Supergravity without gravity and its BV formulation

Julian Kupka,* Charles Strickland-Constable,† and Fridrich Valach‡

Department of Physics, Astronomy and Mathematics, University of Hertfordshire, College Lane, Hatfield, AL10 9AB, United Kingdom

(Received 4 October 2024; accepted 4 February 2025; published 21 February 2025)

The generalized-geometric formulation of 10-dimensional supergravity suggests a particular simple "limit," which results in a theory whose only dynamical degrees of freedom are the dilaton and the dilatino. The theory is still invariant both under generalized diffeomorphisms and a local supersymmetry and in many aspects is structurally similar to the original supergravity, which makes it a convenient playground for understanding more subtle aspects of the full physical setup. In particular, the simplicity and the geometric nature of the dilatonic theory allow us to build a full Batalin-Vilkovisky (BV) extension to all orders in the fermionic variables.

DOI: 10.1103/PhysRevD.111.046020

I. INTRODUCTION

String theory famously features an enormous amount of symmetries and dualities. Part of these is elegantly captured using the framework of generalized geometry [1–4] which, roughly speaking, amounts to the replacement

$$(\text{tangent bundle, Lie derivative}) \longrightarrow (\text{Courant algebroid, generalized Lie derivative}).$$

In this note we focus on performing a particular "limit" of the generalized-geometric formulation of the 10-dimensional $\mathcal{N}=1$ supergravity coupled to vector multiplets with a gauge group [5–7]. For special choices of the gauge group this is the two-derivative part of the heterotic or type I supergravity. After performing the above-mentioned limit the theory becomes topological, in the sense of not containing any (dynamical) metric degrees of freedom. Before describing the limit procedure, let us stress that this is not related to the twist of supergravity in the sense of Costello-Li [8].

It is known [9,10] that, (at least) up to the quadratic order in fermions, the 10-dimensional supergravity admits a generalized-geometric formulation in which the ordinary fields are naturally interpreted as

$$\text{metric} + B\text{-field} + \text{gauge field} + \text{dilaton} \longrightarrow \text{generalized metric}\, \mathcal{G} + \text{half-density}\, \sigma,$$

$$\text{gravitino} + \text{gaugino} + \text{dilatino} \longrightarrow \text{generalized gravitino}\, \psi + \text{generalized dilatino}\, \rho.$$

The supergravity action can be written, up to quadratic order in fermions, in the elegant form [9]

$$S_{\text{quad}} = \int \mathcal{R}(\mathcal{G},\sigma)\sigma^2 - \bar{\psi}_{\bar{a}} \not{D} \psi^{\bar{a}} - \bar{\rho} \not{D} \rho - 2\bar{\rho} D_{\bar{a}} \psi^{\bar{a}}.$$

Let us comment briefly on the details of the construction. (See the Appendices A–E for an introduction to the relevant notions in generalized geometry.) The underlying Courant algebroid here is transitive, given by the bundle

$$E \coloneqq TM \oplus T^*M \oplus \mathrm{ad}(G),$$

where $\mathrm{ad}(G)$ is the adjoint bundle associated to some principal G-bundle, with G a Lie group with an invariant pairing on its Lie algebra. The generalized metric $\mathcal{G}$ can be understood as a symmetric endomorphism of E satisfying $\mathcal{G}^2 = id$, which induces an eigenbundle decomposition $E = C_+ \oplus C_-$. The fields ρ and ψ are an $\mathbb{R}$ and C_--valued spinor half-densities with respect to the subbundle C_+.[11] The index $\bar{a}$ runs over the subbundle C_- and is raised/lowered using the Courant algebroid pairing. In the above physically interesting setup C_+ is taken to be the graph of a map $TM \to T^*M \oplus \mathrm{ad}(G)$, which is equivalent to the data of g, B and A. The half-density σ is given by $\sigma^2 = \sqrt{g}e^{-2\varphi}$, where φ is the dilaton understood as a function on M. (In this work we will, however, refer to σ directly as the dilaton, in accordance with, e.g., [12].)

*Contact author: j.kupka@herts.ac.uk
†Contact author: c.strickland-constable@herts.ac.uk
‡Contact author: f.valach@herts.ac.uk

This note is based on the observation that the formalism is consistent even for other (less immediately physically relevant) choices of Courant algebroids and generalized metrics. We will here investigate the most extreme case, namely taking $\mathcal{G}$ to be the identity operator, while keeping the Courant algebroid constrained only by the condition that its signature (p, q) is either (9, 1) or (5, 5) [13] and the vector bundle itself is spin (so that we obtain 10-dimensional Majorana–Weyl spinors). This leads to several drastic simplifications. First, the generalized metric becomes nondynamical, since there is no nontrivial variation of $\mathcal{G} = id$ which is both symmetric and preserves the constraint $\mathcal{G}^2 = id$. Furthermore, as the subbundle C_+ now spans the entire bundle, there are no generalized gravitinos. The only surviving fields are the dilaton σ and the generalized dilatino ρ, which is the reason we refer to the resulting theory as the *dilatonic supergravity* [19]. In turns out that, in contrast to the physical supergravity, this theory does not require the addition of any terms of higher order in fermions.

The above simplifications allow us to provide a complete generalized-geometric description of the theory to all orders in fermions, together with all its relevant symmetries, and also allow us to write down the full BV action. Our main hope is that some of the insights and structural results can then be carried over to the full physical supergravity.

We can visualize the passage from the physical supergravity to the dilatonic supergravity as follows. In every fiber of a given Courant algebroid, the space $\mathcal{S}$ of possible generalized metrics naturally decomposes into the disjoint union of $\mathcal{S}_n$, corresponding to generalized metrics with $\dim C_+ = n$. In the transitive case, the physically relevant generalized metrics sit inside $\mathcal{S}_{\dim M}$. If G is trivial then $\dim M = \mathrm{rank}(E)/2$, and so the "physical" generalized metrics correspond to the largest $\mathcal{S}_n$ (in terms of dimensionality). The focus of this paper is on the extremal point $\mathcal{S}_{\mathrm{rank}(E)}$ (the top point in Fig. 1), where $\mathcal{G} = id$.

This note is structured as follows: We first describe the dilatonic supergravity and its symmetries. We then provide the full BV construction, count the classical degrees of freedom, and discuss the twisting in the sense of Costello–Li. We look in more detail at two classes of examples, given by exact Courant algebroids and quadratic Lie algebras. Since the latter is particularly simple while still keeping some of the key nontrivial features of

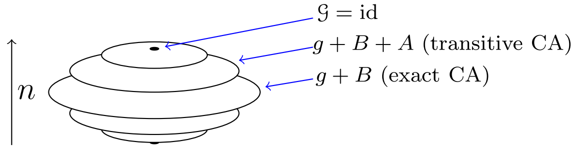

FIG. 1. Decomposition of $\mathcal{S}$ into $\mathcal{S}_n$.

the general setup, we have written the relevant Sec. III B in a self-contained way without the Courant algebroid language, in the hope of making it more accessible for the readers without prior acquaintance with generalized geometry. Finally, we provide an introduction to the relevant geometric notions, including Courant algebroids and generalized geometry, in the collection of Appendices.

II. DILATONIC SUPERGRAVITY

A. The theory and its symmetries

Let $E \to M$ be an arbitrary Courant algebroid with signature either (9, 1) or (5, 5), with Majorana-Weyl spinor bundles S_+ and S_-, and denote by H the line bundle of half-densities on M [20].

