The Hyperbolic Superposition
of a
General Quantum State

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ABSTRACT:

This dissertation documents an inquiry into the geometric structure of the Spaces used to represent a 2-Dimensional Quantum State Space as it relates to Quantum Computation. In particular we investigate the possible applications of Professor NJ Wildberger's Universal Hyperbolic Geometry and discover a genuinely intuitive method for constructing and hence visualising an otherwise abstract algebraic discipline.

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Contents

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Declaration

I confirm that this Thesis is my own work and that any and all materials from other sources have been fully acknowledged.

Signed:

Grant Hubbard

Chapter 1

Introduction

"Anyone who is not shocked by Quantum Theory has not understood it"
Niels Bohr

"I think I can safely say that nobody understands Quantum Mechanics"
Richard P Feynman

"Electrons behave in this respect in exactly the same way as photons; they are both screwy..."
Richard P Feynman

The author is motivated to find some way of conceptualising the mathematical model of the Quantum World as it relates to Quantum Computing; in particular as it is presented in Nielsen and Chuang's seminal work *"Quantum Computation and Quantum Information"* where a single Quantum State is described as a single vector in a Complex Vector Space. The infinite range of positions that this vector may assume, referred to as the Quantum Superposition together form the abstract mathematical object known as the *Qubit* [3]. The author's personal interest in Ancient Greek Geometry is brought to bear on what is in effect an attempt to *paint a picture* of this Qubit Space; a General Quantum State will be taken simply to refer to an arbitrary vector in a Qubit Space.

1.1 Objectives

This Thesis investigates the geometry of the Spaces in which Quantum Computation takes place, with regard to 2-Dimensional Cubit Space. We focus on presenting an overview of the geometries involved: Euclidean, Elliptic and Hyperbolic. In particular, our purpose is to present an intuitive understanding of the geometric framework in which the single 2-State is defined. To this end we investigate certain key aspects of **Universal Geometry**, a system proposed by Professor Norman J Wildberger of the University of New South Wales wherein a constructible, geometric explanation is given for both Elliptic and Hyperbolic Geometry; one that is founded on techniques first developed by the Ancient Greeks and was in existence long before 19^{th} Century Mathematicians such as Bolyai, Lobachevski, Beltrami, Klein, Riemann and Minkowski started formal investigations of what we now refer to collectively as *Non-Euclidean* geometries. [30][33][34][35] We will show that Wildberger's Universal Geometry may be used to present a clear, intuitive and concise method for conceptualising both Elliptic and Hyperbolic 2-Dimensional Planes.

Comparisons will be drawn between Wildberger's approach and the more traditional ones found in the wider literature. This will be done by focusing on the equivalence of the Metrics implicit in each Space. Further, since the Density Matrix is considered by many to be the most versatile and complete description of the single Quantum State, we have chosen an increasingly widely accepted model of ρ to contrast Universal Geometry with; the **Gyrovector Space Method** is an entirely algebraic portrait of a Hyperbolic Space proposed by Professor Abraham A Ungar of North Dakota State University. Whilst contemporary, Ungar's method is founded on the work of Bolyai, Lobachevski, Möbius and Einstein and stresses the inherently *Relativistic* nature of the Hyperbolic State Space. [1][24][25]

1.2 Quantum Considerations

The main motivation for this Thesis is to present geometric interpretations of 2-D Qubit Space; itself being a representation of the Space of possible states of a closed quantum system [3]. Representations of the Quantum State must satisfy the postulates of Quantum Mechanics and since we are concerned solely with modelling a single State in a single moment of time, it is the First Postulate that is of particular relevance: "*The State of a closed Quantum System is described by a Vector in a Hilbert Space*"[3].

In the wider literature, the Pure State is diagrammatically represented as a point on the surface of the Bloch Sphere; the Complex surface of a unit sphere in a 3-dimensional Real Vector Space. We will show that the Bloch Sphere is mathematically equivalent to the Elliptic Riemann Surface $\mathbb{C}_\infty$.

Quantum States may also be represented in Density Operator form, in which case a distinction may be made between the Pure and the Mixed State of a closed quantum system [3]; for Pure States, the Eigenvalues of the Density Matrix must be Real. Ungar tells us that the Mixed State Density Operator Space is a 2-dimensional Hyperbolic Space in terms of the metric extant upon it [1]. We will show that this is consistent with the view that a 3-dimensional Complex Vector Space is the underlying structure of the Hyperbolic Plane. Since the z-component of the Complex Vector Space is conceived of as being purely Imaginary, Real Vectors may only be represented on the Hyperbolic Plane *in the limit* as a Point at Infinity.

1.3 The Structure of the Thesis

Chapter 2
We start by presenting the work done in investigating and defining the various mathematical structures encountered throughout the Thesis before going on to discuss the nature of the relationships that exist between them. A certain amount of inconsistency exists in the exact formalisms and notations used in the wider literature and we take this opportunity to attempt to set out a consistent notation. Whilst it is assumed that the reader will be familiar with most of the definitions given, they are presented, at least in part to document the journey undertaken by the author. More importantly the process has served to clarify the issues at stake and has been instrumental in fulfilling the objectives of the research as a whole.

Chapter 3
We present the traditional model applied to representations of a single 2-level Quantum State and introduce the standard definitions of the State as described by the Vector $|\psi\rangle$ within the context of the Bloch Sphere.
We then investigate the derivation of the Density Matrix ρ as a representation of the Pure or Mixed Quantum State and present the Pauli Operators as being the standard method for so doing.
We also present the relevant aspects of the work of AA Ungar as it relates to the contemporary modelling of the Quantum State. In order to explain Ungar's method, an investigation of the work of WR Hamilton is presented together with a discussion on Spinor Matrices; this generates an alternative explanation of the Gyrovector Method to that given by Ungar himself. We conclude the chapter with a summary of findings so far.

Chapter 4
NJ Wildberger's basic approach to what he refers to as Rational Trigonometry and Universal Geometry is introduced for the first time. We start with the Rational parametrisation of the unit circle since this method is adopted and developed by Wildberger throughout. We include a brief discussion of Rational Trigonometry as a means of outlining the concepts of Quadrance and Spread that Wildberger uses to replace the more familiar notions of Distance and Angle. The use of Quadrance and Spread allows for definition of Euclidean, Spherical, Elliptic and Hyperbolic Geometry over a Rational, and therefore *General* Field. This Chapter is a prelude to the introduction of Universal Elliptic and Universal Hyperbolic Geometry.

Chapter 5
We now present the key aspects of Universal Elliptic Geometry as it pertains to representations of the Pure Quantum State, together with a detailed explanation and proofs of Wildberger's alternative Elliptic Distance Function. We go on to describe a more traditional view of the Elliptic Surface, also related to the derivations of the Riemann Sphere, and prove the equivalence between the traditional and Universal Elliptic Metric.

Chapter 6
We now present the key aspects of Universal Hyperbolic Geometry as it pertains to representations of the Mixed Quantum State, together with a detailed explanation of Wildberger's alternative Hyperbolic Distance Function. We go on to describe a more traditional view of Hyperbolic Space and compare the traditional to the Universal Metric. We see that the question of whether the underlying structure is Real or Complex becomes interesting with regard to the differentiation of the Pure and the Mixed Quantum State; it is this that is the overall deciding factor as to whether the State is modelled in Elliptic or Hyperbolic Space.

Chapter 7
We now present the main conclusions of the research. We find that Universal Elliptic and Hyperbolic Geometries may be employed as a means of conceiving 2-D Quantum State Space. We suggest that the use of such constructive geometric methods may provide an understanding of the basics of mathematical models of Quantum Mechanics to a wider audience.

Although we are exclusively concerned with Non-Euclidean geometries, we provide an overview of Hilbert's Axioms for Euclidean Geometry as **Appendix A** since the study was a necessary part of the research for the author.
As a point of interest, in **Appendix B** we present a geometric interpretation of Group Structures; in particular that of the group structure on a circle.

1.3.1 Notation

$\mathbf{v}$	Bold, Lower Case signifies a vector
$\mathbf{u} \cdot \mathbf{v}$	The Inner Product
$\mathbf{v} \cdot \mathbf{v}$	The Inner Product equates to $\|\mathbf{v}\|^2$
$\mathbf{M}$	Bold, Upper Case signifies a matrix
$\mathbf{M}^T$	The Transpose Matrix of $\mathbf{M}$
$\mathbf{M}^{-1}$	The Inverse Matrix of $\mathbf{M}$
$\mathbf{M}^\dagger$	The Transpose Conjugate of $\mathbf{M}$
$\|\mathbf{M}\|$	The Magnitude of $\mathbf{M}$
$\text{Det}\mathbf{M}$	The Determinant of $\mathbf{M}$
$\mathbb{I}$	The Identity Matrix
ρ	A Hermitian Matrix known as the Density Operator
$\|\psi\rangle$	A Column Vector associated to a Quantum State
$\langle\psi\|$	The Transpose Conjugate of $\|\psi\rangle$
$\langle\psi\|\phi\rangle$	The Inner Product
$\langle\psi\|\psi\rangle$	The Inner Product equates to $\|\|\psi\rangle\|^2$
$\|\psi\rangle\langle\psi\|$	The Outer Product is a Square Matrix
z^*	The Complex Conjugate of z
$a * b$	A general and unspecified Binary Operation between elements a and b
$\Longrightarrow$	Implies
$\Longleftrightarrow$	Implies and is Implied by
$\longmapsto$	Maps to: Homomorphic, Injective
$\longleftrightarrow$	Maps to and from: Isomorphic, Bijective, one-to-one
$\sim$	Identifies with
$:$	Such That
$\mathbb{F}$	A general and unspecified Field
$\mathbb{F}^n$	An n-dimensional Space (n-Space) defined over a general Field $\mathbb{F}$
$\mathbb{R}^n$	A Real n-Space or Real n-dimensional Vector Space
$\mathbb{R}_n$	A Real n-dimensional Subspace
$\mathbb{C}^n$	A Complex n-Space or Complex Vector Space
$\mathbb{C}_n$	A Complex n-dimensional Subspace
$\mathbb{E}^n$	A Euclidean n-Space
$\mathbb{E}^n(\mathbb{R})$	A Real Euclidean n-Space; etc.
$\mathbb{S}^n$	An Elliptic n-Space
$\mathbb{H}^n$	A Hyperbolic n-Space
$\mathcal{H}$	A Hilbert Space
$\mathbb{P}$	A Projection
n	Always signifies a non-negative Integer: $n \in \mathbb{N}^+$
λ	Always signifies a Scalar Quantity: $\lambda \in \mathbb{F}$

Chapter 2

Preliminaries

The purpose of this Chapter is to review and standardise the definitions of the various mathematical constructs encountered throughout the Thesis. Although it is to be assumed that the reader will be familiar with these definitions, the act of summarising and gathering them together in this way has made up a reasonably sized percentage of the research and has played no small part in the understanding of the subject under consideration.

For example, we have the First Postulate of Quantum Mechanics which asserts that the state of a closed quantum system is described by a vector in a Hilbert Space [3][18]; there is an obvious requirement to describe exactly what is meant by this. Since the vectors in question are generally to be thought of as Complex vectors we should go further and elaborate on what is meant by defining a vector, within a Space, over a Field.

The Preliminaries are by no means an exhaustive study, rather an account of the preconditions required for this research. With regard to the reference material used throughout this Chapter, it is obviously the case that many and varied sources exist. Rather than pepper the document with repeated reference notes which make the overall content difficult to read, each section begins with a statement as to the specific source used for the particular definitions that follow. We have attempted to introduce a standardised approach to the notation at this point.

2.1 Fields

Since we will later introduce a geometric treatment of Euclidean, Elliptic and Hyperbolic Spaces that may be defined over a general Field, we start with a definition of *Field* taken from pages 8 & 9 of John M Howie's *Real Analysis* [11].

We assume the existence of the following Identities:

1) The Identity for Addition : $\exists\ 0 \in \mathbb{F}$ such that $\forall a \in \mathbb{F}$ we have $a + 0 = a$
2) The Identity for Multiplication : $\exists\ 1 \in \mathbb{F}$ such that $\forall a \in \mathbb{F}$ we have $a \times 1 = a$

Fields may also possess what is referred to as the Archimedean Property:

$$\forall a \in \mathbb{F}\ \exists\ n \in \mathbb{N} \text{ such that } n > a$$

This is also referred to as the *Axiom of Archimedes.*

Definition 1. *A Field $\mathbb{F}$ is a Set, of which the elements $a, b, c \in \mathbb{F}$ obey the following Axioms for the Binary Operations of Addition and Multiplication.*

(F1) *The Law of Closure:* $(a + b) \in \mathbb{F}$ *and* $(a \cdot b) \in \mathbb{F}$
(F2) *The Law of Associativity:* $a + (b + c) = (a + b) + c$ *and* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
(F3) *The Law of Commutativity:* $a + b = b + a$ *and* $a \cdot b = b \cdot a$
(F4) *The Existence of Inverse:* $\exists\ b \in \mathbb{F} : a + b = 0$ *and* $\forall a \in \mathbb{F}, a \neq 0, \exists\ c \in \mathbb{F} : a \cdot c = 1$
(F5) *The Distributive Law:* $a \cdot (b + c) = ab + ac$

2.1.1 Ordered Fields

In addition to Axioms *(F1)-(F5)* Fields may possess a natural order relation and are therefore known as Ordered Fields [11].

Definition 2. *An Ordered Field is a Field satisfying the Axioms relating to Order in addition to (F1)-(F5):*

(O1) *The Transitive Law:* $a < b$ *and* $b < c$ *implies* $a < c$
(O2) *The Trichotomy Law: Either* $a < b$, $a > b$ *or* $a = b$
(O3) *The Law of Additive Compatibility:* $a < b \Longrightarrow (a + c) < (b + c)$
(O4) *The Law of Multiplicative Compatibility: If* $a < b$ *and* $c > 0$ *then* $(a \cdot c) < (b \cdot c)$

Commentary on Fields

We see from the existence of Identity and from *(F4)* that we have $b = (-a)$ and $c = a^{-1}$ that generate the Inverse Operations of Subtraction and Division from the same set of Axioms. The inclusion of the Multiplicative Inverse (Division) carries with it the *coda* that $\frac{a}{0}$ is generally considered to be undefinable.

Example 1. *It can be seen from the definitions given here that both the Rational Numbers* $\mathbb{Q}$ *and the Real Numbers* $\mathbb{R}$ *form Ordered Archimedean Fields.*

2.2 Spaces

Spaces may be 1-dimensional, 2 or 3-dimensional, n-dimensional or have infinite dimensions. Spaces may be Euclidean or Non-Euclidean, they may be Real or Complex. The motivation for this discussion on mathematical Space is twofold: firstly to define a Hilbert Space, secondly to lay down definitions required for the different geometries encountered. We begin with a definition of a general Space taken from page 1 of Erwin Kreyszig's *Introductory Functional Analysis* [14]

Definition 3. *A Space is a Set* $\mathbb{X}$ *with some added structure which defines the relationship between the elements of* $\mathbb{X}$

The Set $\mathbb{X}$ may, for example, be a Scalar Field such as $\mathbb{R}$. The structure applied to $\mathbb{X}$ may take the form of a set of Axioms; in the case of Metric Spaces for example, the structure is a defined notion of *distance*.

Example 2. *The Real Number Line is the 1-dimensional Space denoted* $\mathbb{R}^1$ *(or just* $\mathbb{R}$*) where the applied structure is the set of Field Axioms (F1) to (O4).*

Example 3. $\mathbb{R} \times \mathbb{R}$ *forms a Real 2-dimensional Space denoted* $\mathbb{R}^2$

where $\mathbb{R} \times \mathbb{R}$ is the Cartesian Product of two Real Number Lines [11]. More will be said with regard to what is meant by *dimension* in Section 2.3

Definition 4. *The Cartesian Product of sets $\mathbb{X}$ and $\mathbb{Y}$ is the Set given by:*

$$\mathbb{X} \times \mathbb{Y} = \{(x, y) : x \in \mathbb{X}, y \in \mathbb{Y}\} \tag{2.1}$$

This Thesis focuses on geometric interpretations as far as possible; we would like to know for example if a Space supports a Euclidean or Non-Euclidean geometry. We cannot know whether $\mathbb{R}^2$ is a 2-dimensional Euclidean Space denoted $\mathbb{E}^2$ from the information given so far. Since Vector Spaces form an intrinsic part of further discussions we will first define a Vector Space.

2.2.1 Vector Spaces

The following definition is taken from page 1 of Paul Halmos's *Finite Dimensional Vector Spaces* [9] see also page 13 of [19] for notation.

