial_\theta\phi) \\
&\quad + \sin^2\theta(-48\Delta^{-1}\partial_\theta^2\Delta - 45(\partial_\theta\phi)^2 + 90\Delta^{-1}\partial_\theta\Delta\partial_\theta\phi + 75\Delta^{-2}(\partial_\theta\Delta)^2 - 320F^2\Delta^3e^{3\phi}), \\
0 &= 80(\Delta^{-3/2}e^{\frac{7\phi}{2}} - 1) + 12\sin(2\theta)(-3\Delta^{-1}\partial_\theta\Delta + 7\partial_\theta\phi) \\
&\quad + \sin^2\theta(16\partial_\theta^2\phi - 33(\partial_\theta\phi)^2 + 18\Delta^{-1}\partial_\theta\Delta\partial_\theta\phi - 9\Delta^{-2}(\partial_\theta\Delta)^2 - 64F^2\Delta^3e^{3\phi} + 144).
\end{aligned} \tag{3.36}$$

In these fields, the conserved charge (3.35) takes the form

$$\begin{aligned}
V &= \frac{1}{32}e^{-3\phi}(18\Delta^{-1}\partial_\theta\Delta(\partial_\theta\phi - 4\cot(\theta)) - 9\Delta^{-2}\partial_\theta\Delta^2 + 15(\partial_\theta\phi - 4\cot(\theta))^2) \\
&\quad - \frac{15}{2}\Delta^{-3/2}e^{\phi/2}\csc^2\theta - 2F^2\Delta^3 = -\frac{3}{\ell_4^2}.
\end{aligned} \tag{3.37}$$

Expanding the system of ordinary differential equations (3.36) near the boundaries of the interval $\theta \in [0, \pi]$ exhibits the singularities. Imposing regularity of the solutions ϕ, Δ at the boundary requires both to be even functions in θ with the lowest coefficients in their Taylor expansion restricted by

$$\begin{aligned}
\Delta|_{\theta=0} &= e^{7\phi/3}|_{\theta=0}, \\
\partial_\theta^2\Delta|_{\theta=0} &= \frac{1}{33}[e^{7\phi/3}(36 + 81\partial_\theta^2\phi - 16e^{10\phi}F^2)]_{\theta=0}.
\end{aligned} \tag{3.38}$$

A solution regular at $\theta = 0$ thus is determined by two integration constants, which may be chosen to be $\phi(0)$ and $\partial_\theta^2\phi|_{\theta=0}$. A generic solution of this type will be singular at the other end $\theta = \pi$ of the interval. Inducing regularity at both endpoints of the interval thus reduces the set of solutions to a discrete set.

Before analysing possible regular solutions in more detail, let us note that we may recover two analytic solutions of the system (3.36), both corresponding to known solutions living within the consistent truncation to $\mathcal{N} = 8$ supergravity [21]

$$\begin{aligned}
&\mathrm{SO}(8): \ \phi = 0, \qquad \Delta = 1, \qquad F = \frac{3}{2}, \qquad \ell_4 = \frac{1}{2}, \\
&\mathrm{SO}(7)_+: \ \phi = -\frac{1}{4}\ln 5, \qquad \Delta = \frac{5^{1/12}}{(3 + 2\cos(2\theta))^{2/3}}, \\
&\qquad F = \frac{5^{3/4}}{2}, \qquad \ell_4 = \frac{3^{1/2}}{2 \times 5^{3/8}}.
\end{aligned} \tag{3.39}$$

The first solution is the round sphere S^7, the second one corresponds to the SO(7)-squashed S^7 found in [20].

C. G_2 truncation and uplift to $D = 11$

Having described in detail the simplification of the consistent truncation to SO(7)-singlets, eventually described by the simple $D = 5$ Lagrangian (3.34), we can now extend the discussion to the full sector of G_2-singlets. Recall, that the Lagrangian obtained from ExFT is given by (3.12), (3.13) in terms of six scalar fields parametrizing the coset space (3.5). Following the previous discussion, we apply the coordinate transformation (3.28) together with a field redefinition

$$\{\phi, \chi, \phi_1, \chi_{1a}, \chi_{1b}, \chi_2\} \longrightarrow \{\Phi, \mathcal{X}, \Phi_1, \mathcal{X}_A, \mathcal{X}_B, \mathcal{X}_2\}, \tag{3.40}$$

by a nonlinear θ-dependent $\mathrm{SL}(2) \times \mathrm{SU}(2,1)$ transformation, generalizing (3.29). After this redefinition, the action

(3.12) takes the compact form

$$\mathcal{L}_{\rm pot} = 3e^{-3\Phi}(\partial_u\Phi)^2 - 2e^{-3\Phi}(\partial_u\Phi_1)^2 - 2e^{-3\Phi-4\Phi_1}M_{mn}D_u\mathcal{X}^mD_u\mathcal{X}^n + \frac{15}{2}e^{-\Phi-2\Phi_1} - 6e^{-\Phi-4\Phi_1}\mathcal{X}_A^2, \tag{3.41}$$

with $\{\mathcal{X}^m\} = \{\mathcal{X}_A, \mathcal{X}_B, \mathcal{X}_2\}$, the matrix

$$M_{mn} = \begin{pmatrix} e^{2\Phi_1}+\mathcal{X}_B^2 & -\mathcal{X}_A\mathcal{X}_B & -\mathcal{X}_B \\ -\mathcal{X}_A\mathcal{X}_B & e^{2\Phi_1}+\mathcal{X}_A^2 & \mathcal{X}_A \\ -\mathcal{X}_B & \mathcal{X}_A & 1 \end{pmatrix}, \tag{3.42}$$

and the "covariant" derivatives D_u defined as

$$D_u\mathcal{X}_A = \partial_u\mathcal{X}_A, \qquad D_u\mathcal{X}_B = \partial_u\mathcal{X}_B - 3\mathcal{X}, \qquad D_u\mathcal{X}_2 = \partial_u\mathcal{X}_2 - 3\mathcal{X}\mathcal{X}_A. \tag{3.43}$$

The kinetic term (3.8) is invariant under the transformation (3.40). The full truncation to G_2-singlets can then be written in manifestly five-dimensional form upon extending (3.31) to

$$S = \int d^4x du\sqrt{|G_{(5)}|}\Big(\mathcal{L}_{\rm 5D,min} - 2\partial_{\hat\mu}\Phi_1\partial^{\hat\mu}\Phi_1 - 2e^{-4\Phi_1}M_{ij}D_{\hat\mu}\mathcal{X}^iD^{\hat\mu}\mathcal{X}^j + \frac{15}{2}e^{-2\Phi_1} - 6e^{-4\Phi_1}\mathcal{X}_A^2\Big), \tag{3.44}$$

with

$$D_{\hat\mu}\mathcal{X}_A = \partial_{\hat\mu}\mathcal{X}_A, \qquad D_{\hat\mu}\mathcal{X}_B = \partial_{\hat\mu}\mathcal{X}_B - 3A_{\hat\mu}, \qquad D_{\hat\mu}\mathcal{X}_2 = \partial_{\hat\mu}\mathcal{X}_2 - 3A_{\hat\mu}\mathcal{X}_A, \tag{3.45}$$

with $\{x^{\hat\mu}\} = \{x^\mu, u\}$, $\hat\mu = 0,\ldots,4$. The first term in (3.45) is the bosonic sector of minimal supergravity in $D = 5$, i.e. describes a vector field A_μ with $D = 5$ Chern-Simons term coupled to $D = 5$ gravity. The remaining part of (3.45) describes the coupling to one hypermultiplet with target space $SU(2,1)/U(1)$ and gauging of a shift isometry according to (3.43) (upon identification of $\mathcal{X}$ with the fifth component A_u of the gauge field). The Lagrangian (3.45) results from the consistent truncation of $D = 11$ supergravity to the G_2-singlets around the six-sphere $S^6 = G_2/SU(3)$. Its form is consistent with the fact that this truncation retains one $D = 5$ gravitino, thus describes the bosonic sector of a $D = 5$, $\mathcal{N} = 1$ supergravity.