The fields of the dilatonic supergravity are the fermionic spinor half-density $\rho \in \Gamma(\Pi S_+ \otimes H)$ and the bosonic positive half-density $\sigma \in \Gamma(H)^+$, where Π denotes the parity shift. Note that this does not include any metric and hence the theory will be naively topological. The action is

$$S(\sigma, \rho) \coloneqq \int_M \mathcal{R}\sigma^2 - \bar{\rho} \not{D} \rho, \tag{1}$$

where $\not{D}$ is the Dirac operator and $\mathcal{R} \coloneqq \not{D}^2 \in C^\infty(M)$. The equation of motion for ρ is

$$\not{D}\rho = 0, \tag{2}$$

while the situation for σ is more degenerate. For instance, if $\mathcal{R}$ is nonvanishing (almost) everywhere then there are no critical points at all (since σ is everywhere positive). On the other hand, if $\mathcal{R} = 0$ everywhere (as happens for instance on exact Courant algebroids) then the extrema of S are given by any σ and ρ satisfying (2).

In contrast to this almost trivial on-shell structure of the theory, we will now see that the action (1) still exhibits interesting symmetries analogous to those of the original physical theory, and admits a rich BV extension. For starter, we note that S is invariant under the generalized diffeomorphisms [21] and supersymmetry transformations

$$\delta_\zeta \rho = \mathcal{L}_\zeta \rho, \qquad \delta_\zeta \sigma = \mathcal{L}_\zeta \sigma, \qquad \zeta \in \Gamma(E), \qquad \delta_\epsilon \rho = \not{D}\epsilon,$$
$$\delta_\epsilon \sigma = \frac{1}{\sigma}\bar{\rho}\epsilon, \qquad \epsilon \in \Gamma(\Pi S_- \otimes H), \tag{3}$$

where $\mathcal{L}$ is the generalized Lie derivative. Note the usefulness of the half-density picture—replacing the spinor half-densities ρ, ϵ by the more conventional spinors $\rho' \coloneqq \sigma^{-1}\rho$ and $\epsilon' \coloneqq \sigma^{-1}\epsilon$, the action and the supersymmetry transformations would take the more complicated form

$$S(\sigma,\rho') = \int_M (\mathcal{R} - \bar\rho' \not{D}_\sigma \rho')\sigma^2,$$
$$\delta_{\epsilon'}\rho' = \not{D}_\sigma \epsilon' - \frac{1}{96}(\bar\rho'\gamma_{\alpha\beta\gamma}\rho')\gamma^{\alpha\beta\gamma}\epsilon',$$
$$\delta_{\epsilon'}\sigma = \sigma(\bar\rho'\epsilon'),$$

where we defined the "dressed" Dirac operator $\not{D}_\sigma\epsilon' \coloneqq \sigma^{-1}\not{D}(\sigma\epsilon')$ and used a Fierz identity. Indeed, one of the simplifications of using the half-densities is that the Dirac operator $\not{D}$ in the action (1) is independent of the fields, as opposed to $\not{D}_\sigma$.

Next, a quick calculation with (3) gives

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\rho = 0, \quad [\delta_{\epsilon_1},\delta_{\epsilon_2}]\sigma = \mathcal{L}_\zeta\sigma, \quad \zeta^\alpha \coloneqq 2\sigma^{-2}\bar\epsilon_2\gamma^\alpha\epsilon_1. \tag{4}$$

We see that we can (but of course will not) in principle consider the symmetries of the theory to be the local supersymmetry together with the diffeomorphisms which only act on σ and leave ρ invariant. However, in order to make connection with the original supergravity and its usual supersymmetry algebra analysis, we will consider the action of diffeomorphisms on ρ as well, and compensate this contribution by another supersymmetry transformation, to obtain a symmetry algebra which closes on-shell. In other words, we can equally well write

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\sigma = \delta_\zeta\sigma + \delta_\epsilon\sigma \tag{5}$$

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\rho = \delta_\zeta\rho + \delta_\epsilon\rho - \frac{1}{2}\zeta_\alpha\gamma^\alpha\not{D}\rho, \tag{6}$$

where

$$\zeta^\alpha \coloneqq 2\sigma^{-2}\bar\epsilon_2\gamma^\alpha\epsilon_1, \qquad \epsilon \coloneqq -\frac{1}{2}\zeta_\alpha\gamma^\alpha\rho.$$

Note that the right-hand side (rhs) of (6) vanishes due to (E2), and the second term on the rhs of (5) is zero since $\bar\rho\gamma^\alpha\rho = 0$. The rest of the algebra satisfies the usual (off-shell) relations

$$[\zeta,\epsilon] = \mathcal{L}_\zeta\epsilon, \qquad [\zeta_1,\zeta_2] = \mathcal{L}_{\zeta_1}\zeta_2. \tag{7}$$

The form (5)–(7) of the local supersymmetry algebra is much simpler than the expressions found in the literature. For instance, in contrast to [5] we only obtain half of the supersymmetric variations (and no Lorentz transformations) on the rhs of (5) and (6).

B. BV action

The BV description (which in this case uses $\mathbb{Z}_2\times\mathbb{Z}$-grading) uses the following fields:

	σ	ρ^a	ξ^α	e_a	f	σ^*	ρ^*_a	ξ^*_α	e^{*a}	f^*
Density	Half	Half	Zero	Half	Zero	Half	Half	One	Half	One
$\mathbb{Z}_2$ degree	[0]	[1]	[0]	[1]	[0]	[0]	[1]	[0]	[1]	[0]
$\mathbb{Z}$ degree	0	0	1	1	2	−1	−1	−2	−2	−3
Total	Even	Odd	Odd	Even	Even	Odd	Even	Even	Odd	Odd

These are all densities valued in either the spinor bundles $S_\pm$, the generalized tangent bundle E, the trivial bundle, or their duals (this is marked by the corresponding index which is here Latin for spinors and Greek for sections of E), the antifields being decorated by a star. The whole BV space can be described as

$$\mathcal{M}_{\rm BV} = T^*[1](\Gamma(H)^+ \times \Gamma(\Pi S_+ \otimes H) \times \Gamma(E[1]) \times \Gamma(\Pi S_-[1] \otimes H) \times C^\infty(M)[2]).$$

The fields ξ and e correspond to the anticommuting ghost for the generalized diffeomorphism symmetry and the commuting ghost for the local supersymmetry, respectively. The field f is a "ghost for ghost" corresponding to the fact that $\mathcal{L}_{\mathcal{D}h} = 0$ for any function $h \in C^\infty(M)$.