Definition 5. *A Vector Space $\mathbb{F}^n$ is a Set of elements $\boldsymbol{x}$,$\boldsymbol{y}$,$\boldsymbol{z}$,..., called Vectors defined over a Field $\mathbb{F}$ where for every $\boldsymbol{x}$,$\boldsymbol{y}$,$\boldsymbol{z} \in \mathbb{F}^n$ there exists $\boldsymbol{z} \in \mathbb{F}^n$ such that:*

(V1) *Addition is commutative:* $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x} = \boldsymbol{z}$
(V2) *Addition is associative:* $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$
(V3) *Additive Identity: There exists a vector $\boldsymbol{0} \in \mathbb{F}^n$ such that $\forall\ \boldsymbol{x} \in \mathbb{F}^n$, $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$*
(V4) *Additive Inverse: For all $\boldsymbol{x} \in \mathbb{F}^n$ there exists $\boldsymbol{y} \in \mathbb{F}^n$ such that $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{0}$*

For every pair α (a scalar $\mathbb{F}$) and $\boldsymbol{x}$ (a vector in $\mathbb{F}^n$) there exists a vector $\boldsymbol{z} \in \mathbb{F}^n$ called the Product $\boldsymbol{z} = \alpha\boldsymbol{x}$ such that:

(V5) *Multiplication is distributive wrt vector addition:* $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y} = \boldsymbol{z}$
(V6) *Multiplication is distributive wrt scalar addition:* $(\alpha_1 + \alpha_2)\boldsymbol{x} = \alpha_1\boldsymbol{x} + \alpha_2\boldsymbol{x} = \boldsymbol{z}$
(V7) *Multiplication is associative:* $\alpha_1(\alpha_2\boldsymbol{x}) = (\alpha_1\alpha_2)\boldsymbol{x} = \boldsymbol{z}$
(V8) *Multiplicative Identity: There exists $e \in \mathbb{F}$ such that $e\boldsymbol{x} = \boldsymbol{x}$*

Example 4. *$\mathbb{R}^3$ is the 3-dimensional Real Vector Space consisting of ordered triples $\boldsymbol{x} = (x_1, x_2, x_3)$ of Real Numbers*

Example 5. *$\mathbb{C}^2$ is the 2-dimensional Complex Vector Space consisting of ordered pairs $\boldsymbol{z} = (z_1, z_2)$ of Complex Numbers*

Next, we would like to know whether our Vector Space is Euclidean or otherwise. See Section 2.2.1 together with Appendix A for a fuller description of Euclidean Spaces.

2.2.2 Euclidean n-Space $\mathbb{E}^n$

A general model of $\mathbb{E}^n$ given on page 13 of John Ratcliffe's *Foundations of Hyperbolic Manifolds* [19] is the n-dimensional Vector Space $\mathbb{F}^n$ that supports the following Euclidean Inner Product:

Definition 6. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be vectors in $\mathbb{R}^n$, then the Euclidean Inner Product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is the Real Number:*

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_1y_1 + x_2y_2 + ... + x_ny_n \tag{2.2}$$

The Euclidean Inner Product is used to define the Euclidean Norm (length) of $\boldsymbol{x}$:

Definition 7. *Let $\boldsymbol{x}$ be a vector in $\mathbb{R}^n$, then the Euclidean Norm is the Real Number:*

$$|\boldsymbol{x}| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{y}} \tag{2.3}$$

2.2.3 Spherical n-Space $\mathbb{S}^n$

The following definition is taken from page 35 of John Ratcliffe's *Foundations of Hyperbolic Manifolds* [19]

Definition 8. *The Spherical Space $\mathbb{S}^n$ is the surface of a unit sphere parametrised by a Vector Space supporting the Euclidean Norm:*

$$\mathbb{S}^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : |\boldsymbol{x}| = 1\} \tag{2.4}$$

Example 6. *The unit sphere $x^2 + y^2 + z^2 = 1$ is a sphere in the Vector Space $\mathbb{R}^3$, the surface of which is the Space $\mathbb{S}^2$*

For Euclidean Geometry, there is a *unique* Line passing through two distinct Points; see Appendix A. If we wish to retain this property for the Surface $\mathbb{S}^2$ we have a problem; since any sphere is antipodally symmetric, a certain amount of geometric information is duplicated, see page 41 of [19]. To see the consequence of duplication, imagine two Great Circles on a sphere; they meet in two places at points antipodal to each other. Therefore no unique Line exists between them; there would be infinitely many such Lines. We should stress that we refer here to *straight lines* which on a sphere correspond to Great Circles. This duplication is eradicated by treating the vectors $\boldsymbol{x}$ and $\boldsymbol{-x}$ both $\in \mathbb{R}^3$ as a single entity $\pm\boldsymbol{x}$. In this case we refer to the Space generated as an Elliptic Space. Elliptic Geometry retains the Euclidean property of there existing a unique Line through any two distinct Points [19]. From this point on we will refer to $\mathbb{S}^2$ as being a 2-D Elliptic Surface, which we will discuss in further detail in Sections 2.2.6 and 2.4 and in Chapter 5

2.2.4 Lorentzian n-Space $\mathbb{L}^n$

Pages 54 & 55 of John Ratcliffe's *Foundations of Hyperbolic Manifolds* [19] defines a Lorentzian Space as the n-dimensional Vector Space $\mathbb{F}^n$ that supports the following Lorentzian Inner Product:

Definition 9. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be vectors in $\mathbb{R}^n$, then the Lorentzian Inner Product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is the Real Number:*

$$\boldsymbol{x} \circ \boldsymbol{y} = x_1 y_1 + x_2 y_2 + ... - x_n y_n \tag{2.5}$$

The Lorentzian Inner Product is used to define the Lorentzian Norm (length) of $\boldsymbol{x}$:

Definition 10. *Let $\boldsymbol{x}$ be a vector in $\mathbb{R}^n$, then the Lorentzian Norm is the Complex Number:*

$$||\boldsymbol{x}|| = \sqrt{\boldsymbol{x} \circ \boldsymbol{y}} \tag{2.6}$$

2.2.5 Hyperbolic n-Space $\mathbb{H}^n$

The following definition is taken from page 61 of John Ratcliffe's *Foundations of Hyperbolic Manifolds* [19]

Definition 11. *The Hyperbolic Space $\mathbb{H}^n$ is the surface of a unit sphere of Imaginary radius parametrised by a Vector Space supporting the Lorentzian Norm:*

$$\mathbb{H}^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : ||\boldsymbol{x}||^2 = -1\} \tag{2.7}$$

Example 7. *The surface of the sphere $x^2 + y^2 - z^2 = -1$ is a Hyperbolic Surface.*

Note the similarity between Spherical and Hyperbolic Spaces; the further classification of Space requires the definition of a *Distance Function* and leads to the development of a *Metric Space.*

2.2.6 Metric Spaces

The following axioms for a general Metric Space are taken from page 3 of Erwin Kreyszig's *Introductory Functional Analysis* [14] and from MM & E Deza's *Encyclopaedia of Distances* [16]

Definition 12. *A Metric Space is a pairing of a Set $\mathbb{X}$ with a distance function d such that for all $x, y, z \in \mathbb{X}$ the following Axioms are satisfied:*

(M1) d *is Real, Finite and non-negative*
(M2) $d(x, y) = 0 \iff x = y$
(M3) $d(x, y) = d(y, x)$
(M4) $d(x, y) \leq d(x, z) + d(z, y)$

The phrase *Euclidean Metric Space* is a general term referring to a Space supporting a Euclidean Distance Function. Non-Euclidean Metric Spaces are then primarily defined by Non-Euclidean distance functions. The three key Metrics encountered in this Thesis are the *Euclidean*, the *Elliptic* and the *Hyperbolic* distance functions.

2.2.7 Euclidean Metric Spaces

Example 8. *The Real Number Line is the Real Euclidean Metric Space $\mathbb{R}^1$ defined over the Field $\mathbb{R}$ together with the Euclidean Distance Function between Points a and b given by the equation:*

$$d_1(a, b) = |b - a| \tag{2.8}$$

This gives the *shortest distance*, along the Line $\mathbb{R}^1$ from the Point $a \in \mathbb{R}$ to the Point $b \in \mathbb{R}$

Example 9. *The Real 2-dimensional Euclidean Plane is the Metric Space $\mathbb{R}^2$ where the set of ordered pairs of Real Numbers $a = (x_1, y_1)$ and $b = (x_2, y_2)$ is considered, together with the Euclidean Metric:*

$$d_2(a, b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{2.9}$$

This gives the *shortest distance*, across the plane $\mathbb{R}^2$ from the Point $a \in (\mathbb{R} \times \mathbb{R})$ to the Point $b \in (\mathbb{R} \times \mathbb{R})$

2.2.8 Elliptic Metric Spaces

For the Elliptic case, taken from page 159 of [16] we have the *Fubini-Study Metric* ξ also referred to as the *Elliptic Distance Function* d_{ell} and the *Hermitian Elliptic Metric.* Chapter 5 of this Thesis looks at the underlying nature of Elliptic Geometry and provides further context for the definition of the distance given as:

Definition 13. *The Elliptic Distance d_{ell} between ψ and ϕ is:*

$$d_{ell}(\psi, \phi) = \xi(\psi, \phi) = \arccos \sqrt{\frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}} \tag{2.10}$$

where $|\psi\rangle$ is the Dirac Notation for the unit vector tied to the origin of a Real 3-dimensional Euclidean Space that parametrises a unit sphere and ψ is a Point on the surface; so that d_{ell} gives the *shortest distance* across the 2-dimensional Elliptic Surface $\mathbb{S}^2$, between Points ψ and ϕ both on the Surface. Note that $|\psi\rangle$ does not refer to a Quantum State in this case. Inner Products and Norms are Euclidean.

2.2.9 Hyperbolic Metric Spaces

For the Hyperbolic case, from page 114 of [16] we have *Hyperbolic Metric d_{hyp}*. Chapter 6 of this Thesis looks at the underlying nature of Hyperbolic Geometry and provides further context for the definition of distance given as:

Definition 14. *The Hyperbolic Distance d_{hyp} between ψ and ϕ is:*

$$d_{hyp}(\psi, \phi) = arccosh \sqrt{\frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}} \tag{2.11}$$

where $|\psi\rangle$ is the Dirac Notation for the unit vector tied to the origin of a 3-dimensional Euclidean Space that parametrises an Imaginary unit sphere [19] and ψ is a Point on the surface; so that d_{hyp} gives the *shortest distance* across the 2-dimensional Hyperbolic Surface $\mathbb{H}^2$, between Points ψ and ϕ both on the Surface. Note that $|\psi\rangle$ does not refer to a Quantum State in this case. Inner Products and Norms are Lorentzian.

2.2.10 Complete Metric Spaces

Section 1.4 (pages 24-28) of [14] deals with the *Completeness* of a Metric Space; Kreyszig states that a Metric Space $\mathbb{M}$ is complete if every *Cauchy Sequence* in $\mathbb{M}$ converges to a limit that is also in $\mathbb{M}$.

Definition 15. *A Cauchy Sequence x_n is a sequence for which the following statement is true:*

$$\forall\ \epsilon > 0\ \exists\ N \textit{ dependent on } \epsilon : |x_m - x_n| < \epsilon\ \forall\ m, n > N$$

Where ϵ is any arbitrarily small positive value and N is an arbitrary point in the sequence that makes the statement true; if such an N cannot be found, the sequence is not Cauchy. Cauchy Sequences in any Metric Space $\mathbb{M}$ that is defined over $\mathbb{R}$ or $\mathbb{C}$ clearly converge to a limit also in $\mathbb{M}$. We should stress however that there are Cauchy Sequences defined over more general Fields $\mathbb{F}$ that do not converge to a limit in $\mathbb{F}$. Kreyszig's Theorem 1.4-4 on page 28 of [14] sums up the criteria relevant to this Thesis:

The Real Number Line and the Complex Plane are Complete Metric Spaces.

2.2.11 Hilbert Space $\mathcal{H}$

On page 128 of Kreyszig's *Functional Analysis* of [14] we are told that a Hilbert Space is a Complete Inner Product Space. An Inner Product Space is a Subspace that sustains an Inner Product:

Definition 16. *Let $\mathbb{F}_n$ be a subset of $\mathbb{F}^n$: Then $\mathbb{F}_n$ is a Subspace of $\mathbb{F}^n$ if and only if:*

(S1) *There exists a vector $\boldsymbol{0} \in \mathbb{F}_n$ so that the Subspace is non-empty*
(S2) *For all $\boldsymbol{u},\boldsymbol{v} \in \mathbb{F}_n$ we have $(\boldsymbol{u}+\boldsymbol{v}) \in \mathbb{F}_n$ hence closure under vector addition*
(S3) *For all $\boldsymbol{u} \in \mathbb{F}_n$ and $\alpha \in \mathbb{F}$ we have $\alpha\boldsymbol{u} \in \mathbb{F}_n$ hence closure under scalar multiplication*

The following definition is taken from page 129 of Erwin Kreyszig [14]

Definition 17. *An Inner Product Space, is a Subspace $\mathbb{F}_n$ defined over a field $\mathbb{F}$ that sustains the following Map, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors in $\mathbb{F}_n$ and α is a scalar in the Field $\mathbb{F}$:*

$$(\boldsymbol{x},\boldsymbol{y}) \longmapsto \langle \boldsymbol{x}|\boldsymbol{y}\rangle = \alpha \tag{2.12}$$

where $\langle \boldsymbol{x}|\boldsymbol{y}\rangle \equiv \boldsymbol{x}\cdot\boldsymbol{y}$ is the Inner Product of $\boldsymbol{x}$ and $\boldsymbol{y}$ and the following Axioms are met:

(IP1) $\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{y}\cdot\boldsymbol{x} = \alpha \in \mathbb{R}$ *hence Commutative*
(IP2) $(\alpha\boldsymbol{x}+\beta\boldsymbol{y})\cdot\boldsymbol{z} = (\alpha\boldsymbol{x}\cdot\boldsymbol{z}) + (\beta\boldsymbol{y}\cdot\boldsymbol{z})$ *hence Distributive*
(IP3) $\boldsymbol{x}\cdot\boldsymbol{x} \geq 0$
(IP4) $\boldsymbol{x}\cdot\boldsymbol{x} = 0 \Longleftrightarrow \boldsymbol{x} = 0$

Example 10. *Let $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ be vectors in the Real Subspace $\mathbb{R}_3$ then the Euclidean Inner Product is given by: $\boldsymbol{x}\cdot\boldsymbol{y} = (x_1y_1 + x_2y_2 + x_3y_3) = \alpha \in \mathbb{R}$*

On Page 129 of [14] Kreyszig states that Inner Product Spaces *pre*-Hilbert Spaces and that:

Definition 18. *A Hilbert Space is a Complete Inner Product Space denoted $\mathcal{H}$*

However, Appendix III of Paul Halmos's *Finite Dimensional Vector Spaces* [9] we have the following statement:

> *The definition of Hilbert Space is easy: it is an infinite dimensional unitary space satisfying one extra condition - namely completeness.*

This thesis is primarily concerned with the projection of a 3-Space onto a 2-D Surface; an in-depth enquiry as to the distinction between Kreyszig and Halmos on the matter of infinite dimensionality is a matter for further research; we will use Definition 18.

2.3 Groups

Group Structures play an important part in the formalism of Quantum Mechanics due to the use of Vectors and Matrices as well as the rotational properties of the Complex numbers. The reader may be interested to refer at this stage to Section B.1 for a completely geometric interpretation of a Group Structure.

The following definitions relating to a general Group $\mathbb{G}$ are taken from page 622 of Erwin Kreyszig's *Introductory Functional Analysis* [14] where * is an arbitrary *Binary Operation.*

Definition 19. *A Group* $\mathbb{G} = (\mathbb{G}, *)$ *is a Set of elements:* $\mathbb{G} = \{x, y, z, ...\}$ *such that* $\mathbb{G} * \mathbb{G} \longmapsto \mathbb{G}$*, and for all* $x, y, z \in \mathbb{G}$ *we have the following:*

(G1) *The Law of Associativity:* $(x * y) * z = x * (y * z)$
(G2) *The Existence of Identity:* $x * \mathbb{I} = \mathbb{I} * x = x$
(G3) *The Existence of Inverse:* $\exists\ y \in \mathbb{G} : x * y = y * x = \mathbb{I}$

Definition 20. $\mathbb{G}$ *is an Abelian Group if we also have:*

(G4) *The Commutative Law:* $\forall\ x, y \in \mathbb{G}\ x * y = y * x$

2.3.1 Orthogonal Groups

Orthogonal Matrices are important elements of Linear Algebra and hence Quantum Mechanics; the following definitions are taken from page 18 of John Ratcliffe's *Foundations of Hyperbolic Manifolds* [19]

Definition 21. *The General Orthogonal Group* $\boldsymbol{O}(n\mathbb{F})$ *is the set of all* $n \times n$ *Orthogonal Matrices denoted* $\boldsymbol{O}$ *such that* $\det \boldsymbol{O} = \pm 1$*, with elements defined over a Real Field* $\mathbb{F}$

A Subgroup of $\mathbf{O}(n\mathbb{F})$ is the Special Orthogonal Group:

Definition 22. *The Special Orthogonal Group* $\boldsymbol{SO}(n\mathbb{F})$ *is the set of all* $n \times n$ *Orthogonal Matrices denoted* $\boldsymbol{O}$ *such that* $\det \boldsymbol{O} = 1$*, with elements defined over a Field* $\mathbb{F}$

The Columns or the Rows of $\mathbf{O}$ form an Orthonormal Basis for $\mathbb{R}^n$ This is a useful way of establishing the *dimension* of a Real Space denoted $\mathbb{R}^n$ since the number of Basis Vectors in a Space is the Dimension of the Space [9].