In the search for AdS_4 solutions, we again impose (3.7) and require scalar fields to be constant in AdS_4 spacetime, such that the system is described by the ordinary differential equations obtained from variation of the Lagrangian (3.41). Variation with respect to $\mathcal{X}_2$ implies that

$$e^{-3\Phi-4\Phi_1}(\partial_u\mathcal{X}_2 + \mathcal{X}_A\partial_u\mathcal{X}_B - \mathcal{X}_B\partial_u\mathcal{X}_A - 6\mathcal{X}_A\mathcal{X}) = F, \tag{3.46}$$

with some constant F, generalizing equation (3.33). Moreover, the field $\mathcal{X}$ appears only algebraically in (3.41), entering the covariant derivatives (3.43). It can thus be eliminated by its own field equation

$$3\mathcal{X} = \partial_u\mathcal{X}_B + 2e^{3\Phi+2\Phi_1}F\mathcal{X}_A, \tag{3.47}$$

where we have already used (3.46) for simplification. Upon integrating out $\mathcal{X}$ and $\mathcal{X}_2$, we are thus left with the one-dimensional Lagrangian

$$\mathcal{L}_{\rm pot,red} = 3e^{-3\Phi}(\partial_u\Phi)^2 - 2e^{-3\Phi}(\partial_u\Phi_1)^2 - 2e^{-3\Phi-4\Phi_1}(\partial_u\mathcal{X}_A)^2 + \frac{15}{2}e^{-\Phi-2\Phi_1} - 6e^{-\Phi-4\Phi_1}\mathcal{X}_A^2 + 2e^{3\Phi+4\Phi_1}F^2 + 8e^{3\Phi+2\Phi_1}F^2\mathcal{X}_A^2, \tag{3.48}$$

describing all the equations that define an AdS_4 solution. In particular, we note that the field $\mathcal{X}_B$ has also disappeared from the Lagrangian, such that we are left with a system of three scalar fields. This is a remnant of the gauge freedom and the Higgs effect in the full $D = 11$ theory.

Before analyzing the field equations and their solutions, let us first spell out the uplift of the model to $D = 11$ dimensions. For the $D = 11$ metric, we have given the result in (3.16) above

$$ds^2 = \Delta^{-1}ds^2_{(4)} + e^{3\phi}\Delta^{-1}d\theta^2 + e^{-\frac{\phi}{2}}\sin^2\theta\Delta^{1/2}ds^2_{S^6}, \tag{3.49}$$

where ϕ and Δ are related to the fields of (3.48) via (3.29)

$$e^{\Phi} = \rho^{4/3}e^{\phi}, \qquad e^{\Phi_1} = \rho^{-1}e^{-3\phi/4}\Delta^{3/4}. \tag{3.50}$$

In particular, the determinant of the metric on the internal space is given by

$$\det g_{(7)} = \rho^{-4}\Delta^2\det g_{S^6}. \tag{3.51}$$

Similarly, one obtains the uplift for the $D = 11$ three-form $C_{(3)}$ and its field strength $F_{(4)}$ as

$$F_{(4)} = 4F\omega_{(4)} + dC_{(3)}, \qquad C_{(3)} = \frac{1}{6}\sin^4\theta A(c_{ijn}c_{kmn}y^mdy^idy^jdy^k - 4Fe^{3\phi/2}\Delta^{3/2}c_{ijk}y^idy^jdy^kd\theta), \tag{3.52}$$

after redefining

$$\mathcal{X}_A = \rho^{-4/3}A, \tag{3.53}$$

and where F is the constant introduced in (3.46). The result is given in terms of the embedding coordinates y^i, $i = 1, \ldots 7$, of the round S^6, $y^i y^i = 1$, while c_{ijk} is the unique totally antisymmetric cubic G_2-invariant tensor, normalized as $c_{ijk}c^{ijk} = 42$. $\omega_{(4)}$ is the AdS_4 volume form.

In turn, the *Ansatz* (3.49), (3.52) is the most general G_2-invariant *Ansatz* for an $AdS_4 \times \Sigma_7$ solution of $D = 11$ supergravity, after gauge fixing of the $D = 11$ tensor gauge symmetries. The metric (3.49) still has the full SO(7) isometry group, which is broken to G_2 by the three-form $C_{(3)}$. An early analysis of G_2-invariant compactifications [16] was restricted to a constant warp factor $\Delta = 1$, which leaves the system with only two solutions, denoted as SO(8) and $SO(7)_-$ below. The subsequent analysis of [17] allowed for a warp factor, but was restricted to solutions that live within the consistent truncation to $\mathcal{N} = 8$ supergravity. In particular, this implies $\phi = \text{const}$, and leaves the system with four solutions, given in the next subsection. Relaxing these restrictions, we will find new numerical solutions in Sec. III E below.

D. Field equations and analytic solutions

For the further analysis, we spell out the equations of motion derived from variation of (3.48) in terms of the coordinate θ and the fields (3.50), (3.53) which directly feature in the expressions for the $D = 11$ fields. Explicitly, these equations are given by

$$
\begin{aligned}
0 = {}& \sin^2\theta(-48\Delta^{-1}\partial_\theta^2\Delta - 45(\partial_\theta\phi)^2 + 90\Delta^{-1}\partial_\theta\Delta\partial_\theta\phi + 75\Delta^{-2}(\partial_\theta\Delta)^2 - 320F^2\Delta^3 e^{3\phi}) \\
&+ 720\Delta^{-3/2}e^{\frac{7\phi}{2}} - 720\cos^2\theta - 36\sin(2\theta)(\Delta^{-1}\partial_\theta\Delta - 5\partial_\theta\phi) \\
&- 16e^{3\phi/2}\Delta^{-3}\sin^2\theta(\Delta^{3/2}(4A\cos\theta + \partial_\theta A\sin\theta)^2 + 84e^{7\phi/2}A^2 + 16e^{3\phi}F^2\Delta^{9/2}A^2\sin^2\theta), \\
0 = {}& \sin^2\theta(16\partial_\theta^2\phi - 33(\partial_\theta\phi)^2 + 18\Delta^{-1}\partial_\theta\Delta\partial_\theta\phi - 9\Delta^{-2}(\partial_\theta\Delta)^2 - 64F^2\Delta^3e^{3\phi} + 144) \\
&+ 80(\Delta^{-3/2}e^{\frac{7\phi}{2}} - 1) + 12\sin(2\theta)(-3\Delta^{-1}\partial_\theta\Delta + 7\partial_\theta\phi) \\
&- 16e^{3\phi/2}\Delta^{-3}\sin^2\theta(\Delta^{3/2}(4A\cos\theta + \partial_\theta A\sin\theta)^2 + 4e^{7\phi/2}A^2 + 16e^{3\phi}F^2\Delta^{9/2}A^2\sin^2\theta), \\
0 = {}& \sin^2\theta(-2\partial_\theta^2 A + 3\partial_\theta A(\partial_\theta\phi + \Delta^{-1}\partial_\theta\Delta) + 32A(1 - e^{3\phi}F^2\Delta^3)) \\
&+ \sin(2\theta)(6A(\partial_\theta\phi + \Delta^{-1}\partial_\theta\Delta) - 8\partial_\theta A) + 24A(e^{7\phi/2}\Delta^{-3/2} - 1).
\end{aligned} \tag{3.54}
$$