The BV action takes the form

$$\begin{aligned} S_{\rm BV} = \int_M & \mathcal{R}\sigma^2 - \bar\rho\not{D}\rho + \sigma^*(\mathcal{L}_\xi\sigma - \sigma^{-1}(\bar\rho e)) + \bar\rho^*(\mathcal{L}_\xi\rho + \not{D}e) \\ & + \bar e^*\left(\mathcal{L}_\xi e + \frac{1}{2}\sigma^{-2}(\bar e\gamma^\alpha e)\gamma_\alpha\rho\right) \\ & + \langle \xi^*, \mathcal{D}f + \frac{1}{2}\mathcal{L}_\xi\xi\rangle - \xi^*_\alpha\sigma^{-2}(\bar e\gamma^\alpha e) \\ & + \frac{1}{2}f^*\left(\mathcal{L}_\xi f + \sigma^{-2}(\bar e\gamma_\alpha e)\xi^\alpha - \frac{1}{6}\langle\xi,\mathcal{L}_\xi\xi\rangle\right) \\ & + \frac{1}{8}\sigma^{-2}(\bar e\gamma_\alpha e)(\bar\rho^*\gamma^\alpha\rho^*). \end{aligned} \tag{8}$$

Before checking this expression, let us note several interesting features of the action. First, note the appearance of a term of higher order in the antifields—this corresponds to the fact that our symmetry algebra only closed on-shell. We also note that in the language of L_∞-algebras the last two terms in the second line of $S_{\rm BV}$ correspond to a 3-bracket, schematically

$$[\text{susy}, \text{susy}, \text{gauge}] = \text{gauge for gauge},$$
$$[\text{gauge}, \text{gauge}, \text{gauge}] = \text{gauge for gauge}.$$

The verification that (8) satisfies the classical master equation $\{S_{\rm BV}, S_{\rm BV}\} = 0$ is straightforward and will not be shown here in full, since for most part it simply corresponds to the preceding calculation of the supersymmetry algebra. We will however explicitly display below some less trivial parts of the calculation, which also exhibit the general pattern [22].

We note that the only nontrivial algebraic identity that is needed in checking the classical master equation is the Fierz identity (A1), and it is due to this identity that we work with spinors in 10 dimensions.

For instance, defining $V\in\Gamma(E)$ by

$$V^\alpha := \sigma^{-2}(\bar{e}\gamma^\alpha e),$$

the terms in $\frac{1}{2}\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\}$ involving ρ^* and two e's (on top of the "classical fields" σ, ρ), combine to give

$$\begin{aligned}
&\int_M \frac{\delta}{\delta\rho^*}\left(\int_M \frac{1}{8}V_\alpha(\bar{\rho}^*\gamma^\alpha\rho^*)\right)\frac{\delta}{\delta\rho}\left(-\int_M \bar{\rho}\not{D}\rho\right)\\
&\quad+\frac{\delta}{\delta e^*}\left(\int_M \frac{1}{2}V^\alpha(\bar{e}^*\gamma_\alpha\rho)\right)\frac{\delta}{\delta e}\left(\int_M \bar{\rho}^*\not{D}e\right)\\
&\quad+\frac{\delta}{\delta\xi^*}\left(-\int_M V^\alpha\xi^*_\alpha\right)\frac{\delta}{\delta\xi}\left(\int_M \bar{\rho}^*\mathcal{L}_\xi\rho\right)\\
&=\int_M \frac{1}{2}V_\alpha(\bar{\rho}^*\gamma^\alpha\not{D}\rho)+\frac{1}{2}V_\alpha(\bar{\rho}\gamma^\alpha\not{D}\rho^*)-\bar{\rho}^*\mathcal{L}_V\rho\\
&=\int_M \frac{1}{2}\bar{\rho}V_\alpha\gamma^\alpha\not{D}\rho^*-\frac{1}{2}\bar{\rho}^*\not{D}V_\alpha\gamma^\alpha\rho=0,
\end{aligned}$$

where toward the end we used (E1), (E2), and the fact that $\overline{\gamma_\alpha\rho}=-\bar{\rho}\gamma_\alpha$. Similarly, using $\gamma_\alpha\gamma^\beta=-\gamma^\beta\gamma_\alpha+2\delta^\beta_\alpha$, the terms involving ξ^*eee give

$$\begin{aligned}
&\int_M -2\sigma^{-2}V^\alpha(\bar{\rho}e)\xi^*_\alpha+\sigma^{-2}V^\alpha(\bar{\rho}\gamma_\alpha\gamma^\beta e)\xi^*_\beta\\
&\quad=-\int_M \sigma^{-2}V^\alpha(\bar{\rho}\gamma^\beta\gamma_\alpha e)\xi^*_\beta,
\end{aligned}$$

which vanishes due to the Fierz identity (A1). One also easily sees that the last two terms in the second line of S_{BV} are needed to ensure the vanishing of the $\xi^*\xi ee$ and $\xi^*\xi\xi\xi$ terms, respectively.

C. Degrees of freedom

We now briefly examine in more detail the on-shell structure of the theory. We will consider the case of Courant algebroids with $\mathcal{R}$ vanishing identically (recall that the other extremal case of everywhere nonvanishing $\mathcal{R}$ admits no classical solutions), and count the corresponding degrees of freedom, i.e., solutions of the equations of motion modulo gauge transformations. We shall do this on the infinitesimal level, in the following sense.

First, note that a classical background, i.e., a bosonic solution to the equations of motion of (1), simply corresponds to

$$\rho=0,\qquad \sigma=\sigma_0,$$

with σ_0 arbitrary. Expanding the BV action around this configuration (we define $\sigma=\sigma_0e^{-\varphi}$), the quadratic part of the action becomes

$$S^{\mathrm{quad}}_{\mathrm{BV}}=\int_M -\bar{\rho}\not{D}\rho+\sigma^*\mathcal{L}_\xi\sigma_0+\bar{\rho}^*\not{D}e+\langle\xi^*,\mathcal{D}f\rangle.$$

From this we read off the linearized BV operator, whose cohomology at degree zero is

$$H^0=H^0_{\mathrm{even}}\oplus H^0_{\mathrm{odd}},\qquad H^0_{\mathrm{even}}\cong\frac{\{\varphi\in C^\infty(M)\}}{\{\sigma_0^{-1}\mathcal{L}_\zeta\sigma_0|\zeta\in\Gamma(E)\}},$$

$$H^0_{\mathrm{odd}}\cong\frac{\{\not{D}\rho=0\}}{\{\not{D}\epsilon|\epsilon\in\Gamma(\Pi S_-\otimes H)\}}.$$

Let us now study in more detail the local degrees of freedom on a transitive Courant algebroid (i.e., we will consider a contractible neighborhood of a point). One can then interpret H^0_{even} as all functions modulo functions which arise as divergences with respect to the density σ_0^2. Since in a contractible region all functions arise in this way, we conclude that $H^0_{\mathrm{even}}=0$. Similarly, one obtains that H^0_{odd}, which coincides with the cohomology of the Dirac operator, is finite-dimensional. [23] Putting things together, we see that H^0 is finite-dimensional.

D. Twist *à la* Costello-Li

The BV form (8) of the dilatonic supergravity provides a simple playground to investigate the twist of supergravity due to Costello-Li [8]. This amounts to the following question: what are the extrema of S_{BV}, or equivalently at which points in $\mathcal{M}^{\mathrm{even}}_{\mathrm{BV}}\subset\mathcal{M}_{\mathrm{BV}}$ does the BV differential $Q_{\mathrm{BV}}:=\{S_{\mathrm{BV}},\cdot\}$ vanish?