Definition 23. *An Orthogonal Matrix* $\boldsymbol{O}$ *has the property:*

$$\boldsymbol{O}^T = \boldsymbol{O}^{-1} \iff \boldsymbol{O}\boldsymbol{O}^T = \boldsymbol{O}^T\boldsymbol{O} = \mathbb{I} \tag{2.13}$$

2.3.2 Unitary Groups

With the introduction of the Complex Field $\mathbb{C}$ we move from Orthogonal Matrices to Unitary Matrices. The following definitions are taken from page 147 of John Ratcliffe's *Foundations of Hyperbolic Manifolds* [19]

Definition 24. *The General Unitary Group* $\boldsymbol{U}(n\mathbb{C})$ *is the set of all* $n \times n$ *Unitary Matrices denoted* $\boldsymbol{U}$ *such that* $\det \boldsymbol{U} = \pm 1$*, with elements in* $\mathbb{C}$

A Subgroup of $\mathbf{U}(n\mathbb{C})$ is the Special Unitary Group:

Definition 25. *The Special Unitary Group* $\boldsymbol{SU}(n\mathbb{C})$ *is the set of all* $n \times n$ *unitary matrices denoted* $\boldsymbol{U}$ *such that* $\det \boldsymbol{U} = 1$*, with elements in* $\mathbb{C}$

The Columns or the Rows of $\mathbf{U}$ form an Orthonormal Basis for $\mathbb{C}^n$ This is a useful way of establishing the *dimension* of a Complex Space denoted $\mathbb{C}^n$ since the number of Basis Vectors in a Space is the Dimension of the Space [9].

Definition 26. *A Unitary Matrix* $\boldsymbol{U}$ *has the property:*

$$\boldsymbol{U}^\dagger = \boldsymbol{U}^{-1} \iff \boldsymbol{U}\boldsymbol{U}^\dagger = \boldsymbol{U}^\dagger\boldsymbol{U} = \mathbb{I} \tag{2.14}$$

2.3.3 Matrices

Important matrices exist other than those cited as Group elements; we present the matrix types discussed in the Thesis next with definitions from [9], [14] and [19]. We are exclusively concerned with 2×2 matrices.

Definition 27. *The Trace of the matrix* $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is given as:* $Trace(\boldsymbol{A}) = (a + d)$

Definition 28. *The Transpose of the matrix* $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is* $\boldsymbol{A}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Definition 29. *The Complex Transpose of the matrix* $\boldsymbol{A}$ *(also referred to as the Hermitian Conjugate) is the matrix* $\boldsymbol{A}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$

Definition 30. *The Inverse of matrix* $\boldsymbol{A}$ *is the matrix* $\boldsymbol{A}^{-1} : \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \mathbb{I}$

Note that the Inverse of Matrix $\boldsymbol{A}$ is defined only where $Det\boldsymbol{A} \neq 0$

From Section 2.3.2 we see that if $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$ then $\boldsymbol{A}$ is Orthogonal.

From Section 2.3.2 we see that if $\boldsymbol{A}^\dagger = \boldsymbol{A}^{-1}$ then $\boldsymbol{A}$ is Unitary.

Note that for matrices with Real-valued entries, $\boldsymbol{A}^T = \boldsymbol{A}^\dagger$

Definition 31. *A Hermitian matrix* $\boldsymbol{H}$ *has the property:* $\boldsymbol{H} = \boldsymbol{H}^\dagger$

Example 11. *The matrix* $\frac{1}{2}\mathbb{I} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\mathbb{I}^\dagger$ *is a Hermitian matrix*

We will introduce the Dirac Notation for matrices at this point:

$$\boldsymbol{A} = |\boldsymbol{A}\rangle \text{ and } \boldsymbol{A}^\dagger = \langle\boldsymbol{A}|$$

Definition 32. *The Inner Product of matrices* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *is given as the Hilbert-Schmidt Inner Product [3]:*

$$\langle\boldsymbol{B}|\boldsymbol{A}\rangle = Trace(\boldsymbol{B}^\dagger\boldsymbol{A}) \tag{2.15}$$

Definition 33. *The Magnitude of matrix* $\boldsymbol{A}$ *is given as:*

$$\langle\boldsymbol{A}|\boldsymbol{A}\rangle = Trace(\boldsymbol{A}^\dagger\boldsymbol{A}) \tag{2.16}$$

Since, for Hermitian matrices we have: $\boldsymbol{H}^\dagger = \boldsymbol{H}$ it is clear that $\langle\boldsymbol{H}|\boldsymbol{H}\rangle = Trace(\boldsymbol{H}^2)$

On page 250 of Steven Roman's *Advanced Linear Algebra* [20] we have the following:

Definition 34. *A Positive Definite Hermitian matrix* $\boldsymbol{H}$ *has the property:*

$$\langle\boldsymbol{H}\boldsymbol{v}, \boldsymbol{v}\rangle > 0 \ \forall \ \boldsymbol{v} \neq 0$$

Definition 35. *The Density Matrix, also known as the Density Operator, denoted* ρ *is a Positive Definite, Hermitian matrix with a Trace equal to 1:*

$$\rho = \boldsymbol{H} \Longrightarrow \langle\boldsymbol{H}|\boldsymbol{H}\rangle = 1 \tag{2.17}$$

2.3.4 The Spinor Matrix

The Spinor is specified by the 4 parameters shown in Figure 2.1: $(r, \theta, \phi, \alpha)$ where the first 3 are ordinary Polar Coordinates that correspond to the 3-dimensional Cartesian Coordinates *(x,y,z)* [22]. The angle α will not be investigated here so that for our purposes the Spinor is the vector $\mathbf{v} = \hat{x} + \hat{y} + \hat{z}$.

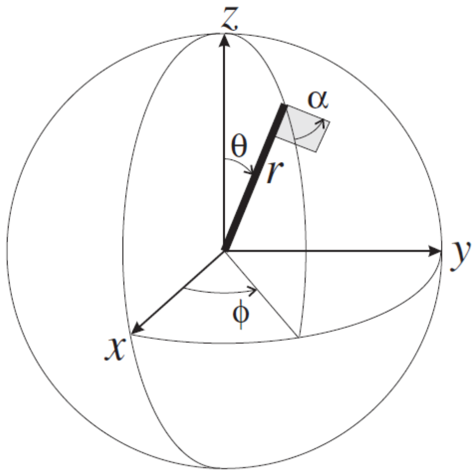

Figure 2.1: The Spinor [22]

Definition 36. *A Spinor Matrix* $\boldsymbol{X}$ *for a vector in a 3-dimensional Subspace is a* 2×2 *matrix generated in the following way:*

$$\mathbf{v}(x,y,z) \longmapsto \mathbf{X} = \begin{bmatrix} z & (x-iy) \\ (x+iy) & -z \end{bmatrix} = \mathbf{X}^{\dagger} \implies z \in \{\mathbb{C}\} \backslash \{\mathbb{R}\}$$

$$\mathbf{X}\mathbf{X}^{\dagger} = \mathbf{X}^{\dagger}\mathbf{X} = \begin{bmatrix} x^2+y^2+z^2 & 0 \\ 0 & x^2+y^2+z^2 \end{bmatrix} = |\mathbf{v}|^2 \mathbb{I} \tag{2.18}$$

The Spinor Matrix $\mathbf{X}$ is therefore Unitary if $|\mathbf{v}| = 1$ since:

$$\mathbf{X}\mathbf{X}^{\dagger} = \mathbf{X}^{\dagger}\mathbf{X} = |\mathbf{v}|^2 \mathbb{I} = \mathbb{I} \tag{2.19}$$

Considering the Determinant of $\mathbf{X}$ we also see that:

$$Det\mathbf{X} = -(z^2) - (x^2+y^2) = -(x^2+y^2+z^2) = -|\mathbf{v}|^2 = -1 \tag{2.20}$$

Therefore we have: $\mathbf{X} \in \mathbf{U}(2\mathbb{C})$ by Definition 24

The Spinor Matrix $\mathbf{X}$ has the following property if $|\mathbf{v}| = i$:

Since now we have $\mathrm{Det}(\mathbf{X}) = -|\mathbf{v}|^2 = 1$ then: $\mathbf{X} \in \mathbf{SU}(2\mathbb{C})$ by Definition 25

The Imaginary nature of $\mathbf{v}$ becomes important for all further discussions related to Hyperbolic Geometry and the work of Abraham A Ungar.

2.4 Projective Geometry

This section is intended to provide a brief overview of the two examples of Projective Geometry that we are concerned with: Elliptic (Spherical) and Hyperbolic. On page 53 of Marcel Berger's *Geometry Revealed* [2] we have the following definition of Projective Space:

Definition 37. *The n-dimensional Projective Space* $\mathbb{P}^n(\mathbb{F})$ *defined over the Field* $\mathbb{F}$ *is the Set of 1-dimensional Subspaces of the Vector Space* $\mathbb{F}^{n+1}$

The 1-D Subspaces mentioned here may be thought of as vectors through the Origin. Although Projective Geometry was known to Greek philosophers such as Archimedes and Apollonius, its nature and properties were not formally studied until the Renaissance Period when Mathematicians such as Girard Desargues (1591-1661) were called upon to formalise the increasingly sophisticated use of perspective in contemporary art [13]. In perspective drawings, whilst both distance and angle are distorted by the projection of the 3-dimensional world onto the 2-dimensional page, crucially, the *ratio* of distances and angles are retained [31]. Projective Geometry also enables the idea of a Point or Line *infinitely* far away to be brought into the viewing plane [31]. The 2-D Euclidean Plane may be thought of as being akin to a chess board with parallel lines stretching away to Infinity; their meeting point never being attained. Compare this to a *perspective* drawing of the same Plane where parallel lines do indeed "meet"at a vanishing point that *represents* a place infinitely far away.

The Stereographic Projection

With reference to Figure 2.2, the Stereographic Projection identifies a Point laying in the Equatorial Plane (shown shaded and referred to as $\mathbb{E}^2$ on page 107 of [19]) with a *unique* Point on the sphere, referred to as $\mathbb{S}^2$. For the sake of clarity, the shaded area $\mathbb{E}^2$ would in fact extend to Infinity in all directions so that Points on the *Southern hemisphere* of $\mathbb{S}^2$ identify with Points in the Plane outside of the sphere. The *South Pole* itself would then be identified with points at Infinity on $\mathbb{E}^2$ and *vice versa*. Riemann refers to this sphere as: $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ where he imagined $\mathbb{E}^2$ to be the entire Complex Plane and where points at Infinity on $\mathbb{C}$ correspond to the point $\boldsymbol{S} \in \mathbb{S}^2$

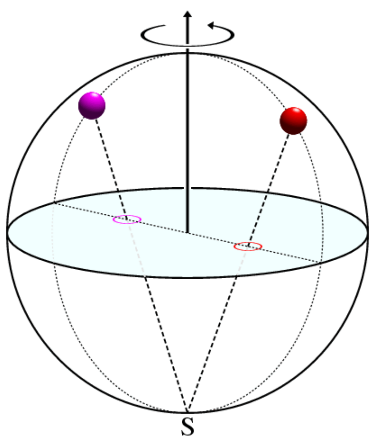

Figure 2.2: The Stereographic Projection [28]

The Gnomonic Projection

The Gnomonic Projection shown in Figure 2.3 identifies the Point $\boldsymbol{A}$ on the surface of the sphere with the Point $\boldsymbol{P}$ in the plane tangent to the North Pole $\boldsymbol{O}$ The Tangent Plane will be referred to as the Hyperbolic Plane $\mathbb{H}^2$ with its Origin at $\boldsymbol{O}$ The Gnomonic Projection is a Bijective Map between the *Northern hemisphere* of $\mathbb{S}^2$ and $\mathbb{H}^2$ [19]. By contrast to the Stereographic Projection, we now have Points on the Equator that are the Projective Points at Infinity with respect to the Tangent Plane. For derivations of Hyperbolic Geometry we adopt the convention that the Equatorial Plane of the sphere is the Cartesian xy-Plane, whilst the Polar Axis is the z-Axis; for the algebraic structure of Hyperbolic Geometry to work, the z-Axis is deemed to be purely Imaginary [19]. See Chapter 6

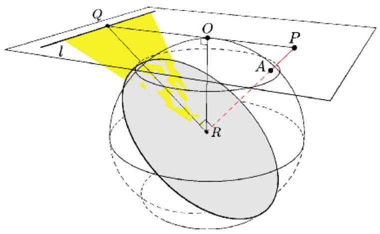

Figure 2.3: The Gnomonic Projection [17]

2.4.1 Consequences of Stereographic/Gnomonic Projections

For a further insight, refer to Figure 2.4 where the image of the North Polar aspect of Earth is presented under both Stereographic (left) [29] and Gnomonic (right) [27] projections. Under the Stereographic Projection (left) the Equator itself plays no special role; it is the slightly darker Parallel, 3^{rd} in from the edge of the image. Under this projection, the South Pole is infinitely far away with respect to the implied Metric and would be represented by the circumference of a circle with an infinitely large radius. An example of a Stereographic Projection is the Riemann Sphere, which is the projective Map between the unit sphere and the entire Complex Plane up to and including Infinity. It is therefore referred to as $\mathbb{C}_\infty$ [19] Turning now to the Gnomonic Projection (right) it is the Equator that is infinitely far away (in terms of the Metric across the disk) and points of the Southern Hemisphere are not present; only the *upper half* of the sphere is projected. Notice also that the distortion of distances that occurs under any projection, is different for each of the methods given here.

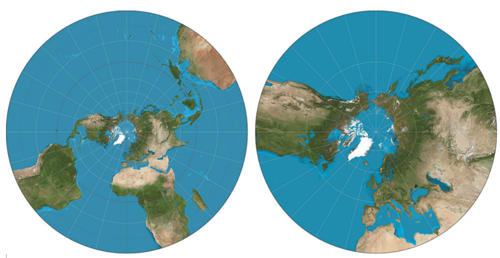

Figure 2.4: Stereographic and Gnomonic Projections of the North Polar Aspect of Earth

Chapter 3

A Quantum Mechanical System

Since the purpose of this Thesis is to investigate different geometric interpretations of the Quantum Mechanical system, we will now present an overview of the relevant aspects of the theory. Quantum Mechanics is essentially a mathematical framework that has been developed over the past 100 years or so for the purpose of modelling the behaviour of sub-atomic particles. This Thesis makes no attempt to speculate on whatever the underlying physical reality governing the behaviour of a quantum system might be. For general referencing purposes we direct the reader to pages 80 and 81 of [3]; currently the most cited work on Quantum Computation.

In this Chapter we describe the particular quantum system under investigation, known as the *qubit*. Then we explore the traditional Bloch Sphere model of the qubit and conclude with an overview of a contemporary model; Abraham A Ungar's Gyrovector Model.

3.1 Introducing the Qubit

Quantum Mechanics is built up from a set of postulates (Axioms), the first of which tells us that the Space in which Quantum Mechanics plays out must be a Hilbert Space. From Section 2.2.11 we see that a quantum mechanical system may therefore be represented within an Inner Product Space that sustains a complete Metric. According to Nielsen and Chuang in [3], the simplest quantum mechanical system is the qubit, which may be represented as a 2-dimensional State Space (Hilbert Space). The qubit will be the only example of a quantum system encountered in this Thesis. An arbitrary state, denoted $|\psi\rangle$ of such a system may be represented as follows:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \Longleftrightarrow |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} : \alpha, \beta \in \mathbb{C} \tag{3.1}$$

Here, $|0\rangle$ and $|1\rangle$ are the *Basis States* of the system and are represented by convention as column vectors. We see that the set of all possible States is infinite; it is the *theoretical possibility* of this infinite variety of States that is referred to as the Quantum Superposition of the State.

It is this property that distinguishes the classical *bit* (Binary Integer) from its quantum counterpart, the Quantum Binary Integer or qubit.

The *bit* is a 2-state system where only 2 possibilities exist; the qubit is also a 2-state system, but it is a system where there are 2 Real plus infinitely many Complex possibilities. We attempt no speculation as to the explanation or physical meaning of this. The unit circle of the familiar Argand Diagram paints a succinct picture of just such a situation: $z = e^{i\theta}$ is an arbitrary Point on the circle that is Complex-valued apart from 1 and (-1) Although infinitely many positions exist, only 2 of them can ever be *Realised*.

A constraint, known as the *normalisation condition* applies to the State $|\psi\rangle$ such that:

$$\langle\psi|\psi\rangle = 1 \iff |\alpha|^2 + |\beta|^2 = 1 \tag{3.2}$$

Here, $\langle\psi|\psi\rangle$ is the Euclidean Inner Product of $|\psi\rangle$ on itself (square of magnitude) and α and β are Complex numbers known as *Probability Amplitudes.* Considering specifically the magnitude of $|\psi\rangle$ we note that:

$$e^{i\gamma}|\psi\rangle = e^{i\gamma}\alpha|0\rangle + e^{i\gamma}\beta|1\rangle \implies \|e^{i\gamma}|\psi\rangle\| = \||\psi\rangle\| \tag{3.3}$$

since, for example, we have:

$$|e^{i\gamma}\alpha|^2 = (e^{i\gamma}\alpha)^*(e^{i\gamma}\alpha) = (e^{-i\gamma}\alpha^*)(e^{i\gamma}\alpha) = \alpha^*\alpha = |\alpha|^2 \tag{3.4}$$

Here, $e^{i\gamma}$ is referred to as the *Global Phase Factor* with γ being the *Azimuth Angle* of a given sphere; we will encounter the Global Phase Factor in more detail when we look at the Bloch Sphere in Section 3.2.

In the formalism of Quantum Mechanics, the state of a quantum system is a unit vector with a Complex structure so that, as has been said, we have:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle : \alpha, \beta \in \mathbb{C}$$

Having both $\alpha, \beta \in \mathbb{C}$ gives rise to a problem if we look for a geometric interpretation of the Space in question; as we effectively have 4 dimensions to deal with, 2 Real and 2 Complex. Intuitively though, we still have a Space given by the Cartesian Product $\mathbb{C} \times \mathbb{C} = \mathbb{C}^2$ as the ambient space in which $|\psi\rangle$ resides and we may specifically refer to this space as a 2-dimensional Complex Hilbert Space which we will denote $\mathcal{H}_2(\mathbb{C})$.