The system admits a conserved charge, corresponding to the invariance of the system (3.48) under translations in u, originally given in (3.14) and related to the AdS_4 radius by (3.15). In terms of the fields $\{\phi, \Delta, A\}$, it takes the explicit form

$$
\begin{aligned}
V = {}& \frac{1}{32}e^{-3\phi}(18\Delta^{-1}\partial_\theta\Delta(\partial_\theta\phi - 4\cot(\theta)) - 9\Delta^{-2}\partial_\theta\Delta^2 + 15(\partial_\theta\phi - 4\cot(\theta))^2) \\
&- \frac{15}{2}\Delta^{-3/2}e^{\phi/2}\csc^2\theta - 2F^2\Delta^3 - \frac{1}{2}e^{-3\phi/2}\Delta^{-3/2}(4A\cos\theta + \partial_\theta A\sin\theta)^2 \\
&+ 2(3e^{2\phi}\Delta^{-3} - 4e^{3\phi/2}\Delta^{3/2}F^2\sin^2\theta)A^2 = -\frac{3}{\ell_4^2},
\end{aligned} \tag{3.55}
$$

generalizing (3.37) to the full G_2 truncation. One may check explicitly, that V is conserved, $\partial_\theta V = 0$, as a consequence of the equations (3.54).

Equations (3.54) are invariant under the scaling symmetry

$$
e^\phi \to \lambda^3 e^\phi, \qquad \Delta \to \lambda^7\Delta, \qquad A \to \lambda^3 A, \qquad F \to \lambda^{-15}F, \qquad \lambda \in \mathbb{R}^*, \tag{3.56}
$$

with constant λ. This is the trombone symmetry of $D = 11$ supergravity [38], under which the AdS_4 radius ℓ_4 (3.55) scales as

$$
\ell_4 \to \lambda^{9/2}\ell_4. \tag{3.57}
$$

For the subsequent numerical analysis, we fix this scaling symmetry (3.56) to set the constant F from (3.46) to

$$
F = \frac{3}{2}. \tag{3.58}
$$

All previously known solutions to the equations (3.54) are analytic, have constant ϕ, and live within the consistent truncation to $\mathcal{N} = 8$ supergravity [21]. They correspond to the four G_2-invariant extremal points of the scalar potential [22]. In our conventions, in particular after having fixed the scaling symmetry by (3.58), they take the form

$$\begin{aligned}
\mathrm{SO}(8):&\ e^{\phi}=1,\quad \Delta=1,\quad A=0,\\
\mathrm{SO}(7)_+:&\ e^{\phi}=3^{-1/5}5^{-1/10},\quad \Delta=5^{13/30}3^{-7/15}(1+4\cos^2\theta)^{-2/3},\quad A=0,\\
\mathrm{SO}(7)_-:&\ e^{\phi}=2^{1/5}3^{-1/5},\quad \Delta=2^{7/15}3^{-7/15},\quad A=2^{-4/5}3^{-1/5},\\
\mathrm{G}_2:&\ e^{\phi}=3^{-3/10},\quad \Delta=3^{-1/30}(1+2\cos^2\theta)^{-2/3},\quad A=3^{1/5}5^{-1/2}(1+2\cos^2\theta).
\end{aligned} \tag{3.59}$$

E. New numerical solutions

In the rest of this paper, we will discuss the equations of motion (3.54) and their solutions by numerical analysis. To this end, we go back to coordinate w from (3.2). As we have already discussed for the SO(7) subsector, cf. (3.38) above, the system of second order differential equations (3.54) is singular at the boundary of the interval $\theta\in[0,\pi]$, i.e. $w=\pm1$. As a consequence, for a regular solution only three of the (*a priori* six) initial conditions can be chosen freely at $w=1$, and we choose these to be

$$\phi(1)\equiv\mathfrak{q},\qquad \phi'(1)\equiv\mathfrak{p},\qquad A(1)\equiv\mathfrak{a}. \tag{3.60}$$

Throughout this section, primes refer to derivatives with respect to w: $\phi'=\partial_w\phi$, etc.. Regularity of the solution at $w=1$ then determines the next coefficients in the respective Taylor expansions

$$\begin{aligned}
&\Delta(1)=e^{7\mathfrak{q}/3},\\
&\Delta'(1)=\frac{1}{33}e^{\mathfrak{q}/3}(80\mathfrak{a}^2+16F^2e^{12\mathfrak{q}}+9e^{2\mathfrak{q}}(9\mathfrak{p}-4)),\\
&A'(1)\to\frac{4}{99}\mathfrak{a}e^{-2\mathfrak{q}}(-20\mathfrak{a}^2+40F^2e^{12\mathfrak{q}}+e^{2\mathfrak{q}}(54\mathfrak{p}-35)),
\end{aligned} \tag{3.61}$$

and similarly, all higher coefficients in the Taylor expansion are fixed by expanding the equations (3.54). For generic choice of the boundary conditions (3.60), the solution will however be singular at the other endpoint $w=-1$ of the interval, or even diverge before reaching the endpoint. Further imposing regularity at the opposite boundary $w=-1$ thus imposes three (highly nonlinear) relations among the parameters (3.60) such that a naive counting argument indicates that the system allows for only a discrete set of regular solutions. Indeed, that is what we observe in the following.

Let us also note that the cosmological constant (3.55) is given as a function of the boundary conditions (3.60) as

$$\ell_4^2=\frac{66e^{5\mathfrak{q}}}{180\mathfrak{a}^2+80F^2e^{12\mathfrak{q}}-21e^{2\mathfrak{q}}(9\mathfrak{p}-4)}. \tag{3.62}$$

In the following numerical analysis, we will separate the cases $A=0$, which amounts to truncating to the subsector of SO(7)-singlets discussed in Sec. III B, and $A\neq0$. In total, we find three new numerical solutions on top of the known analytic solutions (3.59). Our findings are summarized in Table II.

1. A = 0

We first discuss the subsector with $A=0$, which is a consistent truncation of the system (3.54), corresponding to the subsector of SO(7)-singlets discussed in Sec. III B. We then scan the two-dimensional parameter space of initial conditions $\{\mathfrak{q},\mathfrak{p}\}$ for solutions regular at $w=1$, searching

TABLE II. List of regular G_2-invariant solutions of the system (3.54). The first four solutions live within the consistent truncation to $\mathcal{N}=8$ supergravity [21] and correspond to the four G_2-invariant extremal points of the scalar potential [22]. They have been known before and can be given in analytic form (3.59). The three last lines are the new numerical solutions. All digits displayed are within the numerical accuracy.