Here $\mathcal{M}^{\mathrm{even}}_{\mathrm{BV}}$ is the space of configurations with vanishing odd fields, i.e.,

$$\mathcal{M}^{\mathrm{even}}_{\mathrm{BV}}=\{\rho=\xi=\sigma^*=e^*=f^*=0\}\subset\mathcal{M}_{\mathrm{BV}}.$$

Since there are no terms with odd number of odd fields in S_{BV}, to find the extrema on $\mathcal{M}^{\mathrm{even}}_{\mathrm{BV}}$ it suffices to vary the action along the subspace $\mathcal{M}^{\mathrm{even}}_{\mathrm{BV}}$ itself. Since

$$\begin{aligned}
S_{\mathrm{BV}}|_{\mathcal{M}^{\mathrm{even}}_{\mathrm{BV}}}=&\int_M \mathcal{R}\sigma^2+\bar{\rho}^*\not{D}e-\sigma^{-2}(\bar{e}\gamma^\alpha e)\xi^*_\alpha\\
&+\frac{1}{8}\sigma^{-2}(\bar{e}\gamma_\alpha e)(\bar{\rho}^*\gamma^\alpha\rho^*)+\langle\xi^*,\mathcal{D}f\rangle,
\end{aligned}$$

the extrema correspond to

$$\begin{aligned}
&\mathcal{R}\sigma^2+\left(\xi^*_\alpha-\frac{1}{8}\bar{\rho}^*\gamma_\alpha\rho^*\right)\mathcal{D}^\alpha f=0,\qquad d[a(\xi^*)]=0,\\
&\sigma^{-2}(\bar{e}\gamma^\alpha e)=\mathcal{D}^\alpha f,\qquad \not{D}e=-\frac{1}{4}(\mathcal{D}^\alpha f)\gamma_\alpha\rho^*,\\
&\not{D}\rho^*=2\sigma^{-2}\xi^*_\alpha\gamma^\alpha e+\frac{1}{4}\sigma^{-2}(\bar{\rho}^*\gamma^\alpha\rho^*)\gamma_\alpha e,
\end{aligned}\tag{9}$$

where $\mathcal{D}^\alpha f=(\mathcal{D}f)^\alpha$ and we have used the third equation to simplify the other ones. To understand the second equation, we (locally) pick any orientation and identify [24]

$$a(\xi^*) \in \Gamma(TM \otimes H^2) \cong \Gamma(TM \otimes \Lambda^{\dim M} T^*M) \cong \Omega^{\dim M - 1}(M).$$

Costello-Li twisting corresponds to expanding the theory around a background with a nontrivial value of the supersymmetry ghost, i.e., around a solution of (9) with $e \neq 0$. Note that if we set $f = 0$, the supersymmetry ghost must satisfy $\not{D} e = 0$ and $\bar{e}\gamma^\alpha e = 0$.[25] Since this corresponds [via (3)] to the supersymmetry of the (bosonic) background $(\sigma \in \Gamma(H)^+, \rho = 0)$ given by a pure spinor, it can be seen (roughly) as an analog of the Calabi-Yau condition in the present case. The system (9) can then be understood as a generalization of the Calabi-Yau condition. In particular, the theory obtained by expanding around a solution to this system can be regarded, following the conjecture of [8], as an analog of the BCOV theory [26].

III. EXAMPLES

We will now look in more detail at two classes of examples, lying in a sense at the opposite ends of the spectrum of transitive Courant algebroids—these are exact Courant algebroids (where the gauge group is trivial) and quadratic Lie algebras (where the manifold is trivial).

A. Exact Courant algebroids

Let M be a 5-dimensional real manifold and $H \in \Omega^3(M)$ a closed 3-form [27]. Let E be the corresponding exact Courant algebroid. For simplicity we will also assume that M is oriented. The bundle of spinor half-densities then corresponds to the bundle of all forms, and chirality translates to the parity of the form degree. In particular, we can choose to identify $S_+ \otimes H$ and $S_- \otimes H$ with even and odd forms, respectively, or vice versa. The spinor pairing is given by the Mukai pairing

$$(\alpha, \beta) \coloneqq (-1)^{\lceil \frac{\deg \alpha}{2} \rceil} (\alpha \wedge \beta)^{\mathrm{top}},$$

where $(\ldots)^{\mathrm{top}}$ extracts the top form part of the expression. The Dirac operator is

$$\not{D}\rho = d\rho + H \wedge \rho.$$

Since for exact Courant algebroids $\mathcal{R} = 0$, the classical theory is

$$S(\sigma, \rho) = -\int_M (\rho, d\rho + H \wedge \rho),$$

where σ is a positive bosonic half-density and ρ a fermionic collection of either purely even or purely odd polyforms (depending on which one we pick to correspond to S_+ and S_-)

$$\sigma \in \Gamma(H)^+, \qquad \rho \in \Pi\Omega^{\mathrm{even/odd}}(M). \tag{10}$$

The supersymmetry parameter and transformations are

$$\epsilon \in \Pi\Omega^{\mathrm{odd/even}}(M), \qquad \delta_\epsilon \rho = d\epsilon + H \wedge \epsilon,$$
$$\delta_\epsilon \sigma = \frac{1}{\sigma}(\rho, \epsilon).$$

Note that here σ naturally decouples both from the action and the transformation of ρ. Using integration by parts and taking into account the fermionic nature of ρ, we obtain the two theories

$$\frac{1}{2} S_1 = \int_M \rho_0 \wedge d\rho_4 - \frac{1}{2}\rho_2 \wedge d\rho_2 + H \wedge \rho_0 \wedge \rho_2,$$
$$\frac{1}{2} S_2 = \int_M \rho_1 \wedge d\rho_3 - \frac{1}{2} H \wedge \rho_1 \wedge \rho_1.$$

These theories can be regarded as H-twisted versions of the bc-ghost system.

In either case we can now consider the BV extension given by (8). We can however also consistently remove all terms containing σ or σ^*, to get

$$S'_{\mathrm{BV}} = \int_M -(\rho, \not{D}\rho) + (\rho^*, \not{D}e + \mathcal{L}_\xi \rho) + (e^*, \mathcal{L}_\xi e) + \left\langle \xi^*, \frac{1}{2}\mathcal{L}_\xi \xi + \mathcal{D}f \right\rangle + f^* \left(\frac{1}{2}\mathcal{L}_\xi f - \frac{1}{12}\langle \xi, \mathcal{L}_\xi \xi \rangle \right).$$

Note that all the terms are at most linear in antifields, which corresponds to the fact that, with terms with σ removed, the supersymmetry closes off-shell. In fact, one can also remove ξ, ξ^*, f, f^* to get simply

$$S''_{\mathrm{BV}} = \int_M -(\rho, \not{D}\rho) + (\rho^*, \not{D}e),$$

describing a theory acted upon by the supergroup whose Lie superalgebra is purely odd and given by

$$\Gamma(\Pi S_-) \cong \Pi\Omega^{\mathrm{odd/even}}(M).$$

Each of S_{BV}, S'_{BV}, and S''_{BV} thus describe a (different) BV extension of the starting actions S_1 and S_2. Note that due to $\not{D}^2 = 0$, the supersymmetry of S'_{BV} and S''_{BV} is (infinitely [28]) reducible. Thus, if we were to quantize these theories, we would need to add a corresponding tower of ghosts to compensate for the reducibility. However, we are not concerned about this in the present text, as our main goal is to gain insight into the symmetry structure of the original physical theory (where this issue does not arise).