The normalisation condition applies, so that $|\psi\rangle$ is a unit vector from the origin of $\mathbb{C} \times \mathbb{C}$ to a Point on the unit circle of $\mathbb{C} \times \mathbb{C}$. We will refer to this unit circle as $\mathbb{S}^1$ and the unit vector $|\psi\rangle$ as being representative of the Pure Quantum State due to its unitary magnitude.

Definition 38. *Let the Set* $\mathbb{S}^1$ *be:*

$$\mathbb{S}^1 = \{|\psi\rangle \in \mathbb{C}^2 : \||\psi\rangle\| = 1\} \tag{3.5}$$

Proposition 1. *The elements of* $\mathbb{S}^1$ *may be used to generate the Special Orthogonal Group* $\boldsymbol{SO}(2\mathbb{R})$

Proof. To every element $|\psi\rangle$ of the Set $\mathbb{S}^1$ associate a 2×2 matrix called Ψ in the following way:

$$|\psi\rangle \mapsto \Psi = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{3.6}$$

Here, θ is an angle measured from some arbitrary, and we must assume, Real-axis; we see that Ψ has the following properties:

$$\Psi^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \implies \Psi\Psi^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \mathbb{I} \iff \Psi^T = \Psi^{-1}$$

So that, by Section 2.3 we have Ψ as an Orthogonal Matrix. Further, we have:

$$Det(\Psi) = \cos^2\theta + \sin^2\theta = 1 \tag{3.7}$$

Hence Ψ is an element of $\mathbf{SO}(2\mathbb{R})$. $\square$

The Second Postulate of Quantum Mechanics refers to the time evolution of a closed quantum system and asserts that this is described by a *unitary transformation* [3] so that we have:

$$U|\psi\rangle = |\psi'\rangle \tag{3.8}$$

where $|\psi\rangle$ is the State of the system at time t_1 and $|\psi'\rangle$ is the State at time t_2. We know therefore that U must be a 2×2 Unitary Matrix.

A Unitary Matrix generated by $|\psi\rangle$ would then be very useful, from which we see the motivation for Ψ; however, Ψ is Real-valued. To arrive at the proper definition of the Quantum State as a 2×2 Complex (Unitary) Matrix we will first investigate the Bloch Sphere representation of $|\psi\rangle$.

3.2 Introducing The Bloch Sphere

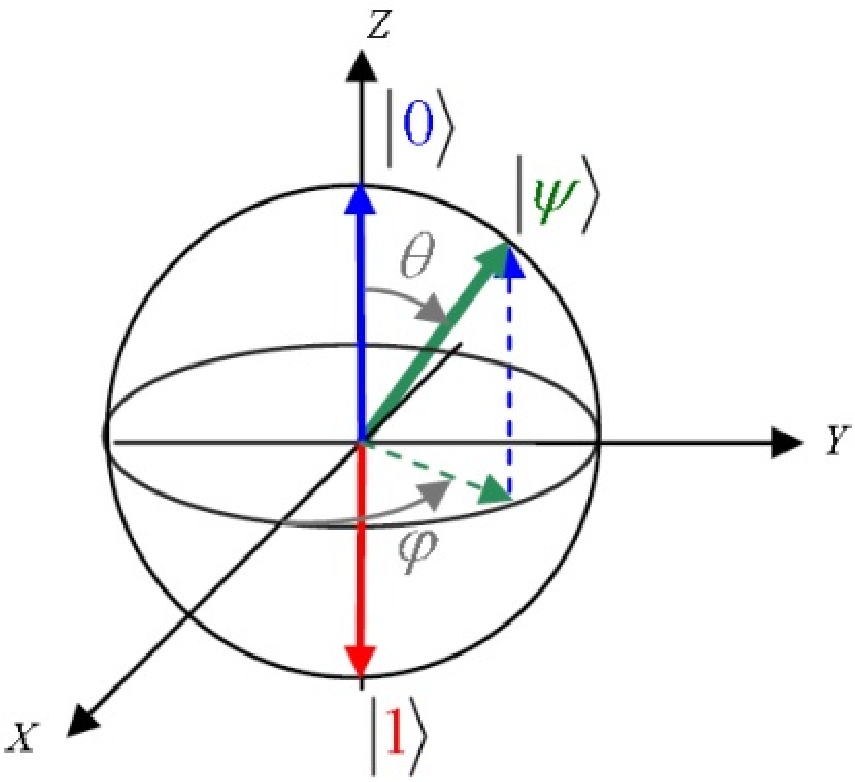

Figure 3.1: The Bloch Sphere

Referring to Figure 3.1 we see that the Bloch Sphere is the surface of a unit sphere in a Real 3-dimensional ambient Space with equation:

$$x^2 + y^2 + z^2 = 1 \tag{3.9}$$

We know that the Cartesian Coordinates are related to Polar Coordinates as follows:

$$x = r \sin\theta \cos\varphi \tag{3.10}$$
$$y = r \sin\theta \sin\varphi \tag{3.11}$$
$$z = r \cos\theta \tag{3.12}$$

The task is to demonstrate that the representation previously given for the Quantum Superposition of States also describes the surface of the Bloch Sphere, where the surface itself has a Complex algebraic structure. Recall that we have:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle : \alpha, \beta \in \mathbb{C} \Longleftrightarrow |\psi\rangle = r_\alpha e^{i\phi_\alpha}|0\rangle + r_\beta e^{i\phi_\beta}|1\rangle \tag{3.13}$$

Multiplying by a Global Phase Factor $e^{-i\phi_\alpha}$ gives:

$$e^{-i\phi_\alpha}|\psi\rangle = r_\alpha|0\rangle + r_\beta e^{i(\phi_\beta - \phi_\alpha)}|1\rangle \tag{3.14}$$

We have the coefficient $r_\beta e^{i(\phi_\beta - \phi_\alpha)}$ which is the exponential form of an arbitrary Complex Number; the rectangular form of which is $(x+iy)$. Using this form and applying the normalisation condition to Equation 3.14 gives:

$$|r_\alpha|^2 + |x+iy|^2 = r_\alpha^2 + (x-iy)(x+iy) = r_\alpha^2 + x^2 + y^2 = 1 \tag{3.15}$$

This is the equation of the Bloch Sphere shown in Figure 3.1 with z defined as being r_α. Hence, switching once again to the Polar form of x,y,z and recalling that $r=1$ we now have:

$$e^{-i\theta_\alpha}|\psi\rangle = z|0\rangle + (x+iy)|1\rangle = \cos\theta|0\rangle + \sin\theta(\cos\phi + i\sin\phi)|1\rangle$$

Which may be written as:

$$e^{-i\theta_\alpha}|\psi\rangle = |\psi\rangle = \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle \tag{3.16}$$

Referring again to Figure 3.1 with the angle θ as shown, we note the following:

$$\theta = 0 \Longrightarrow |\psi\rangle \quad = \quad |0\rangle \tag{3.17}$$

$$\theta = \frac{\pi}{2} \Longrightarrow |\psi\rangle \quad = \quad e^{i\phi}|1\rangle \tag{3.18}$$

We see that ϕ is indeed the *Azimuth Angle* marked φ in Figure 3.1.

Proposition 2. *The Bloch Sphere may be described in its entirety by Zenith and Azimuth Angles in the range* $0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$

Proof. Let $|\psi\rangle = \cos(\theta)|0\rangle + e^{i(\varphi+\pi)}\sin(\theta)|1\rangle$

then $|\psi'\rangle = \cos(\pi-\theta)|0\rangle + e^{i(\varphi+\pi)}\sin(\pi-\theta)|1\rangle$ is antipodal to $|\psi\rangle$

$$|\psi'\rangle \quad = \quad \cos(\pi-\theta)|0\rangle + e^{i(\varphi+\pi)}\sin(\pi-\theta)|1\rangle \tag{3.19}$$

$$= \quad -\cos\theta|0\rangle + e^{i\varphi}e^{i\pi}\sin\theta|1\rangle \tag{3.20}$$

$$= \quad -\cos\theta|0\rangle - e^{i\varphi}\sin\theta|1\rangle \tag{3.21}$$

$$= \quad -|\psi\rangle \tag{3.22}$$

We see that allowing the full revolution of θ from 0 to 2π is unnecessary. We may therefore define the Pure State on the Bloch Sphere as:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle : 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi \tag{3.23}$$

□

3.2.1 From the Bloch Sphere to Operator Space

Referring still to Figure 3.1 we now introduce the Bloch Vector, which will be denoted $\mathbf{v}$ and is a vector in a Real 3-dimensional Subspace:

$$\mathbf{v} = \alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}} : |\mathbf{v}| \leq 1 \tag{3.24}$$

For the Pure State we have the following condition:

$$|\mathbf{v}| = 1 \Longleftrightarrow \mathbf{v} \longleftrightarrow \pm|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \tag{3.25}$$

Note that no such mapping can exist for the case:

$$|\mathbf{v}| < 1 \tag{3.26}$$

By Definition 36 we can see that the Bloch Vector may be used to generate a Spinor Matrix $\mathbf{X}$ such that:

$$\mathbf{v} = \alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}} \longleftrightarrow \mathbf{X} = \begin{bmatrix} \gamma & \alpha - i\beta \\ \alpha + i\beta & -\gamma \end{bmatrix} \tag{3.27}$$

and from this, a Density Matrix may be generated as follows:

$$\frac{1}{2}\Big[\mathbb{I} + \mathbf{X}\Big] = \frac{1}{2}\begin{bmatrix} 1+\gamma & \alpha - i\beta \\ \alpha + i\beta & 1-\gamma \end{bmatrix} \tag{3.28}$$

3.3 Introducing the Density Matrix

An alternative representation of the Quantum State is that of the *Density Operator* or *Density Matrix* denoted ρ. The Hilbert-Schmidt Function defines the Inner Product of ρ to be the Trace of ρ^2

$$\langle\rho|\rho\rangle = Tr(\rho^2) \leq 1 \tag{3.29}$$

The case where the normalisation condition $\langle\rho|\rho\rangle = 1$ is met is called the Pure State. When considering $|\psi\rangle \in \mathbb{S}^1$ we have the Pure State Density Matrix as:

$$\rho = |\psi\rangle\langle\psi| \tag{3.30}$$

Proposition 3. *Every $|\psi\rangle \in \mathbb{S}^1$ generates a Density Operator ρ_ψ*

Proof. Take ρ_ψ as being the *Outer Product* of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$\rho_\psi = |\psi\rangle\langle\psi| = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} : \rho_\psi = \rho_\psi^\dagger : \tag{3.31}$$

We see that ρ_ψ has the following Properties:

$$\rho_\psi = \rho_\psi^\dagger \text{ and } Trace(\rho_\psi) = |\alpha|^2 + |\beta|^2 = \langle\psi|\psi\rangle = 1$$

□

Proposition 4. *Every $|\psi\rangle \in \mathbb{S}^2$ generates a Density Operator ρ_ψ*

Proof. Take ρ_ψ as being the *Outer Product* of $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

$$\rho_\psi = |\psi\rangle\langle\psi| = \begin{bmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix} : \rho_\psi = \rho_\psi^\dagger : \tag{3.32}$$

We see that ρ_ψ has the following Properties:

$$\rho_\psi = \rho_\psi^\dagger \text{ and } Trace(\rho_\psi) = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = \langle\psi|\psi\rangle = 1$$

□

Proposition 5. *The Map from $|\psi\rangle \in \mathbb{S}^1 \mapsto \rho$ is not unique.*

Proof. It is enough to demonstrate that:

$$|-\psi\rangle\langle-\psi| = |\psi\rangle\langle\psi| = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} = \rho \tag{3.33}$$

□

Proposition 6. *The Map from $|\psi\rangle \in \mathbb{S}^1 \mapsto \rho$ is not Bijective.*

Proof. We know that every $|\psi\rangle \in \mathbb{S}^1 \mapsto \rho$ but does every $\rho \mapsto |\psi\rangle$..? Choose ρ such that:

$$\rho = \begin{bmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{bmatrix} \tag{3.34}$$

Clearly ρ is a Hermitian Matrix yet $\rho \neq |\psi\rangle\langle\psi|$. That is to say that there is no $|\psi\rangle \in \mathbb{S}^1$ or $\mathbb{S}^2$ whose Outer Product is ρ as described in Equation 3.34 □

3.3.1 From The Pauli Operators to the Density Matrix

The Basis States for $\mathbb{S}^1$ and $\mathbb{S}^2$ are the 1×2 column vectors $|0\rangle$ and $|1\rangle$ as previously mentioned. The Basis States for the Bloch Sphere, when seen as a 3-dimensional object with both a Surface and an interior Space, are the Pauli Operators; see page 174 of [3]. We now present a different derivation of ρ using the 4 Pauli Operators $\mathbb{I}, X, Y, Z$:

$$\sigma_0 = \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Making use of this Basis Set, Abraham Ungar defines ρ as follows [1]:

$$\rho = \frac{1}{2}(\mathbb{I} + \sigma \cdot \mathbf{v}) \tag{3.35}$$

Here, $\mathbf{v}$ is the Bloch Vector described in Equation 3.24 and σ denotes the set of Pauli Operators X, Y, Z.

Equation 3.35 may then be restated as follows:

$$\rho = \frac{1}{2}\Big[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \Big] \tag{3.36}$$

where α, β, γ are the coefficients of the Basis Vectors. Expanding this gives:

$$\rho = \frac{1}{2}\Big[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Big] \tag{3.37}$$

which in turn simplifies to give:

$$\rho = \frac{1}{2}\Big[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \gamma & \alpha - \beta i \\ \alpha + \beta i & -\gamma \end{bmatrix} \Big] \tag{3.38}$$

$$\iff \rho = \frac{1}{2} \begin{bmatrix} 1 + \gamma & \alpha - \beta i \\ \alpha + \beta i & 1 - \gamma \end{bmatrix} \tag{3.39}$$

This derivation is presented as the "standard" form given in the main texts such as [3] and [18] as well as being referred to throughout by Ungar [1],[24]. However, we also note here the equivalence between this and the Spinor Matrix version given in Equation 3.28

We have now presented the traditional and widely accepted view of the various representations of the single Quantum State as they appear in the wider literature in relation to Quantum Computing. The Pauli Operator derivation of ρ dates back to the work of Von Neumann and has many applications in physics [1].

We are now ready to present, in outline, an alternative treatment of the Bloch Vector approach to ρ.

3.4 Introducing Gyrovector Space

We look now at the work of Professor Abraham A Ungar with respect to representations of the Quantum State as a Density Operator.

Our investigation of Ungar's method hinges on the his observation that the appropriate *Distance Function* to apply to Density Operators is the Bures Fidelity Metric; and that this is Hyperbolic in nature [1],[24],[26]

Recall that so far we have seen representations of $|\psi\rangle$ for Euclidean and Elliptic Spaces only. We will study Hyperbolic Spaces in detail in Chapter 6 but for now the salient point is that the Hyperbolic 2-Space $\mathbb{H}^2$ may be seen as a projection of a *Sphere of Imaginary radius* [19] (see pages 54 and 61 in particular)

$$\mathbb{H}^2 = \{\mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}|^2 = -1\} \tag{3.40}$$

This would imply that the vector $\mathbf{v}$ is not the Real-valued Bloch Vector as it has so far been defined for Elliptic Space. Indeed, this is the basis for Ungar's assertion that the Bloch Vector should more accurately be treated as a **Gyrovector** [1]

In this respect, we believe that the work of Sir William R Hamilton (1805-1865) is the correct starting point from which to view Gyrovector Space. We will therefore commence the discussion on Hyperbolic Operator Spaces with a presentation of Hamilton's 19^{th} Century work on **Quaternions**. An in depth enquiry into the finer detail of Quaternions would fall outside of the scope of this Thesis; here we look simply at those aspects that imply a Complex Vector Space.