Solution	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{a}$	ℓ_4	Comments
SO(8)	0	0	0	0.500000	[18], $\mathcal{N}=8$, round S^7
$\mathrm{SO}(7)_-$	−0.0810930	0	0.461054	0.497590	[19], “parallelized” S^7
$\mathrm{SO}(7)_+$	−0.380666	0	0	0.489270	[20]
G_2	−0.329584	0	0.185703	0.489049	[17], $\mathcal{N}=1$
$\mathrm{SO}(7)'$	−0.250533	−0.137962	0	0.499467	New, preserves SO(7)
G_2'	−0.202438	−0.105189	0.225857	0.504244	New
G_2''	0.0544548	0.892275	0.658650	0.512668	New

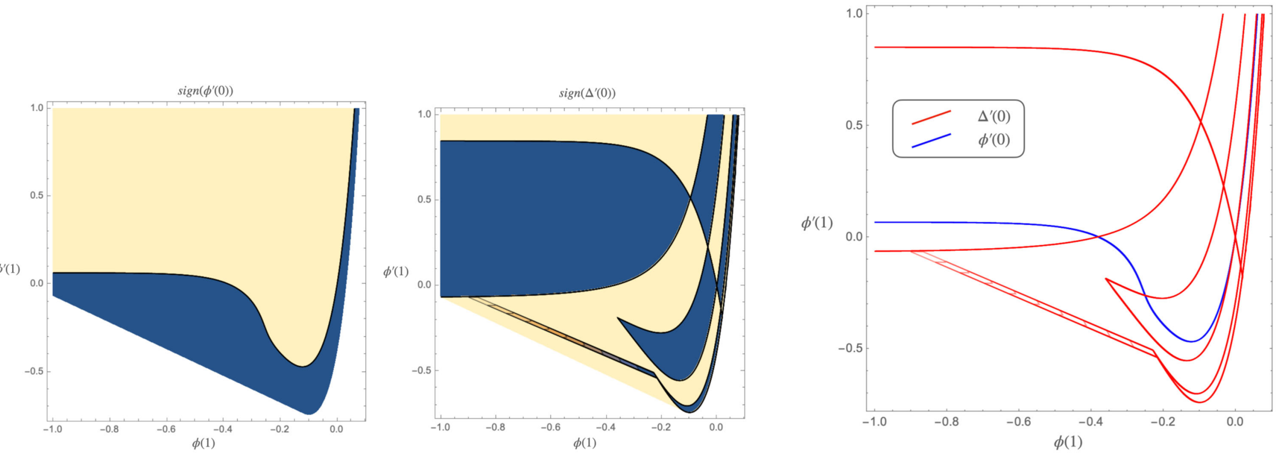

FIG. 1. Initial values for the regular solutions in the SO(7) system with vanishing $A = 0$. The first two plots show the regions in the two-dimensional parameter space $(\mathfrak{q}, \mathfrak{p})$ in which the signs of $\phi'(0)$ and $\Delta'(0)$ are positive (yellow) and negative (blue), respectively. The third plot extracts the lines of vanishing $\phi'(0)$ (blue) and vanishing $\Delta'(0)$ (red). The three intersection points of the red and blue lines in this plot correspond to the solutions SO(8), $\mathrm{SO}(7)_+$, and $\mathrm{SO}(7)'$ from Table II.

for solutions regular throughout the interval $w \in [-1, 1]$. While regularity at $w = -1$ is hard to control, we note that the problem can be simplified for even solutions satisfying

$$\phi(-w) = \phi(w), \qquad \Delta(-w) = \Delta(w), \tag{3.63}$$

corresponding to a $\mathbb{Z}_2$-symmetry of the system (3.54). Starting from a solution regular at $w = 1$, we search for initial conditions such that the solution satisfies

$$\phi'(0) = 0 = \Delta'(0), \tag{3.64}$$

at $w = 0$. This implies the symmetry (3.63) and regularity at the other endpoint $w = -1$ becomes a simple consequence of this symmetry. The conditions (3.64) can be straightforwardly implemented into a numerical search. For a regular solution to exist, however, both conditions (3.64) must hold exactly, not just approximately.

To this end, we first identify the lines in the parameter space of initial conditions $(\mathfrak{q}, \mathfrak{p})$, along which $\phi'(0)$ and $\Delta'(0)$ vanish separately. Even before optimizing the numerical accuracy, we can infer the existence of such lines by identifying the regions in parameter space in which the signs of $\phi'(0)$ and $\Delta'(0)$ are positive and negative, respectively. Concretely, we depict in the first plot of Fig. 1 the yellow region in which $\phi'(0)$ is positive and the blue region in which $\phi'(0)$ is negative. The interface between the two regions then defines a line along which $\phi'(0)$ vanishes. In the second plot of Fig. 1, we depict the analogous information for $\Delta'(0)$. We then extract the lines of vanishing $\phi'(0)$ (blue) and vanishing $\Delta'(0)$ (red) in the third plot, which shows the existence of three intersection points at which both conditions (3.64) are satisfied.[9] Once, we have established the existence of such intersection points, we can work on improving the numerical accuracy of the corresponding solutions. Two of these points correspond to the known SO(8), and $\mathrm{SO}(7)_+$ solutions from (3.59), the third one represents a new SO(7)-invariant solution, which we will denote as $\mathrm{SO}(7)'$. We plot the fields ϕ and Δ for the new numerical solution in Fig. 2. Extending the search along the blue line of vanishing $\phi'(0)$, we find that there are no other intersection with any red lines, i.e. no other solution to (3.64) in the parameter space.

2. $A \neq 0$

We now extend the search of solutions to the full truncation of G_2-invariant singlets, i.e. we allow for non-vanishing A. In this case we scan the three-dimensional parameter space (3.60) for regular solutions. Similar to our discussion of the SO(7) sector, we start by restricting the search to even solutions

$$\phi(-w) = \phi(w), \qquad \Delta(-w) = \Delta(w), \qquad A(-w) = A(w), \tag{3.65}$$

again corresponding to a $\mathbb{Z}_2$-symmetry of the system (3.54). Accordingly, starting from a solution regular at $w = 1$, the symmetry (3.65) is implemented by the following conditions

[9] A better resolution of the hatched zone in the second plot of Fig. 1 would require to improve the numerical accuracy. However, the third plot shows that this region is not close to any blue line, thus irrelevant for the search of solutions.