B. Quadratic Lie algebras

One can restrict the analysis of Sec. II B to the particular case when the Courant algebroid is simply a quadratic Lie algebra. However, there is also a way to build a different (but related) BV extension of (1) for a quadratic Lie algebra, which is what we now turn to. We shall spell out the details, making it self-contained and accessible to readers without prior knowledge of generalized geometry or Courant algebroids.

The setup now corresponds to a given Lie algebra $\mathfrak{g}$ with invariant pairing of signature either (9, 1) or (5, 5). Some nontrivial examples include [29]

(i) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ with inner product of signature $(3,0)+(3,0)+(3,0)+(0,1)$,

(ii) $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ with inner product of signature $(2,1)+(3,0)+(3,0)+(1,0)$, and

(iii) the semi-Abelian Drinfeld doubles $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{a}^*$, for $\mathfrak{a}$ any 5-dimensional Lie algebra (here $\mathfrak{a}^*$ is Abelian and acted upon by $\mathfrak{a}$, and $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{a}^*$ and $\mathfrak{a}$).

Denoting the structure coefficients of $\mathfrak{g}$ by $c_{\alpha\beta\gamma}$, we have the Dirac operator

$$\not{D} = -\frac{1}{12} c_{\alpha\beta\gamma} \gamma^{\alpha\beta\gamma}.$$

One easily verifies, using the Jacobi identity, that

$$\begin{aligned}\mathcal{R} &:= \not{D}^2 = \frac{1}{2}\{\not{D}, \not{D}\} = \frac{1}{144} c^{\alpha\beta\gamma} c_{\delta\epsilon\zeta} (9\delta^{\delta}_{\alpha}\gamma_{\beta\gamma}{}^{\epsilon\zeta} - 6\delta^{\delta\epsilon\zeta}_{\alpha\beta\gamma}) \\ &= -\frac{1}{24} c^{\alpha\beta\gamma} c_{\alpha\beta\gamma} \in \mathbb{R}.\end{aligned}$$

The classical fields of our theory are now $\sigma \in \mathbb{R}^+$ and $\rho \in \Pi S_+$, where again Π denotes the parity shift and S_+ stands for positive chirality Majorana spinors with respect to the pairing on $\mathfrak{g}$. The action for the theory is

$$S(\sigma, \rho) = \mathcal{R}\sigma^2 - \bar{\rho}\not{D}\rho. \qquad (11)$$

This is invariant under the generalized diffeomorphism (3)

$$\delta_\zeta \rho = \frac{1}{4} \zeta^\alpha c_{\alpha\beta\gamma} \gamma^{\beta\gamma} \rho, \qquad \delta_\zeta \sigma = 0, \qquad \zeta \in \mathfrak{g},$$

as well as under the supersymmetry transformations

$$\delta_\epsilon \rho = \not{D}\epsilon, \qquad \delta_\epsilon \sigma = \sigma^{-1}\bar{\rho}\epsilon, \qquad \epsilon \in \Pi S_-.$$

However, we note that in this case we have [cf. (4)]

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 0,$$

i.e., the symmetry algebra closes off-shell into the Lie superalgebra

$$\mathfrak{g} \ltimes \Pi S_-,$$

$$[\zeta_1 + \epsilon_1, \zeta_2 + \epsilon_2] = [\zeta_1, \zeta_2]_{\mathfrak{g}} + \frac{1}{4} \zeta_1^\alpha c_{\alpha\beta\gamma} \gamma^{\beta\gamma} \epsilon_2 - \frac{1}{4} \zeta_2^\alpha c_{\alpha\beta\gamma} \gamma^{\beta\gamma} \epsilon_1,$$

which leads to a simple BV description with fields

$$\begin{aligned}&\sigma \in \mathbb{R}^*, \qquad \sigma^* \in \Pi\mathbb{R}, \qquad \rho \in \Pi S_+, \qquad \rho^* \in S_-, \\ &e \in S_-, \qquad e^* \in \Pi S_+, \qquad \xi \in \Pi\mathfrak{g}, \qquad \xi^* \in \mathfrak{g},\end{aligned}$$

and the BV action

$$\begin{aligned}\tilde{S}_{\mathrm{BV}} &= \mathcal{R}\sigma^2 - \bar{\rho}\not{D}\rho + \sigma^{-1}(\bar{\rho}e)\sigma^* + \bar{\rho}^*\not{D}e + \frac{1}{4}\xi^\alpha c_{\alpha\beta\gamma}(\bar{\rho}^*\gamma^{\beta\gamma}\rho) \\ &\quad - \frac{1}{4}\xi^\alpha c_{\alpha\beta\gamma}(\bar{e}^*\gamma^{\beta\gamma}e) + \frac{1}{2} c^\gamma{}_{\alpha\beta}\xi^\alpha\xi^\beta\xi^*_\gamma.\end{aligned}$$

Let us stress again that this is a different BV extension of the same classical theory (11) than the one obtained by restricting the analysis in Sec. II B to this case. Similarly, one can consider the action (8) with both f and f^* set to zero, i.e., restricting to the subspace $\{f = f^* = 0\} \subset \mathcal{M}_{\mathrm{BV}}$. Analogously to the preceding subsection, these three actions define consistent BV theories [30], and should be regarded as different BV extensions of the same starting theory (11) [31].

The special property of $\tilde{S}_{\mathrm{BV}}$ is that in checking the classical master equation we do not need to use the Fierz identity (A1) and so the theory in fact makes sense on any quadratic Lie algebra with inner product of signature (p, q) with

$$p + q \equiv 10 \pmod 8, \qquad p - q \equiv 0 \pmod 8,$$

so that we have Majorana-Weyl spinors. Again, taking the fields to be complex-valued we can extend this further to any even-dimensional quadratic Lie algebra. Thus $\tilde{S}_{\mathrm{BV}}$ provides a large class of simple finite-dimensional BV theories which serve as toy models for the original supergravity.

IV. CONCLUSIONS

Although the ultimate goal is to perform a similar analysis in the fully "physical" case with (generalized) metrics, the present model already allows us to draw several interesting conclusions that might apply to the structure of the full/physical supergravity.

First, the generalized geometry/Courant algebroid framework tells us that the Dirac operator naturally acts on spinor half-densities [2,32]. Intending to keep the simple formula from [9]

$$\delta_\epsilon \rho = \not{D}\epsilon + \ldots,$$

in our analysis we were consequently forced to treat both ϵ and ρ as spinor half-densities. This in turn leads to the following consequences:

(i) Higher order ρ-terms in the supersymmetric variation of ρ drop out. Quite intriguingly, we note that a similar observation was made in [5], which however uses a different version of dilatino. It is not immediately clear to the authors how these two facts are related.

(ii) Half of the supersymmetry variations in the commutator $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ drop out [see (5) and (6)]. In addition there are no terms with Lorentz transformations.

(iii) The Dirac operator $\not{D}$ appearing in the action is independent of the dilaton.

Furthermore, we see that this restricted setup still keeps some nontrivial aspects of the full "physical" supergravity story (such as a roughly anticipated form of the BV extension of supergravity), even in the seemingly trivial case of Courant algebroids over a point, i.e., quadratic Lie algebras. In fact, if the quadratic Lie algebra $\mathfrak{g}$ contains a coisotropic subalgebra $\mathfrak{h}$ then one can canonically construct [33] a transitive Courant algebroid on the quotient G/H of the corresponding Lie groups—the bracket on this algebroid will arise from the bracket on $\mathfrak{g}$. Thus, roughly speaking, nontrivial geometry of G/H, reflected in the bracket of its vector fields and hence encoded in the Courant algebroid bracket, corresponds to a nontrivial Lie algebra structure on $\mathfrak{g}$. This is one of the basic ideas which allow us to extract geometrically interesting results and ideas by working in a purely algebraic framework.