3.4.1 Quaternions

A geometric interpretation of Complex numbers was first introduced by Jean-Robert Argand (1768-1822) as being the Vector Space of the 2-dimensional Argand Plane. For example, it may be seen that the binary operation of "addition" for Complex numbers is exactly vector addition in the Plane:

$$z_1 + z_2 = (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) = (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2) = z_3 \tag{3.41}$$

Similarly, multiplication of two Complex numbers of unit length may be viewed as a vector rotation about the origin of a 2-dimensional Plane.

$$z_1 z_2 = e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)} = z_3 : |z| = 1 \tag{3.42}$$

Hamilton was motivated by wondering whether such a geometric interpretation extended for 3-dimensions; his investigations culminated in his own development of the Quaternion and to the closely related Spinor Matrix [21]

Hamilton defined a quaternion $\mathbf{Q}$ to be a vector in a Real 4-dimensional Space in the following way [10][21]:

Definition 39. *Let $Q = ai + bj + ck + w$ such that $a, b, c, w \in \mathbb{R}$ and where $i^2 = j^2 = k^2 = ijk = -1$*

Where the element $(ai + bj + ck)$ is the rotational aspect of $\mathbf{Q}$ and w is the scalar aspect. It turns out that there is an alternative and already familiar way of defining Quaternions: they are the Pauli Operators multiplied by -i [36]

$$q_0 = \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; q_1 = I = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \; q_2 = J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; q_3 = K = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

Recall from Section 3.2 that the Pauli Operators form a Basis for a Real 3-dimensional Inner Product Space:

$$\sigma_0 = \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \sigma_1 = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \sigma_2 = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \; \sigma_3 = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where we have:

$$X^2 = Y^2 = Z^2 = iXYZ = \mathbb{I} \tag{3.43}$$

displaying the following similarity to the properties of Hamilton's Quaternions:

$$I^2 = J^2 = K^2 = IJK = -\mathbb{I} \tag{3.44}$$

We may say therefore that Hamilton's Quaternion acting on a vector $\mathbf{v}(\alpha, \beta, \gamma)$ may be expressed as "Imaginary" Pauli Operators acting on $\mathbf{v}$:

$$\mathbf{Q} = \Big[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \Big] \tag{3.45}$$

which simplifies as:

$$\mathbf{Q} = \Big[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -i\gamma & -\beta - i\alpha \\ \beta - i\alpha & i\gamma \end{bmatrix} \Big] = \begin{bmatrix} 1 - i\gamma & -\beta - i\alpha \\ \beta - i\alpha & 1 + i\gamma \end{bmatrix} \tag{3.46}$$

Recall that from Equation 3.27 we have the Spinor Matrix for $\mathbf{v}$ as:

$$\mathbf{X} = \begin{bmatrix} \gamma & \alpha - i\beta \\ \alpha + i\beta & -\gamma \end{bmatrix}$$

Multiplying $\mathbf{X}$ by -i gives:

$$-i\mathbf{X} = \begin{bmatrix} -i\gamma & -\beta - i\alpha \\ \beta - i\alpha & i\gamma \end{bmatrix} \tag{3.47}$$

This means that we can also describe $\mathbf{Q}$ as:

$$\mathbf{Q} = [\mathbb{I} - i\mathbf{X}] \tag{3.48}$$

Although it is clearly the case, it is important to note that $\mathbf{Q}$ acting on $\mathbf{v}$ has exactly the same effect as the standard Pauli Operators acting on $-i\mathbf{v}$

$$\Big[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} I \\ J \\ K \end{bmatrix} \Big] = \Big[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -i\alpha \\ -i\beta \\ -i\gamma \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \Big] \tag{3.49}$$

In other words, conceptually, we are free to do either of the following:

1) View the 3-dimensional Vector Space in which the Bloch Sphere resides as being a *purely Imaginary Space*, in which our Basis States may still be seen as being the Pauli Operators. Or;

2) View the Space as Real-valued, in which case we replace the Pauli Operators with Quaternions.

The consequence of either standpoint is that the Projected 2-Space has become a Hyperbolic Space [19]; the geometry of which will be discussed in Chapter 6

This is of particular relevance since it is Ungar's observation that the Metric for Density Operator Spaces is a Hyperbolic Metric; indeed Ungar states in many of his papers that "*Gyrovector Spaces provide the setting for Hyperbolic Geometry just as Vector Spaces provide the setting for Euclidean Geometry*" [26]

3.4.2 Hyperbolic and Relativistic Aspects of Gyrovectors

Ungar cites the work of Llewellyn H Thomas (1903-1992) throughout his discussions of Gyrovector Spaces [23] as further evidence of the Hyperbolic and therefore Relativistic properties of the Density Operator. The Thomas Rotation and more particularly the sequence of Rotations referred to as the Thomas Precession introduces *Relativistic* considerations into the modelling of the quantum state, the consequence which is to create a non-Euclidean Metric; specifically a Hyperbolic Metric [12]. An analysis of the Thomas Rotation is outside the scope of this Thesis; we mention it here as a possible avenue for further research. See also Ungar [23].

As an example of the Relativistic nature of the Gyrovector, we include a brief discussion from [24] on Möbius Addition, an intrinsic part of Gyrovector formalism:

Definition 40. *Möbius Addition* $\oplus$*: Let* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$$z \oplus z = \frac{z + z}{1 + z^* z} = \frac{2z}{1 + |z|^2} = \frac{2}{1 + |z|^2} \times z = \lambda z \tag{3.50}$$

This implies that $z \oplus z$ sends $z = re^{i\theta}$ to $z = \lambda re^{i\theta}$; in other words the *argument* of z is unchanged and no rotation results. It is also of interest to note:

$$\lim_{|z| \to 1} z \oplus z \to z \tag{3.51}$$

And further that for $|z| \ll 1$, $z \oplus z \approx 2z$ which, together with Equation 3.51 is entirely in accordance the *Relativistic* nature of Möbius Addition. Given that Complex Numbers may be viewed as vectors and that *velocity* is also a vector quantity, $\oplus$ reflects the different treatment required for velocities both *close to* and *far from* the Relativistic limit. However, in general it is the case:

$$|z| < |z \oplus z| < |2z| \tag{3.52}$$

3.4.3 The Gyrometric

Ungar states that the Density Operator ρ is the complete description of the Quantum State and that Density Operators are the elements of what he refers to as a Gyrovector Space [25]. Ungar defines ρ as follows:

$$\rho = \frac{1}{2}\begin{bmatrix} 1+\gamma & \alpha-\beta i \\ \alpha+\beta i & 1-\gamma \end{bmatrix} \tag{3.53}$$

From page 61 of [24] we have:

Definition 41. *The Gyrodistance $d_\oplus$ from point $\boldsymbol{a}$ to point $\boldsymbol{b}$ is given by:*

$$d_\oplus(\boldsymbol{a},\boldsymbol{b}) = \|\ominus\boldsymbol{a}\oplus\boldsymbol{b}\| = \|\boldsymbol{b}\ominus\boldsymbol{a}\| \tag{3.54}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are elements of a Gyrovector Space.

The notation $\oplus$ and $\ominus$ here stands for Gyroaddition and Gyrosubtraction respectively; both Relativistic operations that have the following properties $\forall$ $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbf{G}$ [24]:

$$\begin{aligned} d_\oplus(\mathbf{a},\mathbf{b}) &\geq 0 && (3.55)\\ d_\oplus(\mathbf{a},\mathbf{b}) &= 0 \Longleftrightarrow \mathbf{a}=\mathbf{b} && (3.56)\\ d_\oplus(\mathbf{a},\mathbf{b}) &= d_\oplus(\mathbf{b},\mathbf{a}) && (3.57)\\ d_\oplus(\mathbf{a},\mathbf{c}) &\leq d_\oplus(\mathbf{a},\mathbf{b})\oplus d_\oplus(\mathbf{b},\mathbf{c}) && (3.58) \end{aligned}$$

This set of Axioms are identical to those pertaining to a Metric Space (See Section 2.2.6); Ungar refers to the Bures Fidelity Metric as being the correct distance function across the Hyperbolic Plane.

Definition 42. *The Bures Fidelity Metric $\mathcal{F}$ between Density Matrices $\rho_{\boldsymbol{u}}$ and $\rho_{\boldsymbol{v}}$:*

$$\mathcal{F}(\rho_{\boldsymbol{u}},\rho_{\boldsymbol{v}}) = \frac{1}{2}\left[1+\boldsymbol{u}\cdot\boldsymbol{v}+\sqrt{1-\|\boldsymbol{u}\|^2}\sqrt{1-\|\boldsymbol{v}\|^2}\right] \tag{3.59}$$

To demonstrate the Complex nature of the underlying Vector Space, consider the Bures distance between $\rho_{\mathbf{u}}$ and $\rho_{\mathbf{u}}$; which we have every right to expect should be zero:

$$\mathcal{F}(\rho_{\mathbf{u}},\rho_{\mathbf{u}}) = \frac{1}{2}\left[1+\mathbf{u}\cdot\mathbf{u}+\sqrt{1-\|\mathbf{u}\|^2}\sqrt{1-\|\mathbf{u}\|^2}\right] = 0 \Longleftrightarrow \mathbf{u}\cdot\mathbf{u} = -1 \Rightarrow \|\mathbf{u}\|^2 = 1$$

Note that the condition that $\mathbf{u}\cdot\mathbf{u} = -1$ suggests that $\mathbf{u}$ has an Imaginary magnitude. This point will be further analysed in Chapters 6 and 7

3.5 Summary of Findings for Quantum State Spaces

The long established use of the Bloch Sphere model motivates our investigation of Universal Elliptic Geometry as a possible alternative; whereas, it has been Ungar's Gyrovector model that has motivated our investigations into Universal Hyperbolic Geometry.

3.5.1 The Use of the Spinor Matrix

There is an alternative to the traditional use of the Pauli Operators as a means of deriving ρ since the Spinor Matrix $\mathbf{X}$ of the Bloch Vector $\mathbf{v}$ can be employed to the same effect:

$$\rho = \frac{1}{2}\Big[\mathbb{I} + \mathbf{X}\Big] = \frac{1}{2}\begin{bmatrix} 1+\gamma & \alpha - i\beta \\ \alpha + i\beta & 1-\gamma \end{bmatrix} = \frac{1}{2}(\mathbb{I} + \sigma \cdot \mathbf{v})$$

Quaternions

Hamilton's Quaternions may also be generated by the Spinor Matrix, since we have:

$$\rho = \frac{1}{2}\Bigg[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} I \\ J \\ K \end{bmatrix}\Bigg] = \frac{1}{2}\Bigg[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -i\alpha \\ -i\beta \\ -i\gamma \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\Bigg] = \frac{1}{2}\Big[\mathbb{I} - i\mathbf{X}\Big]$$

3.5.2 The Nature of the Gyrovector

Ungar's observation that the Gyrovector should be seen as an inherently Hyperbolic entity is confirmed:

$$\mathcal{F}(\rho_{\mathbf{u}}, \rho_{\mathbf{u}}) = 0 \Longleftrightarrow \mathbf{u} \cdot \mathbf{u} = -1 \tag{3.60}$$

since we have the following definition of Hyperbolic *2*-Space from Definition 11

$$\mathbb{H}^2 = \{\mathbf{u} \in \mathbb{R}^3 : |\mathbf{u}|^2 = -1\} \tag{3.61}$$

For a more detailed explanation see Chapter 6

Chapter 4

Universal Geometry

We now come to the main topic of this Thesis; the work of Professor Norman J Wildberger and the question of whether this system can be applied to representations of the Quantum State. Wildberger's approach is essentially to go back to the work of the Ancient Greeks; most notably to that Apollonius of Perga (*circa* 262-190 BCE) and Pappus of Alexandria (*circa* 290-350 AD).

Wildberger's motivation in doing this appears to be twofold: firstly, it is the case that these ancient techniques provide very visual derivations for what are otherwise abstract algebraic expressions, and secondly since they can be adequately defined using Integer or Rational values, the system does not require any analysis of infinite sums or other definitions of Irrational and non-computable values [31, 32, 35].

Indeed, a striking feature of Universal Geometry is the fact that Irrational functions such as *Sine, Cosine* etc., are not required. We make no particular use of this fact in the Thesis, although we highlight it here as an avenue for possible further research.

We begin by presenting a method for parametrising a unit circle over an arbitrary Field; one that forms the basis for the Universal distance functions encountered in this Thesis.

4.1 The Rational Parametrisation of the Unit Circle

The following formula would have been known to Euclid and may be used to generate Integer values for side lengths of right triangles; otherwise known as Pythagorean Triples, where $m, n \in \mathbb{N}$:

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2 \tag{4.1}$$

By dividing through by the RHS, this can be restated so as to generate Rational values for right triangles with the hypotenuse of unit length:

$$\frac{(m^2 - n^2)^2}{(m^2 + n^2)^2} + \frac{(2mn)^2}{(m^2 + n^2)^2} = 1 \tag{4.2}$$

This is a rather unwieldy expression; by dividing through by m^2 and substituting $\frac{n}{m} = t$ we may achieve an expression of only one variable:

$$\frac{(1 - t^2)^2}{(1 + t^2)^2} + \frac{(2t)^2}{(1 + t^2)^2} = 1 \tag{4.3}$$

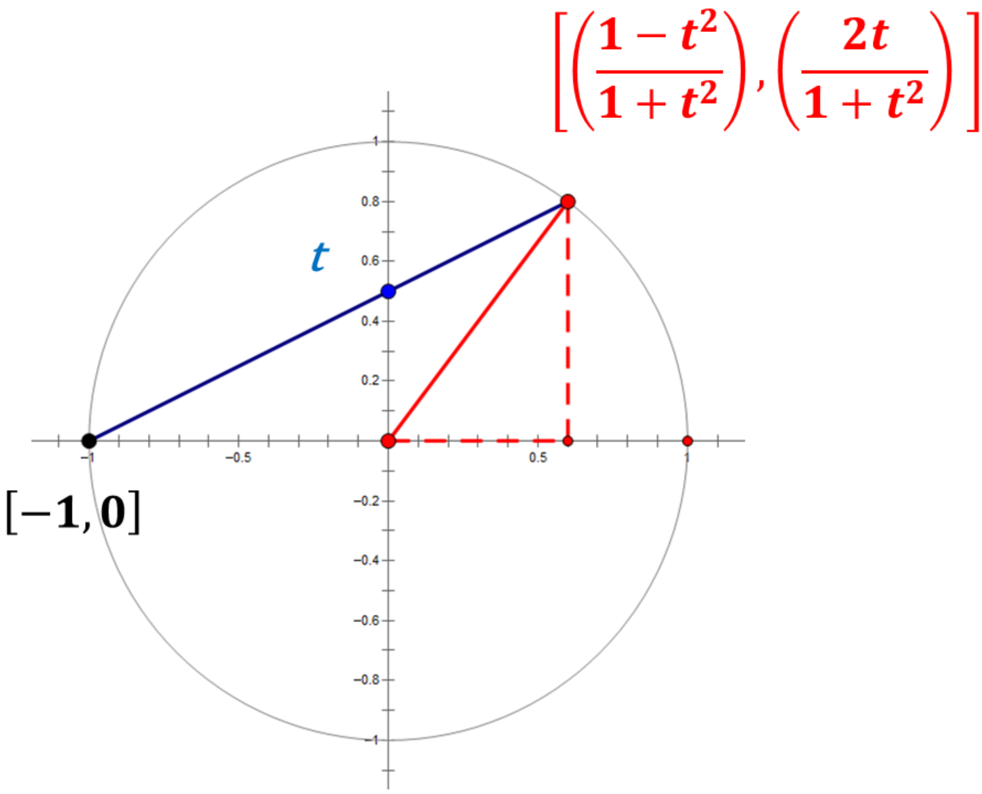

Figure 4.1: The Rational Unit Circle

Figure 4.1 shows a *Stereographic Projection* (see Section 2.4) between Points on the unit circle and Points on the vertical y-axis, with respect to the point $[-1, 0]$

Referring back to Equation 4.3 we see now the significance of t; it is the Point $t \in \mathbb{Q}$ which generates the traditional Cartesian Coordinates of the unit circle centred at [0,0] Given in fact that we have $t \in \mathbb{Q}$ then all points generated lying on the unit circle will also consist on Rational values.

Note that the point [-1,0] is a special case that represents t *at infinity*.

Note also the following Trigonometric Identities:

$$\tan\frac{\theta}{2} = t \Longleftrightarrow \cos\theta = \frac{(1-t^2)}{(1+t^2)} \Longleftrightarrow \sin\theta = \frac{(2t)}{(1+t^2)} \tag{4.4}$$

A recurring feature of Wildberger's work is the achievement of a geometric interpretation of familiar algebraic structures. Here we recall the fact that the angle subtended at the Point [-1,0] between the x-axis and the blue line is half that subtended at the [0,0] between the x-axis and the red line.

This Planar construction is developed to 3-dimensions for Elliptic and Hyperbolic Universal Geometry in subsequent chapters. Another recurring mechanism is the replacement of traditional linear distance measure with the quadratic *area measure*; together with a rational treatment of the notion of angle. This gives rise to a completely different view of Trigonometry which we will now briefly introduce.

4.2 Rational Trigonometry

As opposed to traditional trigonometry, Wildberger's approach does not involve the use of irrational or transcendental quantities such as square-roots or $\cos\theta$ and so no prior development of the Continuum is necessary [33, 35]. Traditional notions of *Distance and Angle* are replaced by the quadratic quantity called *Quadrance* and the proportional quantity called *Spread*; the claim being that the latter are both more general and more powerful [32].

Definition 43. *If A is the Point $A = [x_a, y_a]$ and B is the Point $B = [x_b, y_b]$ with reference to a fixed pair of rectangular axes, then the Quadrance Q(A,B) is given by the Equation:*

$$Q(A,B) = (x_b - x_a)^2 + (y_b - y_a)^2 \tag{4.5}$$

Using this definition we see that Quadrance is really the square of Euclidean Distance, in which case Pythagoras' Theorem can be stated thus:

The triangle ABC in Figure 4.2 has a right vertex at C precisely when:

$$Q(AC) + Q(BC) = Q(AB) \tag{4.6}$$

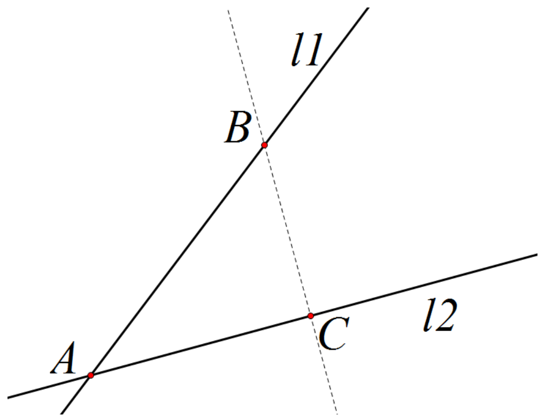

Figure 4.2: Rational Trigonometry

Definition 44. *Given that the Line BC in Figure 4.2 is perpendicular to l2 then the Spread S(l1,l2) between Lines $l1$ and $l2$ is given by the Equation:*

$$S(l1, l2) = \frac{Q(B,C)}{Q(A,B)} \tag{4.7}$$

For the notion of *Perpendicularity* we introduce the notation: $BC \perp l2$

Wildberger's proofs to these definitions rest on the following observations related to the equations of Lines:

In terms of the Cartesian Coordinate system a line in the 2-D Euclidean Plane has the equation $y = mx + c$ or more generally $ax + by + c = 0$. However, this equation is not unique to the line it defines as $\lambda(ax + by + c = 0)$ is the same line for any non-zero λ.