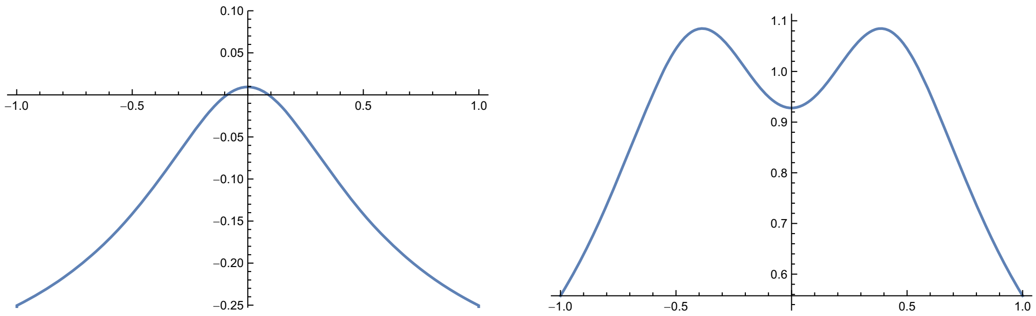

FIG. 2. New solution SO(7)’: fields ϕ and Δ as functions of w.

$$\phi'(0) = 0 = \Delta'(0) = A'(0). \tag{3.66}$$

at $w = 0$. Regularity at the other endpoint $w = -1$ is then implied by the symmetry (3.65). Consequently, we scan the three-dimensional parameter space for points where all three conditions (3.66) hold exactly. Generalizing the analysis of Sec. III E 1, we study the intersections of the hyperplanes, defined by the vanishing of $\phi'(0)$, $\Delta'(0)$, and $A'(0)$, respectively. However, this analysis reveals only the known solutions $SO(7)_-$ and G_2, listed in Table II and (3.59).

Next, we employ another $\mathbb{Z}_2$-symmetry of the system (3.54): $A \to \pm A$, and search for solutions in which ϕ and Δ are even whereas A is odd in w

$$\phi(-w) = \phi(w), \qquad \Delta(-w) = \Delta(w),$$
$$A(-w) = -A(w). \tag{3.67}$$

Similar to (3.66), the symmetry (3.67) can be implemented by the following conditions

$$\phi'(0) = 0 = \Delta'(0) = A(0), \tag{3.68}$$

at $w = 0$, which in turn implies regularity throughout the interval. We search for such solutions with the same method described above. In Fig. 3, we have depicted two slices in the three-dimensional parameter space, defined by fixed neighbored values $\mathfrak{a}_1$, $\mathfrak{a}_2$, of $\mathfrak{a}$. In each slice we plot the three curves defined by the vanishing of $\phi'(0)$, $\Delta'(0)$, and $A(0)$, respectively. The configuration of the lines shows that on some intermediate slice $\mathfrak{a}_1 < \mathfrak{a} < \mathfrak{a}_2$ there must be a common intersection point of the three lines. Having established its existence, we can then zoom in and optimize the numerical accuracy of the solution. The result is the new solution called G_2' in Table II. The corresponding profiles of the fields ϕ, Δ and A are plotted in Fig. 4.

It remains to extend the analysis to the full parameter space by systematically scanning the two-dimensional slices of fixed $\mathfrak{a}$. Zooming into a different area in parameter space, we have also identified the slices shown in Fig. 5. Again, the configuration of lines indicates the existence of an intermediate slice with a common intersection of all three lines, thus another exact solution to (3.68). The resulting solution is given as G_2'' in Table II. The corresponding profiles of the fields ϕ, Δ and A are plotted in Fig. 6.

We have further gone through the slices of the three-dimensional parameter space and not found any other critical region that would indicate another solution. Although we have not attempted a rigorous proof, the

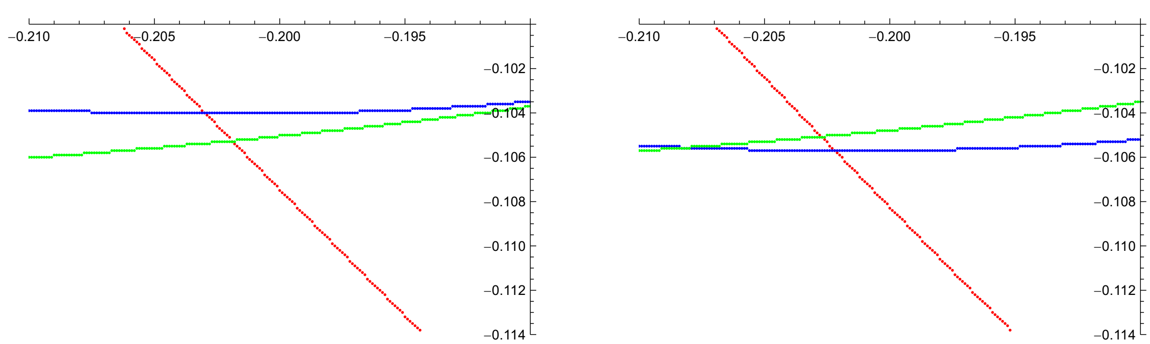

FIG. 3. Lines of vanishing $\phi'(0)$ (red), vanishing $\Delta'(0)$ (blue) and vanishing $A(0)$ (green) on slices in the parameter space of initial conditions, with $\mathfrak{q}$ on the horizontal axis and $\mathfrak{p}$ on the vertical axis. The two slices are given at the values $\mathfrak{a}_1 = -0.226667$ (left) and $\mathfrak{a}_2 = -0.225417$, respectively. The common intersection of red, blue and green line, which must appear on some intermediate slice, corresponds to the solution G_2'.

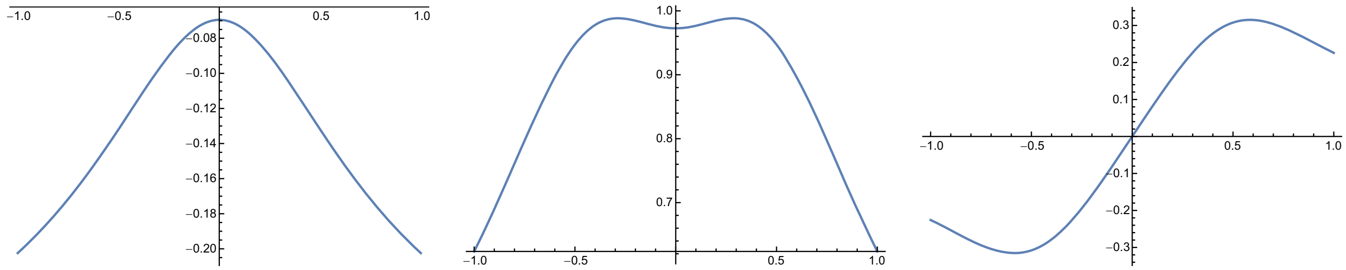

FIG. 4. New solution G_2': fields ϕ, Δ, and A as functions of w.