We conclude the discussion by a brief comment on the relation to the first order formulation of supergravity. In the latter approach one replaces the metric degrees of freedom by the vielbein and the spin connection. This however leads to a further enhancement of the gauge symmetry (by adding the Lorentz transformations), and in particular also adds extra terms to the BV action. A further technical complication stems from the nonuniqueness of the Levi-Civita connections in generalized geometry. Nevertheless, we expect that, if desired, the first order reformulation of this letter should be possible, following [34]. We leave this to a future work.

ACKNOWLEDGMENTS

We are grateful to Pavol Ševera and Dan Waldram for helpful discussions. C. S.-C. and F. V. are supported by an EPSRC New Investigator Award, Grant No. EP/X014959/1. J. K. and F. V. would like to thank the organizers of the conference "Mathematical supergravity" at the UNED, Madrid, where the idea for this work started. We also thank Miguel Pino Carmona and Tancredi Schettini-Gherardini for the discussion regarding the relevant nomenclature.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: SPINORS IN SIGNATURES (9, 1) AND (5, 5)

Let us denote by S the space of Majorana spinors. We will adhere to the conventions (cf. [9])

$$\{\gamma_\alpha, \gamma^\beta\} = 2\delta_\alpha^\beta, \qquad \bar{\psi} \coloneqq \psi^T C, \qquad C\gamma_\alpha C^{-1} = -\gamma_\alpha^T,$$
$$C^T = -C, \qquad \gamma^{\alpha\ldots\beta} \coloneqq \gamma^{[\alpha}\ldots\gamma^{\beta]}.$$

which in particular imply that for any pair of fermionic spinors

$$\bar{\psi}\gamma_{(p)}\chi = (-1)^{[\frac{p+1}{2}]}\bar{\chi}\gamma_{(p)}\psi,$$

where $\gamma_{(p)}$ denotes any $\gamma_{\alpha\ldots\beta}$ with p indices.

Majorana spinors admit the decomposition $S = S_+ \oplus S_-$ into Weyl spinors, where

$$\gamma^{(10)}|_{S_\pm} = \pm\mathrm{id}_{S_\pm}, \qquad \gamma^{(10)} \coloneqq \gamma^0\ldots\gamma^9.$$

We also have the following vanishing bilinears:

$$\bar{\psi}\gamma_{(2k)}\chi = 0 \quad \text{if } \psi, \chi \text{ have the same chirality},$$
$$\bar{\psi}\gamma_{(2k+1)}\chi = 0 \quad \text{if } \psi, \chi \text{ have opposite chirality}.$$

This in particular implies that the only nonzero bilinear of a single chiral fermionic spinor ρ is

$$\bar{\rho}\gamma_{\alpha\beta\gamma}\rho.$$

Two further important things that happen in 10 dimensions (which do not happen in $10 + 8k$ dimensions for $k \geq 1$) are

(i) the fourth antisymmetric tensor power of S_+ (or S_-) does not contain any singlet, implying that one cannot add any quartic terms in the chiral fermionic spinor ρ to the action (1), and

(ii) the third symmetric power of S_- does not contain any S_+ summand, and vice-versa, implying the following Fierz identity for any bosonic chiral spinor e:

$$(\bar{e}\gamma_\alpha e)\bar{e}\gamma^\alpha = 0. \tag{A1}$$

The latter is the reason why we restrict our analysis to 10 dimensions, since (A1) is needed in order that our BV action (8) satisfies the classical master equation.

APPENDIX B: ON λ-DENSITIES

Let M be an n-dimensional manifold and λ a real number. We define the line bundle L^λ as the bundle associated to the frame bundle of M via the 1-dimensional representation of $GL(n,\mathbb{R})$ given by

$$A \mapsto |\det(A)|^{-\lambda}.$$

Practically, this means that every local frame on M induces a local section of L^λ, and changing the frame by a transition matrix A results in multiplying the section by $|\det(A)|^{-\lambda}$. Sections of L^λ are called *λ-densities*. If $\lambda = 1/2$, we simply talk about *half-densities*; for simplicity we will set

$$H \coloneqq L^{1/2}.$$

We also have $L^\lambda \otimes L^{\lambda'} \cong L^{\lambda\otimes\lambda'}$. Owing to the absolute value in their definition, λ-densities enjoy two important properties:

(i) they always exist globally, i.e., the line bundle L^λ is always trivial (though it does not have a canonical trivialization), and
(ii) it makes sense to talk about positive or negative λ-densities (at every point on M).

The space of everywhere positive half-densities will be denoted by $\Gamma(H)^+$.

Finally, 1-densities can be naturally integrated. In fact, if M is orientable, a choice of orientation on M provides an identification of 1-densities with top forms on M. However, integration of 1-densities is well defined even on non-orientable manifolds.

APPENDIX C: ON COURANT ALGEBROIDS

A *Courant algebroid* [1] is a vector bundle $E \to M$, equipped with some additional structure, namely

(i) an $\mathbb{R}$-bilinear operation $[\cdot,\cdot]\colon \Gamma(E)\times\Gamma(E)\to\Gamma(E)$,
(ii) a fiberwise nondegenerate bilinear symmetric form $\langle\cdot,\cdot\rangle$,
(iii) a vector bundle map $a\colon E\to TM$,

satisfying several axioms. First, for all $u, v\in\Gamma(E)$, $f\in C^\infty(M)$ we have

$$[u, fv] = f[u, v] + (a(u)f)v.$$

This allows us to extend the action of u on sections of E (via $[u,\cdot]$) and on functions [via $a(u)$] to a derivation $\mathcal{L}_u$ of the whole tensor algebra on E. For instance, for $v, w\in\Gamma(E)$ we have

$$\mathcal{L}_u(v\otimes w) = (\mathcal{L}_u v)\otimes w + v\otimes(\mathcal{L}_u w).$$

We call $\mathcal{L}$ the *generalized Lie derivative*. The remaining axioms of the Courant algebroid can then be simply stated as

$$\mathcal{L}_u[v,w] = [\mathcal{L}_u v, w] + [v, \mathcal{L}_u w], \qquad \mathcal{L}_u\langle\cdot,\cdot\rangle = 0,$$
$$[u,v]+[v,u] = \mathcal{D}\langle u,v\rangle,$$

for any $u, v, w\in\Gamma(E)$ and $f\in C^\infty(M)$, where the operator $\mathcal{D}\colon C^\infty(M)\to\Gamma(E)$ is defined by

$$\langle \mathcal{D}f, u\rangle \coloneqq a(u)f.$$

Note that the pairing/inner product $\langle\cdot,\cdot\rangle$ allows us to identify $E\cong E^*$, which we will use freely. In other words, the indices on a Courant algebroid are always lowered/raised using this inner product.

The compatibility of $\mathcal{L}$ and the pairing imply that $\mathcal{L}_u$ also acts naturally on any associated spinor bundles. Finally, we have a natural action on λ-densities, given by $\mathcal{L}_u\sigma = L_{a(u)}\sigma$, where L is the ordinary Lie derivative.