It is the *proportion* of the coefficients $a : b : c$ that uniquely specifies the line and this proportion is used throughout Wildberger's Universal Geometries and forms the basis of his observation that Quadrance and Spread are equivalent [33]. Wildberger further notes that the proportion can always be simplified thus:

$$a : b : c \equiv \frac{a}{c} : \frac{b}{c} : 1 \tag{4.8}$$

Two important concepts in any geometric or algebraic system are *Parallel* and *Perpendicular*. Consider the Lines:

$$l1 = a_1x + b_1y + c_1 = 0 \text{ and } l2 = a_2x + b_2y + c_2 = 0$$

Definition 45. *The Lines l1 and l2 are Parallel precisely when:*

$$a_1b_2 - a_2b_1 = 0 \Longleftrightarrow S(l1, l2) = 0 \tag{4.9}$$

As a means of relating this back to the more familiar notion from Linear Algebra, consider the determinant of the matrix A:

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \tag{4.10}$$

where *Det* $A = 0$ implies either infinitely many or no solutions; meaning that the column vectors of A represent either the same Line or two parallel Lines; so parallel in either case.

Definition 46. *The Lines l1 and l2 are Perpendicular precisely when:*

$$a_1a_2 + b_1b_2 = 0 \Longleftrightarrow S(l1, l2) = 1 \tag{4.11}$$

Referring now to Figure 4.3 we present (without proof) three Laws from Rational Trigonometry purely as a point of interest:

$$A, B, C \text{ are Collinear precisely when } (Q_1 + Q_2 + Q_3)^2 = 2(Q_1^2 + Q_2^2 + Q_3^2)$$

$$\text{For any triangle } ABC \text{ with non-zero Quadrances we have } \frac{S_1}{Q_1} = \frac{S_2}{Q_2} = \frac{S_3}{Q_3}$$

$$\text{For any triangle } ABC \ (S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3$$

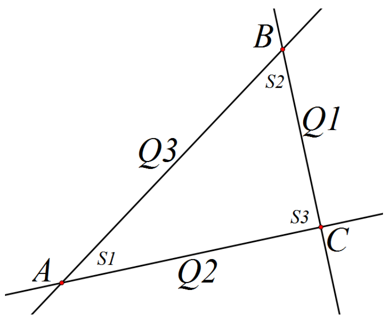

Figure 4.3: Rational Trigonometric Laws

Chapter 5

Spherical and Elliptic Geometry

Both Spherical and Elliptic Geometry are forms of Projective Geometry and may be seen as that geometry resulting from the Stereographic Projection of a Real 3-dimensional Subspace onto a 2-dimensional Plane. See Section 2.4

In Section 3.2 we saw that 2-dimensional Qubit Space may be described as a 2-dimensional Complex Surface referred to as $\mathbb{S}^2$ with an arbitrary State being:

$$\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle = \pm|\psi\rangle \in \mathbb{S}^2 \tag{5.1}$$

This would imply that $\mathbb{S}^2$ is an Elliptic 2-Space. We also know that the Bloch Vector $\mathbf{v}$ may be used to generate an alternative representation of the Quantum State as follows:

$$\mathbf{v} = \alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}} \longleftrightarrow \rho = \frac{1}{2}\begin{bmatrix} 1+\gamma & \alpha - i\beta \\ \alpha + i\beta & 1-\gamma \end{bmatrix} \tag{5.2}$$

Where the condition: $(\alpha, \beta, \gamma) \in \mathbb{R}$ implies the existence of the Real 3-Space required to begin this part of the investigation of Wildberger's method's as they apply to the modelling of Quantum State Space.

5.1 Universal Elliptic Geometry

The Rational parametrisation on the unit circle in the Plane started with an equation for generating Pythagorean Triples and ended with requiring the single parameter t. Wildberger now asks us to consider the fact that a unit sphere in a Real 3-dimensional Euclidean Space may be parametrised in a similar fashion so that we may parametrise the sphere with Points in a Real 2-dimensional Plane. The following formula, which holds for all $a, b, c, d \in \mathbb{Z}$ would have been known to Euclid and may be used to generate Pythagorean Quadruples [35]:

$$(2abd^2)^2 + (2b^2cd)^2 + (b^2d^2 - a^2d^2 - b^2c^2)^2 = (b^2d^2 + a^2d^2 + b^2c^2)^2 \tag{5.3}$$

This equation can in fact be extended to produce Pythagorean n-Tuplets [33]. Its relevance here is the fact that it may be visualised as parametrising a parallelepiped with Integer values on all sides plus the internal space diagonal. If we further imagine this parallelepiped to have one vertex situated at the Origin of a 3-dimensional Cartesian Coordinate System then we may parametrise a sphere with a radius:

$$r = (b^2d^2 + a^2d^2 + b^2c^2) \tag{5.4}$$

The expression given in Equation 5.3 is rather unwieldy; to simplify let $(b, d) \neq 0$ and divide through by b^2d^2 giving:

$$\left(\frac{2a}{b}\right)^2 + \left(\frac{2c}{d}\right)^2 + \left(1 - \frac{a^2}{b^2} - \frac{c^2}{d^2}\right)^2 = \left(1 + \frac{a^2}{b^2} + \frac{c^2}{d^2}\right)^2 \tag{5.5}$$

Now make the following substitution:

$$\text{Let } X = \frac{a}{b} \text{ and let } Y = \frac{c}{d}$$

Hence:

$$(2X)^2 + (2Y)^2 + (1 - X^2 - Y^2)^2 = (1 + X^2 + Y^2)^2 \tag{5.6}$$

Now divide through by the RHS:

$$\left(\frac{2X}{1 + X^2 + Y^2}\right)^2 + \left(\frac{2Y}{1 + X^2 + Y^2}\right)^2 + \left(\frac{1 - X^2 - Y^2}{1 + X^2 + Y^2}\right)^2 = 1 \tag{5.7}$$

So far we have done nothing more than present an interesting way of pinpointing exact points on a sphere of unit radius. We need to look at the significance of the values X and Y but first we draw attention to the fact that we may define a Bloch Vector as:

$$\mathbf{v} = \left(\frac{2X}{1 + X^2 + Y^2}\right)\hat{x} + \left(\frac{2Y}{1 + X^2 + Y^2}\right)\hat{y} + \left(\frac{1 - X^2 - Y^2}{1 + X^2 + Y^2}\right)\hat{z} \tag{5.8}$$

That is to say, the Bloch Vector may be parametrised by a single Point in the Equatorial Plane.

Wildberger Shows us that given the standard 3-dimensional Cartesian Coordinates x,y,z; any Point $[X, Y]$ in the 2-dimensional Equatorial Plane maps uniquely *and exactly* to the Point $\boldsymbol{V}$ on the 2-dimensional Surface of the unit sphere $\mathbb{S}^2$ as described by Equation 5.8 [35] Refer to Figure 5.1 and recall that the Equatorial Plane extends to Infinity in all directions.

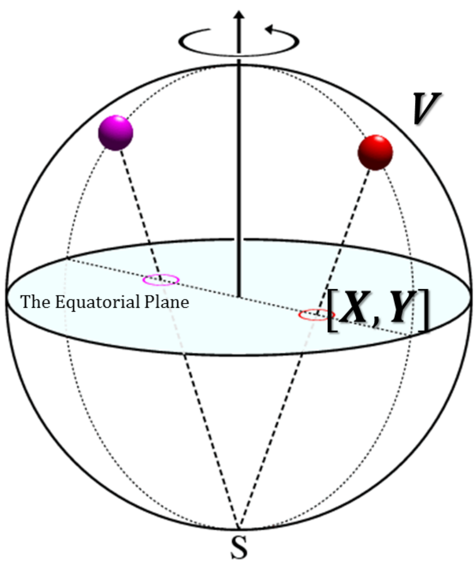

Figure 5.1: The Stereographic Projection [28]

Let us now identify the vector $\mathbf{v}$ in Equation 5.8 with the Bloch Vector $\mathbf{v}$; the sphere depicted in Figure 5.1 is then the Bloch Sphere and will be denoted $\mathbb{S}^2$ as previously. This Surface is analogous to the Riemann Sphere, otherwise known as the Elliptic Riemann Surface $\mathbb{C}_\infty$

In order to show that Wildberger's derivations are consistent with more traditional views of Elliptic Geometry, we will focus on the *Metric* or *Distance Function* given by Wildberger and show that this is related to Metrics given for other Spherical Surfaces.

5.2 The Elliptic Distance Function

Proposition 7. *The Fubini-Study Metric for the Bloch Sphere $\mathbb{S}^2$ and the Elliptic Riemann Surface $\mathbb{C}_\infty$ is related to Wildberger's Elliptic Distance Function Q for a general unit sphere via the Sine Function.*

Proof. From Section 2.2.6 we have the following definition for the Fubini-Study Metric ξ, also known as the Elliptic Distance Function d_{ell} [16] between Points ϕ and ψ on the Surface $\mathbb{S}^2$:

$$d_{ell}(\psi,\phi) = \xi(\psi,\phi) = \arccos\sqrt{\frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}} \tag{5.9}$$

Where $|\psi\rangle$ and $|\phi\rangle$ are to be thought of as unit Bloch Vectors in $\mathbb{R}^3$ defining their respective Points. For the sake of clarity; $|\psi\rangle$ and $|\phi\rangle$ are NOT Quantum State Vectors here.

Referring to Figure 5.1 we have Wildberger's Universal Elliptic Distance Function Q:

Definition 47. *Wildberger's Elliptic Distance Function Q between Points V and V' of the Surface $\mathbb{S}^2$ is given as:*

$$Q(V,V') = 1 - \frac{(XX'+YY'+1)^2}{(X^2+Y^2+1)(X'^2+Y'^2+1)} \tag{5.10}$$

where [X, Y] and [X',Y'] are Points in the Equatorial Plane corresponding to V and V' on the Sphere.

Required to Prove: $\xi \sim Q$; firstly, we see that:

$$\xi(\psi,\phi) = \arccos\sqrt{\frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}} \Longrightarrow \cos^2(\xi) = \frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle} \Longrightarrow \sin^2(\xi) = 1 - \frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}$$

Now looking at Q, recall that:

$$V = [X,Y] = \left[\frac{\alpha}{\gamma},\frac{\beta}{\gamma}\right] \text{ and } V' = [X',Y'] = \left[\frac{a}{c},\frac{b}{c}\right]$$

Then Equation 5.10 becomes:

$$Q(V,V') = 1 - \frac{\left[\frac{\alpha}{\gamma}\frac{a}{c} + \frac{\beta}{\gamma}\frac{b}{c} + 1\right]^2}{\left[\frac{\alpha^2}{\gamma^2} + \frac{\beta^2}{\gamma^2} + 1\right]\left[\frac{a^2}{c^2} + \frac{b^2}{c^2} + 1\right]} \tag{5.11}$$

Make single fractions where possible by multiplying through by the common denominators:

$$Q(V,V') = 1 - \frac{\left[\dfrac{a\alpha + b\beta + c\gamma}{c\gamma}\right]^2}{\left[\dfrac{\alpha^2+\beta^2+\gamma^2}{\gamma^2}\right]\left[\dfrac{a^2+b^2+c^2}{c^2}\right]} \tag{5.12}$$

Inverting the denominator and multiplying gives:

$$Q(V,V') = 1 - \left[\frac{a\alpha + b\beta + c\gamma}{c\gamma}\right]^2 \left[\frac{\gamma^2}{\alpha^2+\beta^2+\gamma^2}\right]\left[\frac{c^2}{a^2+b^2+c^2}\right] \tag{5.13}$$

Cancel and combine as a single fraction:

$$Q(V,V') = 1 - \frac{(a\alpha + b\beta + c\gamma)(a\alpha + b\beta + c\gamma)}{(\alpha^2+\beta^2+\gamma^2)(a^2+b^2+c^2)} \tag{5.14}$$

Now if $|\psi\rangle$ is a Bloch Vector then $|\psi\rangle = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}$
and if $|\phi\rangle$ is a Bloch Vector then $|\phi\rangle = a\hat{x} + b\hat{y} + c\hat{z}$ in which case we have:

$$\langle\psi|\phi\rangle = a\alpha + b\beta + c\gamma \tag{5.15}$$

and, without loss of generality:

$$\langle\psi|\psi\rangle = \alpha^2 + \beta^2 + \gamma^2 \tag{5.16}$$

Therefore:

$$Q(V,V') = 1 - \frac{(a\alpha + b\beta + c\gamma)(a\alpha + b\beta + c\gamma)}{(\alpha^2+\beta^2+\gamma^2)(a^2+b^2+c^2)} = 1 - \frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle} \tag{5.17}$$

Hence:

$$Q(V,V') = \sin^2\xi(\psi,\phi) = \sin^2 d_{ell}(\psi,\phi) \tag{5.18}$$

□

Wildberger makes clear reference to this relationship between the Universal Distance Function and the traditional versions but we have not been able to find an explicit proof in his published material. Now that we have provided this we can conclude that the Universal Elliptic Metric is consistent with traditional Elliptic Metrics. Next we will look at a more traditional derivation of Elliptic Geometry in respect of the Riemann Sphere denoted $\mathbb{C}_\infty$ and referred to elsewhere in this Thesis as $\mathbb{S}^2$.

5.3 Traditional Elliptic Geometry

In this section we present a more traditional derivation of Elliptic Geometry specifically in relation to the Riemann Sphere. Again, we are specifically referring to the *surface* of the sphere; this being a 2-dimensional Surface of Complex Structure that we denote $\mathbb{S}^2$ and Riemann referred to as $\mathbb{C}_\infty$.

We will ascertain the nature of the mappings that exist between the elements of $\mathbb{S}^2$; this time with reference to Richard Earl's 2007 Trinity College lecture notes [4], which have been an invaluable aid in formulating this Section. We present this Section to further demonstrate the authenticity of Wildberger's work.

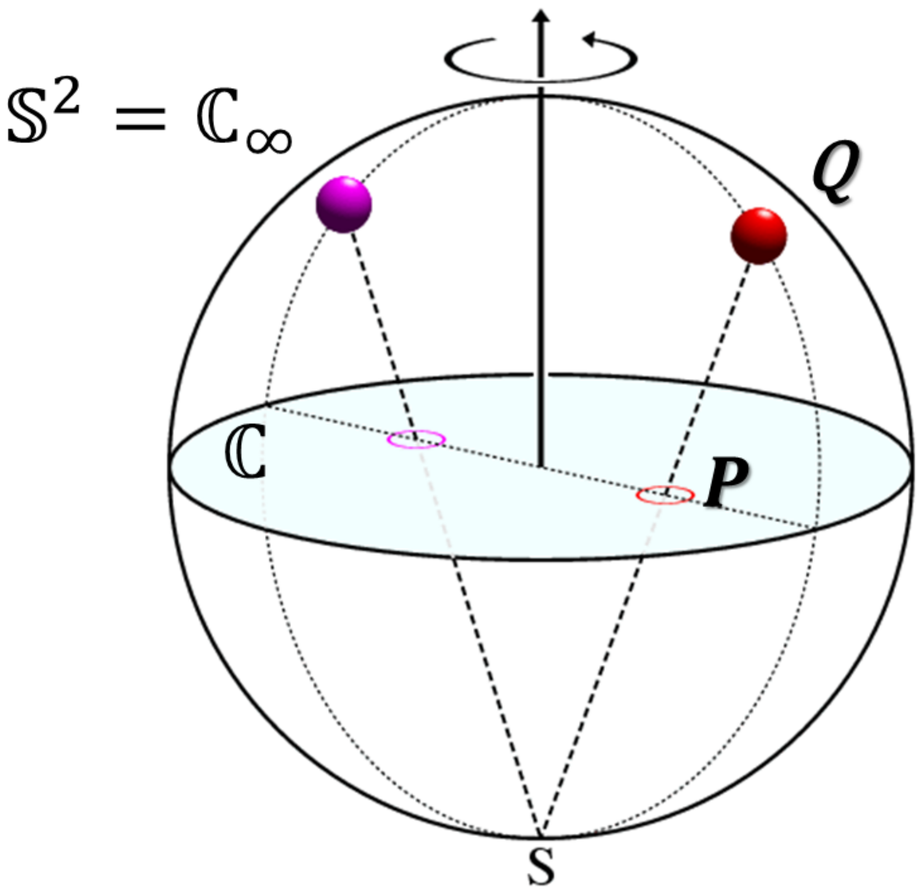

Figure 5.2: The Riemann Sphere [28]

Definition 48. *The Stereographic Projection is the Map* $\mathbb{P} : \mathbb{C} \leftrightarrow \mathbb{C}_\infty \backslash \{\boldsymbol{S}\} : \boldsymbol{P} \leftrightarrow \boldsymbol{Q}$

$$\mathbb{P}[x, y] \in \mathbb{C} = \boldsymbol{Q} = \left[\frac{2x}{(1 + x^2 + y^2)}, \frac{2y}{(1 + x^2 + y^2)}, \frac{(1 - x^2 - y^2)}{(1 + x^2 + y^2)}\right] \tag{5.19}$$