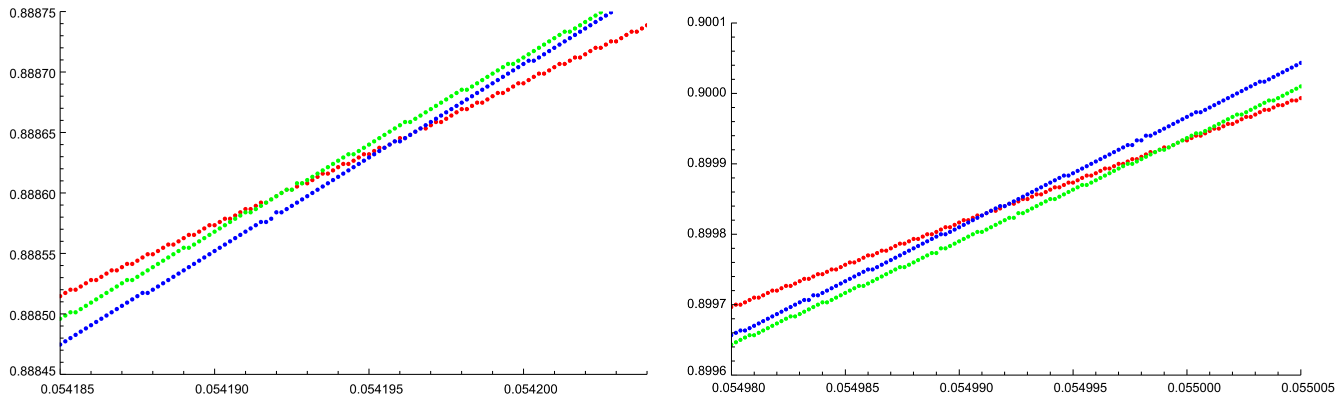

FIG. 5. Lines of vanishing $\phi'(0)$ (red), vanishing $\Delta'(0)$ (blue) and vanishing $A(0)$ (green) on slices in the parameter space of initial conditions, with $\mathfrak{q}$ on the horizontal axis and $\mathfrak{p}$ on the vertical axis. The two slices are given at the values $\mathfrak{a} = 0.658$ (left) and $\mathfrak{a} = 0.66$, respectively. The common intersection of red, blue and green line, which must appear on some intermediate slice, would correspond to the solution G_2''.

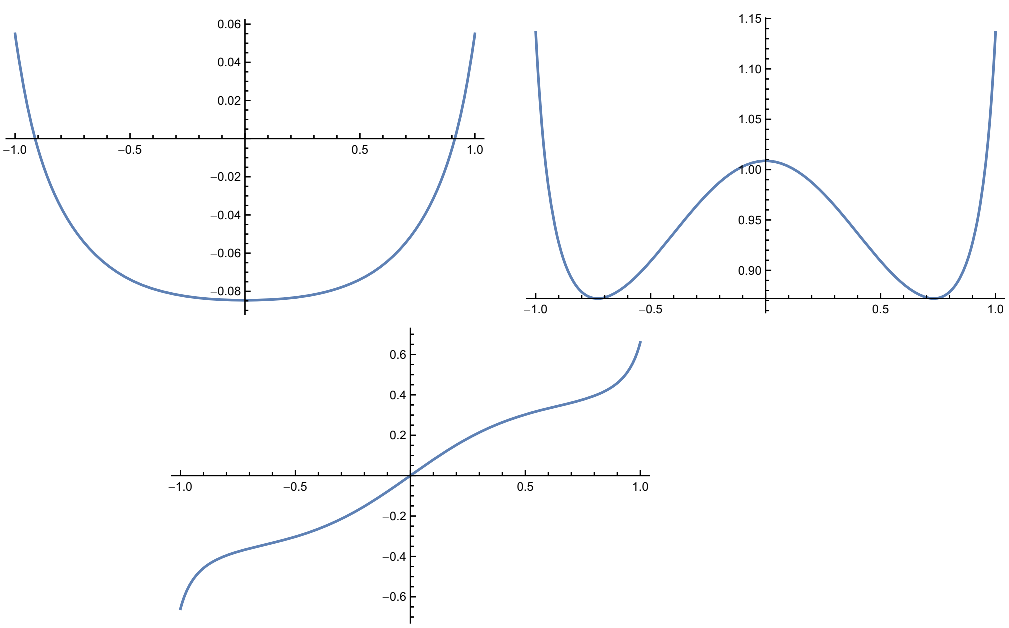

FIG. 6. New solution G_2'': fields ϕ, Δ, and A as functions of w.

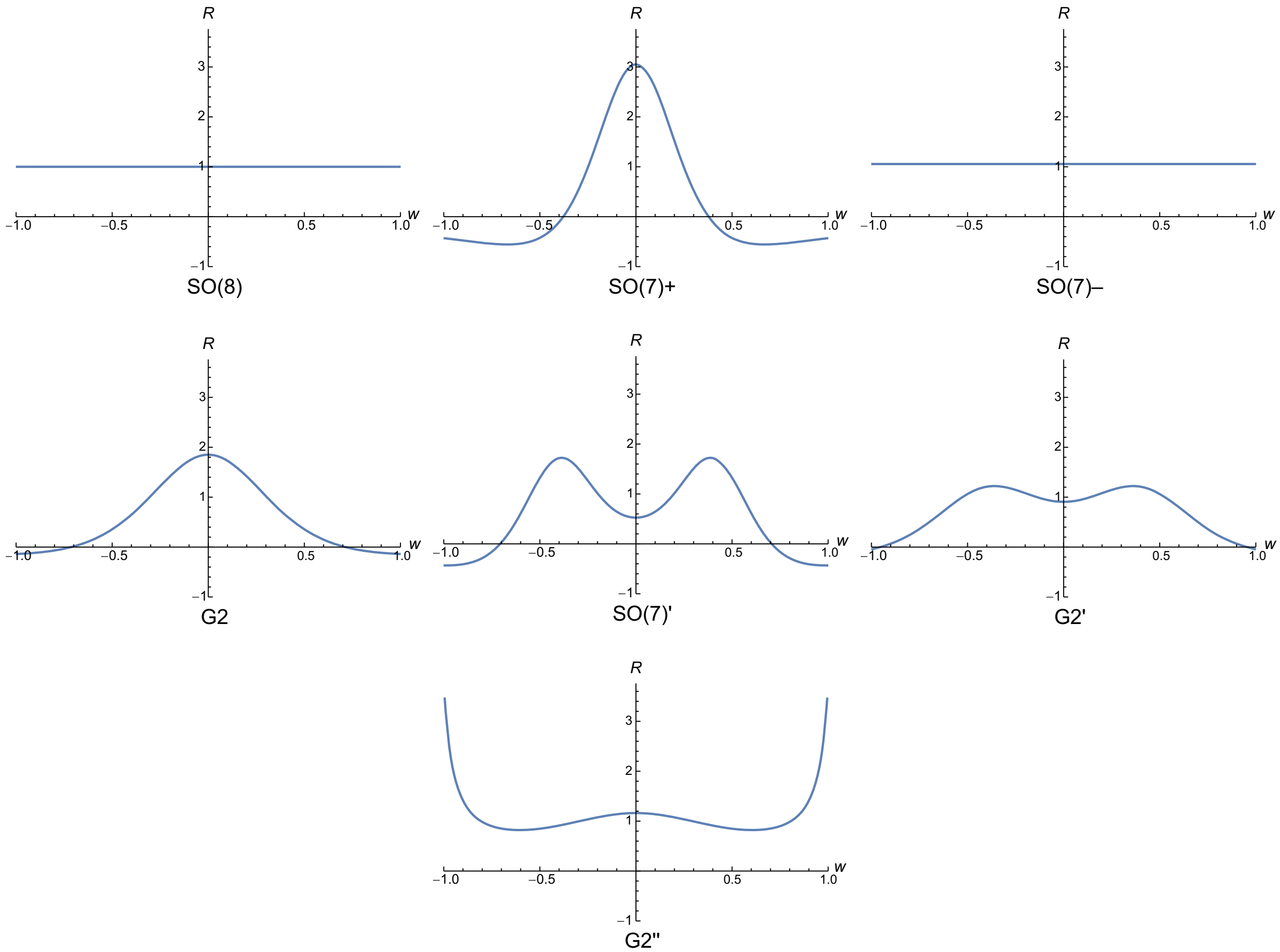

FIG. 7. Curvature scalar R_7 of the internal deformed S^7 as a function of $w \in [-1, 1]$ for the different G_2-invariant solutions collected in Table II.

analysis suggests that the set of regular solutions given in Table II is complete, if one restricts to even (3.66) and odd (3.68) solutions. Relaxing the latter conditions, one may expect yet more regular solutions, but we have not explored this systematically.