Several things can be worked out following this definition. First, for any two sections we get

$$a([u,v]) = [a(u), a(v)],$$

which implies that if $\mathrm{rank}(a)$ is constant (we say the algebroid is *regular*) then $\mathrm{im}(a)\subset TM$ is an integrable distribution. Furthermore, denoting the dual map to a by a^*, we have the important property $a\circ a^* = 0$, which can be restated by saying that

$$0\to T^*M \xrightarrow{a^*} E \xrightarrow{a} TM\to 0 \qquad \text{(C1)}$$

is a chain complex. We say that a Courant algebroid is *exact* if (C1) is an exact sequence. More generally, it is *transitive* if a is surjective. It is known [2,35] that every exact Courant algebroid has the following form

$$E\cong TM\oplus T^*M, \qquad a(X+\alpha) = X,$$
$$\langle X+\alpha, Y+\beta\rangle = \alpha(Y)+\beta(X),$$
$$[X+\alpha, Y+\beta] = L_X Y + (L_X\beta - i_Y d\alpha + H(X,Y,\cdot)),$$

for some closed 3-form H on M. Two exact CAs on M whose 3-forms differ by an exact 3-form can be shown to be isomorphic. In particular, all exact Courant algebroids over M look locally the same.

Similarly [2], every transitive Courant algebroid on M is locally determined by a choice of a *quadratic Lie algebra* $\mathfrak{g}$ (i.e., a Lie algebra together with an invariant nondegenerate symmetric pairing). Explicitly, around any point in M we can find an open subset $U\subset M$ and $\mathfrak{g}$ such that

$$E|_U\cong TU\oplus T^*U\oplus(\mathfrak{g}\times U), \qquad a(X+\alpha+s) = X,$$
$$\langle X+\alpha+s, Y+\beta+t\rangle = \alpha(Y)+\beta(X)+\langle s,t\rangle_{\mathfrak{g}},$$
$$[X+\alpha+s, Y+\beta+t] = L_X Y + (L_X\beta - i_Y d\alpha + \langle ds, t\rangle_{\mathfrak{g}}) + (L_X t - L_Y s + [s,t]_{\mathfrak{g}}).$$

As a special case we can take $M = $ point, so that $E = \mathfrak{g}$, $a = 0$, with the bracket and pairing on E coinciding with those on $\mathfrak{g}$. Any quadratic Lie algebra can thus be seen as a Courant algebroid. Dilatonic supergravity in this particularly simple setup is studied in detail in Sec. III B.

APPENDIX D: GENERALIZED CONNECTIONS AND THE DIRAC OPERATOR

In this section we follow [32] (see also [2]). On any Courant algebroid $E \to M$ we define *generalized connections* to be maps satisfying

$$D\colon \Gamma(E) \otimes \Gamma(E) \to \Gamma(E) \quad \text{s.t.} \quad D_{fu} v = f D_u v,$$
$$D_u(fv) = f D_u v + (a(u) f) v, \qquad D_u \langle \cdot, \cdot \rangle = 0.$$

Here the last property again uses the fact that D_u can be extended (due to the second property) to a derivation of the whole tensor algebra of E. Again, the definition implies that generalized connections also naturally act on spinors with respect to E.

Slightly more surprisingly, generalized connections also naturally act on λ-densities on M via

$$D_u \sigma \coloneqq L_{a(u)} \sigma - \lambda \sigma D_\alpha u^\alpha, \qquad u \in \Gamma(E), \qquad \sigma \in \Gamma(L^\lambda),$$

Thus, both $\mathcal{L}_u$ and D_u act naturally on any E-tensors or spinors valued in λ-densities on M.

For any generalized connection, we can define its torsion by

$$T(u, v) \coloneqq D_u v - D_v u - [u, v] + \langle Du, v \rangle.$$

One can check that this expression is tensorial in both slots, and in fact the torsion is a tensor

$$T \in \Gamma(\Lambda^3 E).$$

Assume now that D is torsion-free (i.e., its torsion vanishes; such connections exist on any Courant algebroid [36]). It then turns out [32], crucially, that the *Dirac operator*

$$\not{D} \coloneqq \gamma^\alpha D_\alpha \colon \Gamma(S \otimes L^{1/2}) \to \Gamma(S \otimes L^{1/2})$$

is in fact independent of the choice of the particular torsion-free connection D and is thus intrinsic to the Courant algebroid structure itself. Note that this independence only holds when $\not{D}$ acts on spinor half-densities. One implication of this fact is the useful formula

$$[\not{D}, \mathcal{L}_u] = 0. \tag{D1}$$

For instance, on an exact Courant algebroid twisted by $H \in \Omega^3_{cl}(M)$, with M oriented, spinor half-densities can be understood as differential forms on M, and the Dirac operator is

$$\not{D}\rho = d\rho + H \wedge \rho,$$

while for a quadratic Lie algebra we obtain

$$\not{D} = -\frac{1}{12} c_{\alpha\beta\gamma} \gamma^{\alpha\beta\gamma},$$

where $c_{\alpha\beta\gamma}$ denotes the structure constants. For a transitive Courant algebroid the Dirac operator is locally a sum of the two above (where we can also take the 3-form H to vanish).

Another important fact about the Dirac operator acting on spinor half-densities is that its square $\not{D}^2$ contains no derivatives and in fact corresponds to the multiplication by a function, which we will denote by

$$\mathcal{R} \in C^\infty(M).$$

As the notation suggests, $\mathcal{R}$ can be understood (up to a prefactor) as the scalar curvature associated to the generalized metric $\mathcal{G} = id$ (cf. the generalized Lichnerowicz formula in [9]).

Note that (D1) implies that $\mathcal{R}$ is preserved by the action of the generalized Lie derivative $\mathcal{L}$, and hence it is constant on the integral leaves of the distribution $im(a) \subset TM$. Explicitly, on a transitive Courant algebroid we obtain the constant function

$$\mathcal{R} = -\frac{1}{24} c_{\alpha\beta\gamma} c^{\alpha\beta\gamma}, \tag{D2}$$

where c are the structure constants of the corresponding quadratic Lie algebra $\mathfrak{g}$. In particular, $\mathcal{R}$ vanishes on exact Courant algebroids.

APPENDIX E: SOME USEFUL FORMULAS

For any bosonic spinor half-densities α, β, we have

$$\bar{\alpha}\not{D}\beta + \bar{\beta}\not{D}\alpha = \mathcal{L}_u \sigma^2, \qquad u^\alpha = \sigma^{-2} \bar{\alpha} \gamma^\alpha \beta$$

for any positive half-density σ. (Note that the rhs is indeed independent of σ—when multiplying σ by any nonzero function the newly created terms involving derivatives of the function cancel against each other.) In particular

$$\int_M \bar{\alpha}\not{D}\beta = -\int_M \bar{\beta}\not{D}\alpha. \tag{E1}$$

Let $R \to M$ be an associated vector bundle to the spin lift of the bundle of the oriented orthonormal frames of E and $\lambda \in \mathbb{R}$. If D is torsion free then for any λ-density valued in R we have

$$\mathcal{L}_u \psi = D_u \psi + A \cdot \psi + \lambda (D_\alpha u^\alpha) \psi, \qquad A_{\alpha\beta} \coloneqq D_\alpha u_\beta - D_\beta u_\alpha.$$