Proof. Let $\boldsymbol{S}$ be the point $\boldsymbol{S} = [0, 0, -1]$ called the South Pole of $\mathbb{S}^2$ then for all points $\boldsymbol{Q} \neq \boldsymbol{S} \in \mathbb{S}^2$ the line through $\boldsymbol{S}$ and $\boldsymbol{Q}$ intersects the Equatorial Plane at $\boldsymbol{P}$ so that we have $\boldsymbol{P} = [x, y, 0] \in \mathbb{C}$ and the Point $\boldsymbol{Q} = [\alpha, \beta, \gamma] \in \mathbb{S}^2$

We know that $\boldsymbol{Q}$ lies on the line $\boldsymbol{SP}$ and that the vector $\overrightarrow{\boldsymbol{SP}} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Hence:

$$\boldsymbol{Q} = [0,0,-1] + \lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = [(\lambda x), (\lambda y), (\lambda - 1)] \tag{5.20}$$

Substituting these values of $\boldsymbol{Q}$ into the equation of the sphere gives:

$$(\lambda x)^2 + (\lambda y)^2 + (\lambda - 1)^2 = 1 \tag{5.21}$$

Expanding and simplifying gives:

$$(\lambda x)^2 + (\lambda y)^2 + \lambda^2 - 2\lambda + 1 = 1 \Longleftrightarrow \lambda^2(x^2 + y^2 + 1) - 2\lambda = 0 \tag{5.22}$$

Factorising gives:

$$\lambda[\lambda(x^2 + y^2 + 1) - 2] = 0 \Longleftrightarrow \lambda = \frac{2}{(x^2 + y^2 + 1)} \tag{5.23}$$

Substitute the non-trivial value of λ into Equation 5.20 gives $\boldsymbol{Q}$ in terms of the x, y coordinates of the Equatorial Plane:

$$(\lambda - 1) = \frac{2}{(x^2 + y^2 + 1)} - 1 = \frac{2}{(x^2 + y^2 + 1)} - \frac{(x^2 + y^2 + 1)}{(x^2 + y^2 + 1)} = \frac{2 - (x^2 + y^2 + 1)}{(x^2 + y^2 + 1)}$$

Hence:

$$\mathbb{P}(x, y) = \boldsymbol{Q} = \left[\frac{2x}{(1 + x^2 + y^2)}, \frac{2y}{(1 + x^2 + y^2)}, \frac{(1 - x^2 - y^2)}{(1 + x^2 + y^2)}\right]$$

Which we see is exactly equivalent to Wildberger's method. □

For the sake of completeness we present the inverse Map $\mathbb{P}^{-1}$ from the Surface $\mathbb{S}^2$ to the Plane $\mathbb{C}$ without proof:

Definition 49. *The Inverse Stereographic Projection is the Map* $\mathbb{P}^{-1} : \mathbb{C}_\infty \backslash \{\boldsymbol{S}\} \leftrightarrow \mathbb{C} : \boldsymbol{Q} \leftrightarrow \boldsymbol{P}$

$$\mathbb{P}^{-1}[\alpha, \beta, \gamma] \in \mathbb{S}^2 = \boldsymbol{P} = \frac{\alpha + i\beta}{1 - |\gamma|} \in \mathbb{C} \tag{5.24}$$

With regard to the uniqueness of $\mathbb{P}$ we have the following statement from page 88 of Marcel Berger's *Geometry Revealed* [2]:

> *"...the Stereographic Projection... puts the Euclidean Plane in a one-to-one correspondence with the points on the sphere minus the [South] Pole. Thus if we identify the Euclidean Plane with the Complex Plane $\mathbb{C}$ there is a Bijection between points on the sphere in its entirety and $\mathbb{C} \cup \infty$... [with] the [South] Pole now corresponding to the point at infinity..."*

and from [4]:

So as to make $\mathbb{P}$ a *Bijection*, we adopt the *Identification* $\mathbb{P}(\infty) \simeq \mathbf{S}$ and $\mathbb{P}^{-1}(\mathbf{S}) \simeq \infty$

Wildberger makes no particular mention of Quantum Mechanics as an application for the methods presented here; in order to address this possibility, we turn now to the question of what happens when we apply these projections to a Complex Vector Space.

Chapter 6

Hyperbolic Geometry

Since, according to the First Postulate of Quantum Mechanics, a general Quantum State is described by a Complex Vector in a Hilbert Space, we will now investigate the implication that such spaces possess a geometry that is essentially Hyperbolic in nature.

We will see that Hyperbolic Geometry may be seen as a form of Projective Geometry resulting from the Gnomonic Projection of a 3-dimensional Real Subspace onto a 2-dimensional Plane [see Section 2.4] or alternatively as the Complex counterpart of Spherical Geometry [19].

We will look first at the method of deriving Hyperbolic Geometry put forward by NJ Wildberger in [33][34] before comparing the Universal approach to that given in traditional texts such as [19] and finally to the more contemporary *Gyrovector Method* proposed by AA Ungar in [1][24][25].

As in Chapter 5 we will focus on the implied Metric as a means of comparing different models.

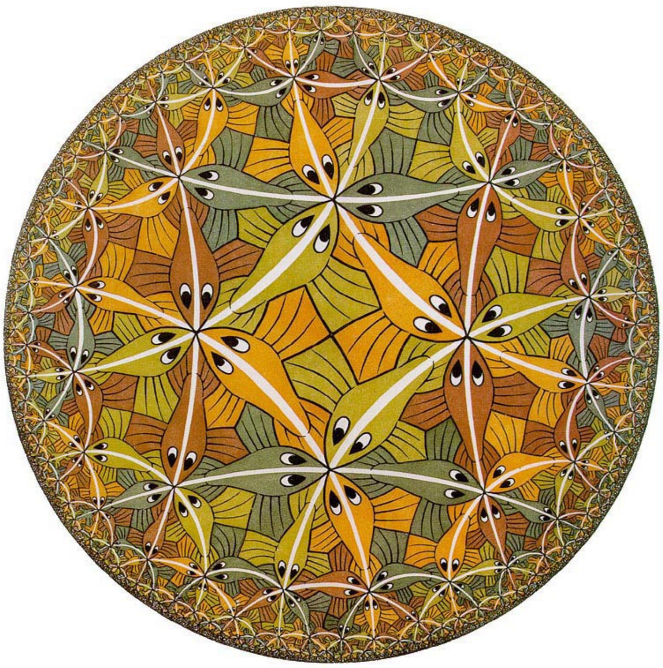

Figure 6.1: MC Esher's Circle Limit III

6.1 Universal Hyperbolic Geometry

Wildberger's approach is to project a 3-dimensional Euclidean Subspace onto a 2-dimensional *viewing plane* and claims that through the incorporation of the *Dual Line* of Apollonius of Perga, together with the *Cross Ratio* of Pappus of Alexandria, he derives a Metric, and hence a geometry that is essentially Hyperbolic in nature.

The method is to consider the intersection between lines through the origin of $\mathbb{E}^3$ and the Plane $z = 1$ which is a Tangent Plane to the North Pole $\boldsymbol{O}$; referring to Figure 6.2. The first thing to notice about the Gnomonic Projection is the duplication of the Northern and Southern hemispheres under this projection.

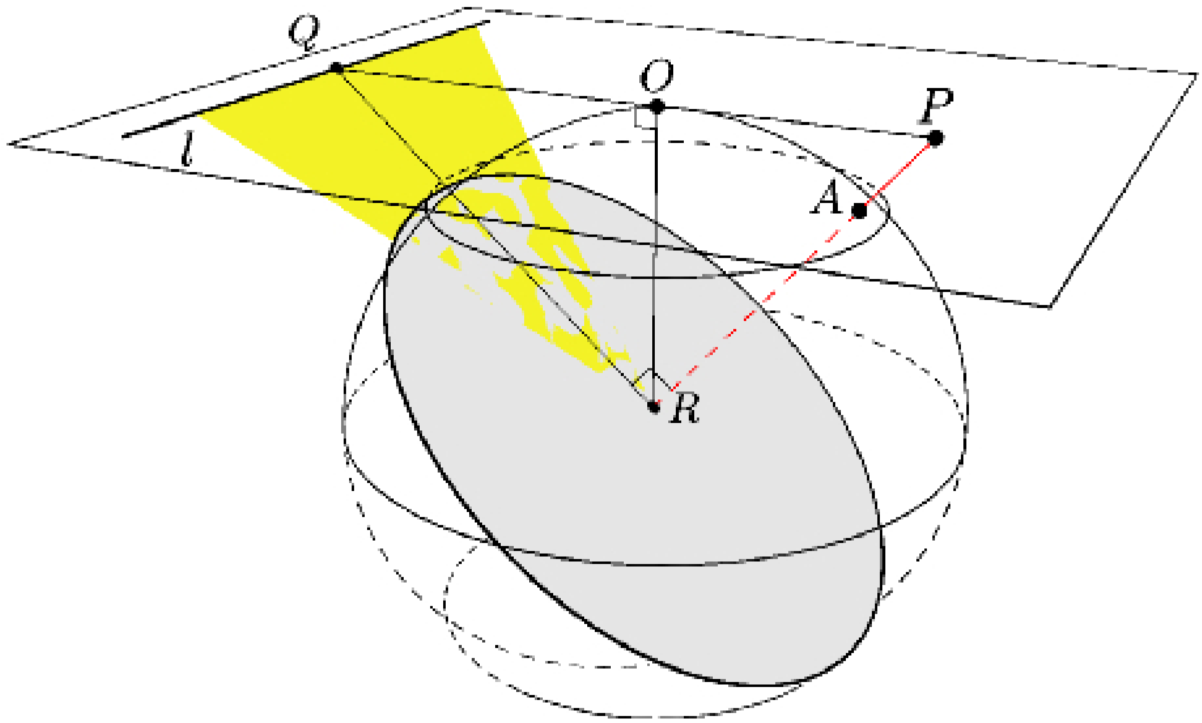

Figure 6.2: The Gnomonic Projection of a Sphere [17]

Referring to Figure 6.2 we see that the line extending through $\boldsymbol{RA}$ intersects the Tangent Plane at the point $\boldsymbol{P}$; this is the *Gnomonic Projection of the Point* $\boldsymbol{A}$.

We see also that the Line $\boldsymbol{l}$ is the Gnomonic Projection of the angled Plane shown shaded. Wildberger Refers to this tangent plane as a 2-dimensional Hyperbolic Plane, which we will denote $\mathbb{H}^2$.

Henceforth, we will call the Point $\boldsymbol{P}$ the Hyperbolic Point $\mathcal{P}$ and the Line $\boldsymbol{l}$ the *Hyperbolic Line* $\mathcal{L}$

We now look at Wildberger's derivation of the Hyperbolic Plane $\mathbb{H}^2$ taken from [33] and [34]

Recall from Chapter 4 the observation that a Line $x + y = z$ can be uniquely defined by the proportion $[x : y : z]$ which is equivalently stated:

$$x + y = z \sim [x : y : z] \sim [\frac{x}{z} : \frac{y}{z} : 1] \tag{6.1}$$

Definition 50. *A* ***Hyperbolic Point*** $\mathcal{P}$ *is defined as the proportion* $[x : y : z]$ *where* $x + y = z$ *is a Line through the Origin of the 3-dimensional Subspace* $\mathbb{E}^3$*:*

$$\mathcal{P} \equiv [X, Y] = \left[\frac{x}{z}, \frac{y}{z}\right] \tag{6.2}$$

Definition 51. *A* ***Hyperbolic Line*** $\mathcal{L}$ *is defined as the proportion* $(l : m : n)$ *where* $lx + my + nz = 0$ *is the equation of a Plane through the Origin of the 3-dimensional Subspace* $\mathbb{E}^3$*:*

$$\mathcal{L} \equiv lX + mY = n \tag{6.3}$$

Definition 52. *A* ***Null Point*** *occurs when a Point lies on the unit circle of* $\mathbb{H}^2$

$$x^2 + y^2 - z^2 = 0 \Longleftrightarrow l^2 + m^2 - n^2 = 0 \tag{6.4}$$

This refers to the situation where the original Line or Plane in the 3-dimensional Subspace is parallel to the *x,y* Plane of $\mathbb{E}^3$

Definition 53. *A* ***Duality*** *exists between Hyperbolic Points and Lines such that:*

$$\mathcal{P} = \mathcal{L}^{\perp} \Longleftrightarrow \mathcal{L} = \mathcal{P}^{\perp} \Longleftrightarrow x : y : z = l : m : n \tag{6.5}$$

Where the notation $\mathcal{P} = \mathcal{L}^{\perp}$ means that $\mathcal{P}$ is the Pole of $\mathcal{L}$ whilst $\mathcal{L}$ is the Polar of $\mathcal{P}$. This is a reference to the Pole/Polar relationship given in the work of Apollonius of Perga which we will look at shortly. Perpendicularity has a particular meaning in the Hyperbolic Plane and is defined thus:

Definition 54. *Lines* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are* ***Perpendicular*** *when* $\mathcal{L}_1$ *passes through* $\mathcal{L}_2^{\perp}$*. This happens exactly when* $\mathcal{L}_2$ *passes through* $\mathcal{L}_1^{\perp}$ *giving:*

$$l_1 l_2 + m_1 m_2 - n_1 n_2 = 0 \Longleftrightarrow x_1 x_2 + y_1 y_2 - z_1 z_2 = 0 \tag{6.6}$$

The consequence of Equation 6.6 is that for Universal Hyperbolic Geometry it is perfectly logical to refer to Points as well as Lines as being perpendicular to one another. See pages 5 and 6 of NJ Wildberger's *Universal Hyperbolic Geometry I* [33] for the source material presented in Section 6.1.

6.1.1 Apollonius of Perga

Since the Metric or Distance Function for $\mathbb{H}^2$ is derived from the Pole/Polar Duality mentioned above, we will now look at this construction in more detail. First devised by Apollonius as a means of exactly locating the Tangent Points on a given circle using only a straight edge [33], the method is as follows:

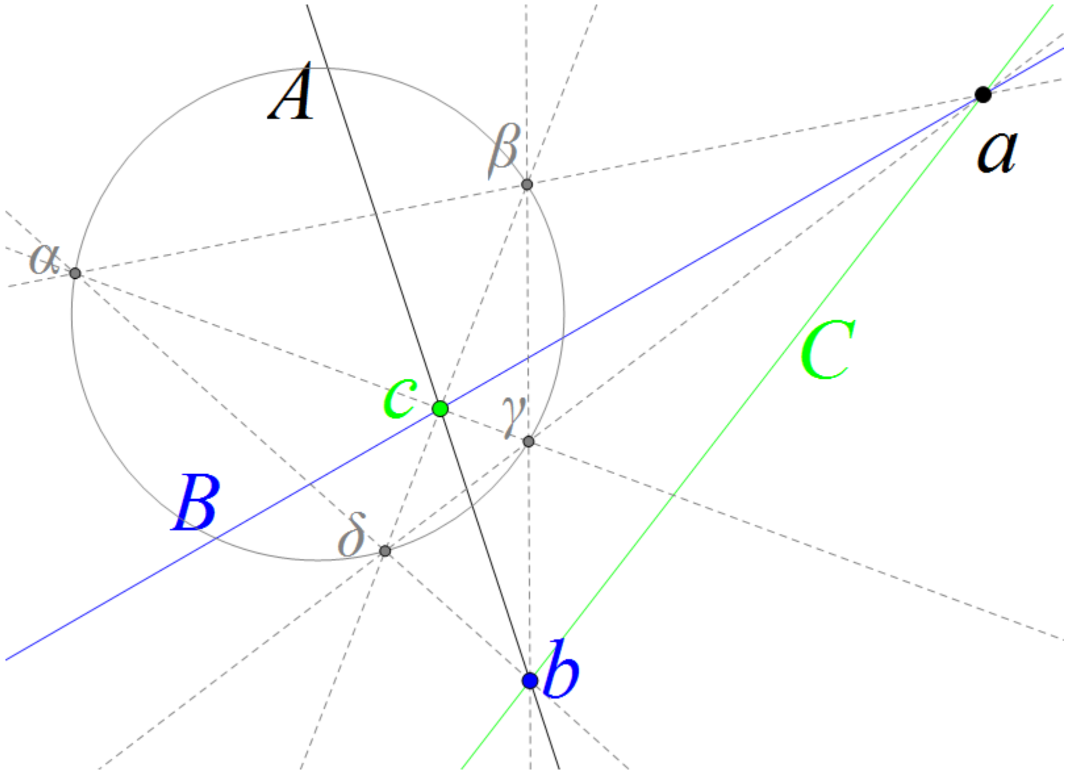

Figure 6.3: The Pole/Polar Duality of Apollonius

Refer to Figure 6.3:

1) Start with the arbitrary Point $\boldsymbol{a}$ and a given circle.
2) Construct 2 arbitrary Lines from $\boldsymbol{a}$ through the Points α, β and γ, δ
3) Connect $\alpha - \gamma$ and $\beta - \delta$ and mark the resulting intersection at $\boldsymbol{c}$
4) Construct the Lines $\alpha\delta$ and $\beta\gamma$ and mark the resulting intersection at $\boldsymbol{b}$
5) Construct the Line $\boldsymbol{C{=}ab}$

Definition 55. *The Line* $\boldsymbol{C}$ *is the Polar Line of the Point* $\boldsymbol{c}$

Definition 56. *The Point* $\boldsymbol{c}$ *is the Pole of the Line* $\boldsymbol{C}$

Wildberger adopts the following notation to describe this relationship (duality):

$$A = a^{\perp} \iff A^{\perp} = a \tag{6.7}$$
$$B = b^{\perp} \iff B^{\perp} = b \tag{6.8}$$
$$C = c^{\perp} \iff C^{\perp} = c \tag{6.9}$$

A property of this construction is that for any point $\boldsymbol{a}$ lying outside the circle, the Polar Line $\boldsymbol{A}$ intersects the circle at the tangent points on the circle to the Point $\boldsymbol{a}$

6.1.2 Pappus of Alexandria

Now that we have defined the duality of Points and Lines, we develop further the method of defining *distance* for Wildberger's Universal Hyperbolic Geometry. We start by presenting 2 facts known from antiquity:

Definition 57. *Pappus' Cross Ratio Theorem: if a,b,c,d are 4 collinear Points in any order, then the Cross Ratio* $\boldsymbol{R}$ *is given by:*

$$\boldsymbol{R}(ab:cd) = \frac{|ac|}{|ad|} \div \frac{|bc|}{|bd|} \tag{6.10}$$

Corollary 58. *a,b,c,d are a Harmonic Range if and only if:*

$$\frac{|ac|}{|ad|} \div \frac{|bc|}{|bd|} = 1 \tag{6.11}$$

We refer to this as the work of Pappus due to the fact that it was Pappus that observed the invariance of this ratio under a projection.