Let us recall that the spacetime geometry for all of them is of the form $AdS_4 \times \Sigma_7$, c.f. (3.49), where the internal space Σ_7 is given by a squashed seven sphere preserving $SO(7)$ isometries. In order to characterize the different geometries, we compute the curvature scalar R_7 of the internal manifold Σ_7. From (3.49), and using the equations of motion (3.54) and (3.55) to simplify the expression, we obtain the following expression

$$\begin{aligned} R_7 = &\, 6e^{-3\phi/2}\Delta^{-1/2}(4wA-(1-w^2)\partial_w A)^2 \\ &+96F^2(1-w^2)e^{3\phi/2}\Delta^{5/2}A^2+72e^{2\phi}\Delta^{-2}A^2 \\ &+40F^2\Delta^4-\frac{12\Delta}{\ell_4^2}-3(1-w^2)e^{-3\phi}\Delta^{-1}(\partial_w\Delta)^2, \end{aligned} \quad (3.69)$$

in terms of the fields ϕ, Δ, and A. As an illustration, we may plot the resulting function for the different solutions of Table II, which is displayed in Fig. 7.

F. Numerics

In the previous section, we have established the existence of regular solutions at certain discrete points in the three-dimensional parameter space. For each solution, once we have proven its existence, we can zoom in to improve the numerical accuracy. To this end, we have finally implemented a simple gradient descent algorithm in Python. As explained above, the problem is set by three initial conditions (3.60). Next, we define a regularization function, or loss function, to assess the regularity of a given solution. Put differently, this function quantifies how far a solution is from being regular.

As discussed above, for even and odd solutions, regularity is conveniently encoded in the conditions (3.68) and (3.68), respectively. Accordingly, we can define the loss function as

$$\mathcal{L} = \ln(\phi'(0)^2 + \Delta'(0)^2). \tag{3.70}$$

when $\mathfrak{a} = 0$, and as

$$\mathcal{L} = \ln(\phi'(0)^2 + \Delta'(0)^2 + A'(0)^2),$$
$$\mathcal{L} = \ln(\phi'(0)^2 + \Delta'(0)^2 + A(0)^2). \tag{3.71}$$

for even (3.65) and odd (3.67) solutions, respectively.

With these loss functions in place, we can perform gradient descent by updating the initial parameters according to

$$(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}) \longleftarrow (\mathfrak{p}, \mathfrak{q}, \mathfrak{a}) - \alpha \nabla \mathcal{L}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}) \tag{3.72}$$

where α is the learning rate, controlling the step size in the gradient descent. This method allows us to verify and refine the previous analysis, enabling a fine-tuning of the initial parameters. The results are collected in Table II where all numbers are accurate to the displayed digits.

IV. CONCLUSIONS

In this paper, we have discussed the consistent truncations to K-singlets with respect to a subgroup K of the isometry group of the internal manifold. We have reviewed how these truncations are described in the framework of generalized geometry and exceptional field theory. As an application, we have worked out the field equations for the most general G_2-invariant AdS_4 solution of $D = 11$ supergravity, with the internal space Σ_7 given by a squashed seven-sphere preserving SO(7) isometries. The ExFT description of this truncation features a scalar sector described by the six-dimensional coset space $(SU(2,1) \times SU(1,1))/(U(2) \times U(1))$ with all scalars still depending on an extra coordinate θ. The latter encodes the description of the infinite Kaluza-Klein towers of G_2-singlets within a four-dimensional field theory. Searching for AdS_4 vacua, we have shown that the system can be simplified to a set of three second-order ordinary differential equations for three scalar fields. Furthermore, we have given the explicit uplift of this sector to $D = 11$ dimensions.

Imposing a compact internal seven-dimensional space restricts the search to solutions regular at the endpoints of the interval $\theta \in [0, \pi] = \mathcal{I}$, with the seven-sphere represented as a foliation of $S^6 = G_2/SU(3)$ over the interval $\mathcal{I}$. The equations of motion are singular at these endpoints and a closer inspection shows that only a discrete set of such regular solutions exists. More precisely, solutions that are regular at one endpoint $\theta = 0$ are characterized by a three-dimensional parameter space. Imposing regularity throughout the interval defines discrete points in this space. We conduct a numerical scan for these points. Importantly, we find that the condition of regularity can be very efficiently implemented by requiring the solutions to be even/odd according to (3.65) or (3.67), such that regularity at the opposite endpoint follows from symmetry. In this sector, we recover in particular the four solutions that were previously known in analytic form [17]. These all live within the consistent truncation to $\mathcal{N} = 8$ supergravity [21] and correspond to the four G_2-invariant extremal points of its scalar potential [22]. On top of these known solutions, we identify three new numerical regular solutions, which we label as $SO(7)'$, G_2', and G_2'', respectively. They all uplift to $D = 11$ geometries of the form $AdS_4 \times \Sigma_7$ together with a nonvanishing three-form flux which preserves $G_2 \subset SO(7)$ symmetry. All these solutions are collected in Table II. Within the sector of even/odd solutions satisfying (3.65) or (3.67), the analysis appears to be complete. Relaxing these additional conditions, one may expect yet more regular solutions, and it would be highly interesting to extend the numerical search to be able to identify all the regular solutions of the system.

The embedding of the new solutions into the ExFT framework allows to directly extract the generalized frames associated to these backgrounds. In turn, that should allow to adapt the techniques of [4–7] for a computation of the Kaluza-Klein spectra around these new backgrounds. It would be particularly interesting to find if supersymmetry is preserved by any of these backgrounds.

Remarkably, most of the solutions we have identified already live within the consistent truncation to $\mathcal{N} = 8$ supergravity. I.e. they only require nonvanishing scalar fields from the lowest Kaluza-Klein multiplet. Allowing for nonvanishing scalars among the infinitely many higher Kaluza-Klein modes somewhat surprisingly only gives rise to three new AdS_4 solutions in this sector. In turn, it is then tempting to speculate that these new solutions might also be related to some particular consistent truncations of the full theory. It may be worth noting that the ω-deformed maximal supergravities constructed in [39] do admit additional G_2-invariant vacua while retaining the $\mathcal{N} = 8$ vacuum of the round sphere [39–41]. Yet, it probably is wishful musing to imagine that these theories might play a role in the description of the new vacua. While that would certainly be an exceptional turn of events, we leave these questions and others for future studies.

ACKNOWLEDGMENTS

We would like to thank Franz Ciceri, Camille Eloy, Hermann Nicolai and Ergin Sezgin for stimulating discussions. The work of M. G. has been supported by the Australian Research Council (ARC) Future Fellowship FT180100353, the ARC Discovery Project DP240101409, and by a Capacity Building Package of the University of Queensland.

DATA AVAILABILITY

No data were created or analyzed in this study.

---

[1] K. Lee, C. Strickland-Constable, and D. Waldram, Spheres, generalised parallelisability and consistent truncations, Fortschr. Phys. **65**, 1700048 (2017).