In particular, for spinor half-densities we have

$$\mathcal{L}_u \psi = D_u \psi + \frac{1}{2} (D_\alpha u_\beta) \gamma^{\alpha\beta} \psi + \frac{1}{2} (D_\alpha u^\alpha) \psi.$$

Using this, one easily shows that for any spinor half-density ψ and $u \in \Gamma(E)$

$$\not{D}(u_\alpha \gamma^\alpha \psi) = 2 \mathcal{L}_u \psi - u_\alpha \gamma^\alpha \not{D}\psi. \tag{E2}$$

[1] Z. J. Liu, A. Weinstein, and P. Xu, Manin triples for Lie bialgebroids, J. Diff. Geom. **45**, 547 (1997).
[2] P. Ševera, Letters to Alan Weinstein about Courant algebroids, arXiv:1707.00265.
[3] N. Hitchin, Generalized Calabi-Yau manifolds, Q. J. Mech. Appl. Math. **54**, 281 (2003).
[4] M. Gualtieri, Generalized complex geometry, Ph.D. thesis, Oxford University, 2004, arXiv:math/0401221.
[5] E. Bergshoeff, M. de Roo, B. de Wit, and P. van Nieuwenhuizen, Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields, Nucl. Phys. **B195**, 97 (1982).
[6] G. F. Chapline and N. S. Manton, Unification of Yang-Mills theory and supergravity in ten-dimensions, Phys. Lett. **120B**, 105 (1983).
[7] M. Dine, R. Rohm, N. Seiberg, and E. Witten, Gluino condensation in superstring models, Phys. Lett. **156B**, 55 (1985).
[8] K. Costello and S. Li, Twisted supergravity and its quantization, arXiv:1606.00365.
[9] A. Coimbra, C. Strickland-Constable, and D. Waldram, Supergravity as generalized geometry I: Type II theories, J. High Energy Phys. 11 (2011) 091.
[10] A. Coimbra, R. Minasian, H. Triendl, and D. Waldram, Generalised geometry for string corrections, J. High Energy Phys. 11 (2014) 160.
[11] These are half-densities with respect to the base M. Note that our formulation with half-densities differs slightly from the treatment in [9]. Although this might seem like a minor point, we will see later that it leads to some drastic simplifications in the formulas, in particular eliminating the need of higher fermionic corrections. We also emphasise here that the way we formulate things here is by using the $O(p,q)$ structure group, as opposed to the equivalent description in [9] in terms of $O(p,q)\times\mathbb{R}^+$.
[12] W. Siegel, Superspace duality in low-energy superstrings, Phys. Rev. D **48**, 2826 (1993).
[13] This condition can be relaxed even more to general Courant algebroids of rank 10, *if* we take all the fields to be complex valued.
[14] C. M. Hull, Generalised geometry for M-theory, J. High Energy Phys. 07 (2007) 079.
[15] P. P. Pacheco and D. Waldram, M-theory, exceptional generalised geometry and superpotentials, J. High Energy Phys. 09 (2008) 123.
[16] A. Coimbra, C. Strickland-Constable, and D. Waldram, $E_{d(d)}\times\mathbb{R}^+$ generalised geometry, connections and M theory, J. High Energy Phys. 02 (2014) 054.
[17] D. Cassani, O. de Felice, M. Petrini, C. Strickland-Constable, and D. Waldram, Exceptional generalised geometry for massive IIA and consistent reductions, J. High Energy Phys. 08 (2016) 074.
[18] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky, The geometry of the master equation and topological quantum field theory, Int. J. Mod. Phys. A **12**, 1405 (1997).
[19] This is not related to the other notions of dilatonic supergravity which can be found in the literature.
[20] For relevant details, definitions, and notation concerning spinors, half-densities, and Courant algebroids, see the Appendices A–D. As briefly mentioned before, the existence of $S_\pm$ requires the bundle to be spin, which boils down to certain mild topological conditions. In particular exact Courant algebroids are always spin [4].
[21] Recall that on transitive Courant algebroids these consist of infinitesimal diffeomorphisms, B-field gauge transformations, and gauge transformations.
[22] We are using the Koszul sign rule and graded-geometric conventions according to which the Lie derivative satisfies $L_X = [d, i_X]$ (where $[\cdot,\cdot]$ is the graded commutator), and the Hamiltonian vector field for a function h is defined by $i_{X_h}\omega = -dh$. Since the BV symplectic form $\omega = dp_i \wedge dq^i$ is odd, a (short) calculation shows that for any *even* function h we have

$$\{h,\cdot\} \coloneqq X_h = \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} + \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p^i},$$

where we used the *left* derivatives.
[23] To see this, we note that on a transitive Courant algebroid over a contractible base the Dirac operator is the sum of the Dirac operators on an exact Courant algebroid and a quadratic Lie algebra—in the first case this is simply the de Rham differential, while in the second case this is an algebraic operator on the spinors with respect to the Lie algebra (see Appendix D)—both with finite-dimensional cohomology.
[24] Recall that $a: E \to TM$ is the anchor map (see the Appendix C).
[25] Note that there are no nontrivial real solutions to $\bar{e}\gamma^\alpha e = 0$ in signature (9, 1). The twisting thus requires either signature (5, 5) or allowing complex spinors. The authors are grateful to Martin Cederwall for bringing this important fact to their attention.
[26] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. **165**, 311 (1994).
[27] Due to the lack of letters in the alphabet(s), we will be using H for both the 3-form flux and the bundle of half-densities. The distinction should always be clear from the context, and thus the reader will not be confused.
[28] Note that if we consider the action S_1 above together with the symmetry transformation (10), the gauge parameter ϵ contains a five-form, which does not generate a variation of ρ. This is much like the situation in exceptional generalised geometry [14–16], where imposing that the gauge parameters assemble into exceptional group representations leads to introducing such parameters. In that situation one can alternatively consider restricting the gauge parameters to forms of those degrees which do generate variations, as one would in supergravity, as implemented in [17], but this may lead to a different quantum theory. This illustrates that the specification of the gauge transformations should be considered part of the definition of the theory (see e.g., [18]).
[29] Again, if we allow the fields to take complex values, we can drop the signature requirement and only impose $\dim\mathfrak{g} = 10$. Then one can consider other examples, such as $\mathfrak{g} = \mathfrak{so}(5)$.
[30] For the last one this uses the fact that the bracket on $\mathfrak{g}$ is Lie.
[31] Again, note that some symmetries might be reducible (e.g., if $\mathfrak{g}$ has a nontrivial centre then there exist ζ's generating

trivial transformations), and so in order to quantise the theory we would need to compensate for this by introducing further ghosts.

[32] A. Alekseev and P. Xu, Derived brackets and Courant algebroids (unpublished).

[33] D. Li-Bland and E. Meinrenken, Courant algebroids and Poisson geometry, Int. Math. Res. Not. **2009**, 2106 (2009).

[34] B. Jurčo, F. Moučka, and J. Vysoký, Palatini variation in generalized geometry and string effective actions, J. Geom. Phys. **191**, 104909 (2023).

[35] P. Ševera, Poisson–Lie T-duality and courant algebroids, Lett. Math. Phys. **105**, 1689 (2015).

[36] M. Garcia-Fernandez, Ricci flow, Killing spinors, and T-duality in generalized geometry, Adv. Math. **350**, 1059 (2019).