[2] O. Hohm and H. Samtleben, Consistent Kaluza-Klein truncations via exceptional field theory, J. High Energy Phys. 01 (2015) 131.

[3] D. Cassani, G. Josse, M. Petrini, and D. Waldram, Systematics of consistent truncations from generalised geometry, J. High Energy Phys. 11 (2019) 017.

[4] E. Malek and H. Samtleben, Kaluza-Klein spectrometry for supergravity, Phys. Rev. Lett. **124**, 101601 (2020).

[5] E. Malek and H. Samtleben, Kaluza-Klein spectrometry from exceptional field theory, Phys. Rev. D **102**, 106016 (2020).

[6] M. Cesàro and O. Varela, Kaluza-Klein fermion mass matrices from exceptional field theory and $\mathcal{N} = 1$ spectra, J. High Energy Phys. 03 (2021) 138.

[7] B. Duboeuf, E. Malek, and H. Samtleben, Kaluza-Klein spectrometry beyond consistent truncations: The squashed S^7, J. High Energy Phys. 04 (2023) 062.

[8] B. Duboeuf, M. Galli, E. Malek, and H. Samtleben, Holographic RG flow from the squashed to the round S^7, Phys. Rev. D **108**, 086002 (2023).

[9] M. J. Duff and C. N. Pope, Consistent truncations in Kaluza-Klein theories, Nucl. Phys. **B255**, 355 (1985).

[10] C. D. A. Blair, M. Pico, and O. Varela, Infinite and finite consistent truncations on deformed generalised parallelisations, J. High Energy Phys. 09 (2024) 065.

[11] M. A. Awada, M. J. Duff, and C. N. Pope, $N = 8$ supergravity breaks down to $N = 1$, Phys. Rev. Lett. **50**, 294 (1983).

[12] M. J. Duff, B. E. W. Nilsson, and C. N. Pope, Spontaneous supersymmetry breaking by the squashed seven sphere, Phys. Rev. Lett. **50**, 2043 (1983); **51**, 846(E) (1983).

[13] D. Cassani, G. Dall'Agata, and A. F. Faedo, Type IIB supergravity on squashed Sasaki-Einstein manifolds, J. High Energy Phys. 05 (2010) 094.

[14] K. Skenderis, M. Taylor, and D. Tsimpis, A consistent truncation of IIB supergravity on manifolds admitting a Sasaki-Einstein structure, J. High Energy Phys. 06 (2010) 025.

[15] J. P. Gauntlett and O. Varela, Universal Kaluza-Klein reductions of type IIB to $N = 4$ supergravity in five dimensions, J. High Energy Phys. 06 (2010) 081.

[16] M. Gunaydin and N. P. Warner, The G_2 invariant compactifications in eleven-dimensional supergravity, Nucl. Phys. **B248**, 685 (1984).

[17] B. de Wit, H. Nicolai, and N. Warner, The embedding of gauged $N = 8$ supergravity into $d = 11$ supergravity, Nucl. Phys. **B255**, 29 (1985).

[18] M. J. Duff and C. N. Pope, Kaluza-Klein supergravity and the seven-sphere, in *Supersymmetry and Supergravity '82*, edited by S. Ferrara, J. Taylor, and P. van Nieuwenhuzen (World Scientific, Singapore, 1983), pp. 183–228.

[19] F. Englert, Spontaneous compactification of eleven-dimensional supergravity, Phys. Lett. **119B**, 339 (1982).

[20] B. de Wit and H. Nicolai, A new SO(7) invariant solution of $d = 11$ supergravity, Phys. Lett. **148B**, 60 (1984).

[21] B. de Wit and H. Nicolai, $N = 8$ supergravity, Nucl. Phys. **B208**, 323 (1982).

[22] N. P. Warner, Some new extrema of the scalar potential of gauged $N = 8$ supergravity, Phys. Lett. **128B**, 169 (1983).

[23] O. Hohm and H. Samtleben, Exceptional form of $D = 11$ supergravity, Phys. Rev. Lett. **111**, 231601 (2013).

[24] O. Hohm and H. Samtleben, Exceptional field theory. II. $E_{7(7)}$, Phys. Rev. D **89**, 066017 (2014).

[25] B. de Wit and H. Nicolai, The consistency of the S^7 truncation in $D = 11$ supergravity, Nucl. Phys. **B281**, 211 (1987).

[26] B. Duboeuf, E. Malek, and H. Samtleben, Cubic and higher-order supergravity couplings for AdS vacua using exceptional field theory, J. High Energy Phys. 05 (2024) 214.

[27] nLab, Coset space structure on n-spheres—table, https://ncatlab.org/nlab/show/coset+space+structure+on+n-spheres+–+table (2024).

[28] D. Cassani and P. Koerber, Tri-Sasakian consistent reduction, J. High Energy Phys. 01 (2012) 086.

[29] J. P. Gauntlett, S. Kim, O. Varela, and D. Waldram, Consistent supersymmetric Kaluza-Klein truncations with massive modes, J. High Energy Phys. 04 (2009) 102.

[30] C. Ahn and S.-J. Rey, Three-dimensional CFTs and RG flow from squashing M2-brane horizon, Nucl. Phys. **B565**, 210 (2000).

[31] D. Cassani and A.-K. Kashani-Poor, Exploiting $\mathcal{N} = 2$ in consistent coset reductions of type IIA, Nucl. Phys. **B817**, 25 (2009).

[32] A.-K. Kashani-Poor, Nearly Kähler reduction, J. High Energy Phys. 11 (2007) 026.

[33] A. Coimbra, C. Strickland-Constable, and D. Waldram, $E_{d(d)} \times \mathbb{R}^+$ generalised geometry, connections and M theory, J. High Energy Phys. 02 (2014) 054.

[34] F. Englert and H. Nicolai, Supergravity in eleven-dimensional space-time, in *Proceedings of the XIIth*

International Colloquium on Group Theoretical Methods in Physics 1983, edited by G. Denardo, G. Ghirardi, and T. Weber, *Lect. Notes Phys.* 201 (Springer, New York, 1984), pp. 249–283.

[35] B. Biran, A. Casher, F. Englert, M. Rooman, and P. Spindel, The fluctuating seven sphere in eleven-dimensional supergravity, Phys. Lett. **134B**, 179 (1984).

[36] E. Sezgin, The spectrum of the eleven-dimensional supergravity compactified on the round seven sphere, Phys. Lett. **138B**, 57 (1984).

[37] A. Casher, F. Englert, H. Nicolai, and M. Rooman, The mass spectrum of supergravity on the round seven sphere, Nucl. Phys. **B243**, 173 (1984).

[38] E. Cremmer, H. Lu, C. N. Pope, and K. S. Stelle, Spectrum generating symmetries for BPS solitons, Nucl. Phys. **B520**, 132 (1998).

[39] G. Dall'Agata, G. Inverso, and M. Trigiante, Evidence for a family of $SO(8)$ gauged supergravity theories, Phys. Rev. Lett. **109**, 201301 (2012).

[40] A. Borghese, A. Guarino, and D. Roest, All G_2 invariant critical points of maximal supergravity, J. High Energy Phys. 12 (2012) 108.

[41] D. Berman, T. Fischbacher, G. Inverso, and B. Scellier, Vacua of ω-deformed $\mathrm{SO}(8)$ supergravity, J. High Energy Phys. 06 (2022) 133.