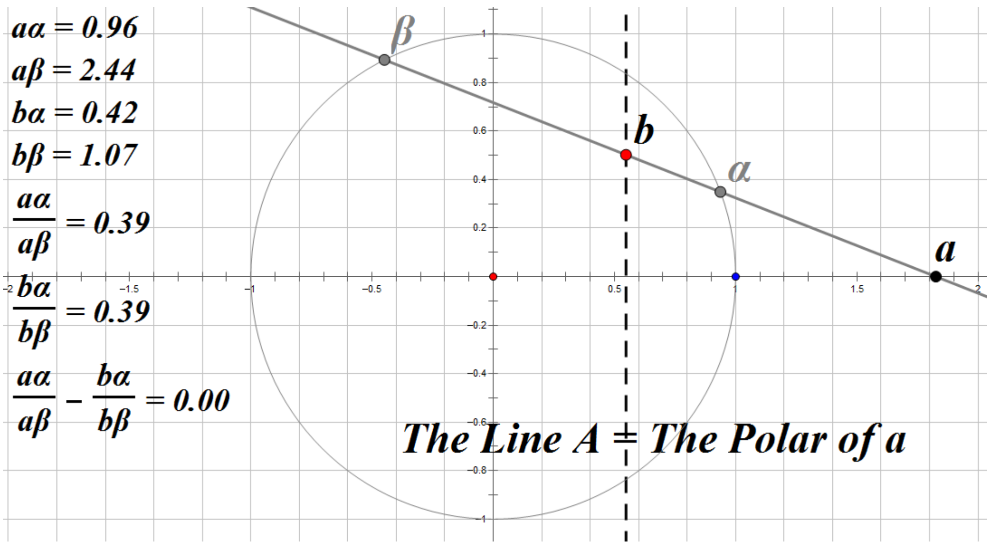

Figure 6.4: Pappus' Harmonic Range

It is Apollonius' observation that the Pole/Polar duality creates a Harmonic Range as illustrated in Figure 6.4 where we see that, for example $\boldsymbol{R}(ab:\alpha\beta) = 1$ for any arbitrary Point $\boldsymbol{a}$

Wildberger uses these facts to define the Hyperbolic Distance between Points $\mathcal{P}$ and $\mathcal{P}'$ in $\mathbb{H}^2$ as the Cross Ratio of the intersections between these Points and their respective Polar Lines. See page 18 of [33]

An elegant algebraic form exists for this ratio due to the fact that the equation of a general Hyperbolic Point $\mathcal{P}$ and the equation of its Polar Line $\mathcal{L} = \mathcal{P}^{\perp}$ share the same coefficients in the Projective Plane $z = 1$ also denoted $\mathbb{H}^2$. See Definitions 53 and 59

Definition 59. *Let the Projective Plane have coordinate axes* $[X, Y]$*, then if* $\mathcal{P}$ *is the Point* $[a : b]$ *then the Line* $\mathcal{L} = \mathcal{P}^{\perp}$ *has the equation* $aX + bY = 1$*.*

Example 12. *In Figure 6.4 we have* $\boldsymbol{a} \approx [1.8 : 0]$ *and* $\boldsymbol{A} = \boldsymbol{a}^{\perp}$ *is the Line* $1.8X + 0Y = 1 \Longrightarrow X \approx 0.55$

6.1.3 Universal Hyperbolic Distance

To arrive at Wildberger's final definition of distance, we repeat the process described in Section 6.1.1 to produce the two Points and two Dual Lines shown in Figure 6.5. The Cross Ratio of the 4 Points $\boldsymbol{P,Q,R,S}$ is used to generate the Universal Hyperbolic Distance between $\boldsymbol{P}$ and $\boldsymbol{Q}$; where we recall that the Hyperbolic Points $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$ would themselves be Lines in an underlying 3-Space, so that:

The Point $\mathcal{P} = [\frac{a}{c} : \frac{b}{c}] = [a : b : c] \in \mathbb{H}^2$ is the Line $(ax + by - cz) \in \mathbb{E}^3$

The Line $\mathcal{P}^{\perp} = (\frac{a}{c}X + \frac{b}{c}Y = 1) = (a : b : c) \in \mathbb{H}^2$ is the Plane $ax + by - cz = 0 \in \mathbb{E}^3$

We see that the Point $\mathcal{R}$ lies on $\mathcal{P}^{\perp}$ just as $\mathcal{S}$ lies on $\mathcal{Q}^{\perp}$ so that the coefficients of the Points are also solutions to the Lines. After Cartesian Coordinates are applied to the intersections $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$ we arrive at the definition given on page 17 of [33] where Wildberger states the distance between Hyperbolic Points to be:

Definition 60. *The Universal Hyperbolic Distance Function between points* $\mathcal{P} = [a : b : c]$ *and* $\mathcal{Q} = [\alpha : \beta : \gamma]$ *is given as:*

$$q(\mathcal{P}, \mathcal{Q}) = 1 - \frac{(a\alpha + b\beta - c\gamma)^2}{(a^2 + b^2 - c^2)(\alpha^2 + \beta^2 - \gamma^2)} \tag{6.12}$$

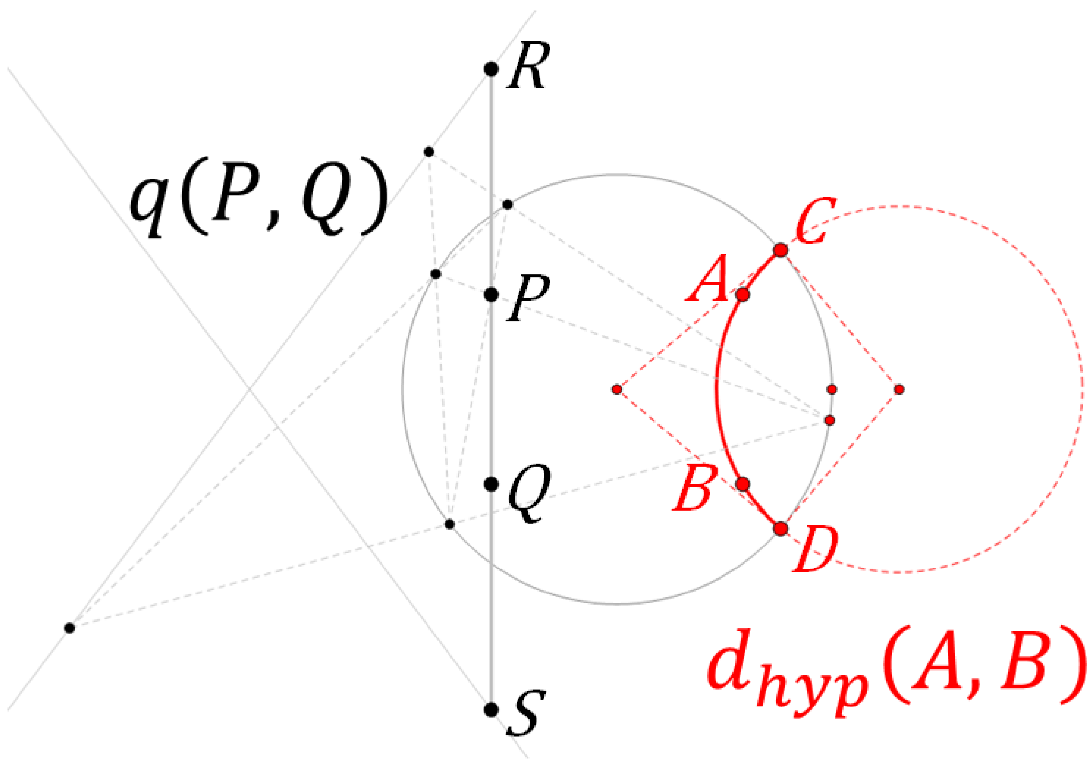

Figure 6.5: The Hyperbolic Distance Function

Figure 6.5 is drawn with a vertical symmetry such that the distance $\boldsymbol{PQ}$ must equal $\boldsymbol{AB}$ by any measure.

6.2 Traditional Hyperbolic Geometry

Before we compare Wildberger's Universal Hyperbolic Distance with the more traditional definitions we will present the description of Hyperbolic Geometry given in John Ratcliffe's *Foundations of Hyperbolic Manifolds* [19]. On page 61 Ratcliffe states that:

> *"... the Hyperbolic n-Space F^n should be a sphere of Imaginary radius..."*

with the formal definition given as:

$$F^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}|^2 = -1\} \tag{6.13}$$

We note that it was consideration of the case for $n = 3$ that led Hamilton to develop the Quaternion methods discussed in Section 3.4.1:

$$F^3 = \{\mathbf{x} \in \mathbb{R}^4 : |\mathbf{x}|^2 = -1 : \mathbf{x} = x + y + z + t\} \tag{6.14}$$

We are considering the projection of a 3-Space onto a 2-dimensional Plane denoted $\mathbb{H}^2$ so that our definition becomes:

$$\mathbb{H}^2 = \{\mathbf{x} \in \mathbb{E}^3 : |\mathbf{x}|^2 = -1\} \tag{6.15}$$

Proposition 8. *A Line $\boldsymbol{L}$ through the Origin of $\mathbb{E}^3$ describes a Unique Point $\mathcal{P} \in \mathbb{H}^2$*

Proof. Any Line $\mathbf{L}$ passing through $[0, 0, 0]$ is uniquely described by the proportion of the coefficients of $\mathbf{L}$:

$$\mathbf{L} = (a : b : c) \tag{6.16}$$

In the 3-dimensional Subspace $\mathbb{E}^3$, the coordinates of any Point $\mathcal{P}$ that lies in the plane $z = 1$ will be of the form $[x : y : 1]$ so that $\mathbf{L}$ can be defined thus:

$$\mathbf{L} = \left(\frac{a}{c} : \frac{b}{c} : 1\right) \Longrightarrow c \neq 0 \tag{6.17}$$

The case where $c = 0$ refers to a Line lying in the x, y Plane of $\mathbb{E}^3$ which does not intersect Projective Plane $z = 1$. For all cases where $c \neq 0$ the Line $\mathbf{L}$ projects onto $\mathbb{H}^2$ at the unique point:

$$\mathcal{P} = \left[\frac{a}{c}X, \frac{b}{c}Y\right] \tag{6.18}$$

where *$[X, Y]$* are the orthogonal coordinate axes of $\mathbb{H}^2$ so that we have $\mathbf{L} = (a : b : c) = \mathcal{P} = [X, Y]$ and we can say that, clearly $\mathbf{L}$ and $\mathcal{P}$ are in a one-to-one relationship. We should make clear that we are talking about a Line in $\mathbb{E}^3$ and NOT a Vector. □

Definition 61. *The Gnomonic Projection $\mathbb{P}$ is the Map: $\mathbb{P} : \mathbb{E}^3 \longleftrightarrow \mathbb{H}^2 : \boldsymbol{L} \leftrightarrow \mathcal{P}$*

On page 3 of [31] Wildberger defines a *bilinear form* between two vectors of a 3-dimensional Subspace which turns out to be the Lorentzian Inner Product given on page 54 of [19]; from both sources, we have:

$$[a, b, c] \circ [\alpha, \beta, \gamma] = a\alpha + b\beta - c\gamma \tag{6.19}$$

which suggests that the z-axis of the underlying 3-space should indeed be thought of as being a purely Imaginary axis. If we consider a general vector $\mathbf{x}$ in $\mathbb{E}^3$ then we have:

$$\mathbf{x}_{x,y,z} : \mathbf{x} \circ \mathbf{x} = x^2 + y^2 - z^2 \tag{6.20}$$

Where $\mathbf{x} \circ \mathbf{x}$ is the Lorentzian Inner Product of $\mathbf{x}$ with itself:

$$\mathbf{x} \circ \mathbf{x} = |\mathbf{x}|^2_{hyp} \tag{6.21}$$

Ratcliffe provides the following two definitions on pages 58 and 59 of [19] which would be useful to include here:

Definition 62. *Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{E}^3$ are Lorentz-Orthogonal* $\iff \boldsymbol{x} \circ \boldsymbol{y} = 0$

Definition 63. *The Lorentzian angle between $\boldsymbol{x}$ and $\boldsymbol{y}$ is the Real number $\eta(\boldsymbol{x},\boldsymbol{y})$:*

$$\boldsymbol{x} \circ \boldsymbol{y} = |\boldsymbol{x}||\boldsymbol{y}| \cosh \eta(\boldsymbol{x},\boldsymbol{y}) \iff \cosh \eta(\boldsymbol{x},\boldsymbol{y}) = \frac{\boldsymbol{x} \circ \boldsymbol{y}}{|\boldsymbol{x}||\boldsymbol{y}|} \tag{6.22}$$

Having noted that Wildberger's bilinear form is just the Lorentzian Inner Product, we look now at the cross product. Again we see that Wildberger's definition of the Hyperbolic Cross Product given on page 7 of [33] is the Lorentzian Cross Product defined by Ratcliffe on page 62 of [19]:

Definition 64. *The Lorentzian Cross Product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{E}^3$ is:*

$$\boldsymbol{x} \otimes \boldsymbol{y} = J(\boldsymbol{x} \times \boldsymbol{y}) : J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.23}$$

Wildberger's notation reads equivalently:

$$J(x_1, y_1, z_1 : x_2, y_2, z_2) = (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_2 y_1 - x_1 y_2) \tag{6.24}$$

We are now ready to compare the Universal distance with the traditional definitions.

6.3 The Hyperbolic Distance Function

It is the method by which measurement in the Hyperbolic Plane is calculated that most clearly demonstrates the difference between Universal Hyperbolic Geometry and the more traditional models. The Hyperbolic Distance Function d_{hyp} given in Equation 2.11 involves calculating the coordinates of the 4 points A, B, C, D shown in red in Figure 6.5 where the circle $ABDC$ is centred at the intersection of the Tangent Lines from C and D. On page 121 of Deza and Deza's *Encyclopaedia of Distances* [16], the Hyperbolic Distance Function is given also in term of a Cross Ratio as:

$$d_{hyp}(A, B) = \frac{1}{2} \ln \frac{(C - A)(D - B)}{(C - B)(D - A)} \tag{6.25}$$

which has been slightly re-stated here so as to be read with reference to Figure 6.5.

On page 17 of [33] and again on page 14 of [34] Wildberger gives the relationship between q and d_{hyp} as:

$$q(\mathcal{P}, \mathcal{Q}) = -\sinh^2 d_{hyp}(A, B) \tag{6.26}$$

Before we look closer at the differences in these definitions of distance we highlight a major difference in the geometry implied in Figure 6.5; the traditional approach shown in red is contained within the bounding circle, whereas the Universal method extends beyond it.

From page 61 of John Ratcliffes's *Foundations of Hyperbolic Manifolds* [19] we have the Hyperbolic Distance Function $d_H(A, B) \in \mathbb{H}^2$ given as the Real Number $\eta(\mathbf{x},\mathbf{y})$ such that:

$$d_H(A, B) = \eta(\mathbf{x},\mathbf{y}) \tag{6.27}$$

for vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{E}^3$

By Definition 63 this would imply that:

$$d_H(A, B) = \cosh^{-1} \frac{\mathbf{x} \circ \mathbf{y}}{|\mathbf{x}||\mathbf{y}|} \tag{6.28}$$

$$\iff \cosh d_H(A, B) = \frac{\mathbf{x} \circ \mathbf{y}}{|\mathbf{x}||\mathbf{y}|} \tag{6.29}$$

$$\iff \cosh^2 d_H(A, B) = \frac{(\mathbf{x} \circ \mathbf{y})^2}{|\mathbf{x}|^2|\mathbf{y}|^2} \tag{6.30}$$

Using Osborne's Rule: $\cosh^2 x - \sinh^2 x = 1$ we have:

$$1 + \sinh^2 d_H(A, B) = \frac{(\mathbf{x} \circ \mathbf{y})^2}{|\mathbf{x}|^2|\mathbf{y}|^2} \tag{6.31}$$

$$\iff -\sinh^2 d_H(A, B) = 1 - \frac{(\mathbf{x} \circ \mathbf{y})^2}{|\mathbf{x}|^2|\mathbf{y}|^2} \tag{6.32}$$

By Equation 6.19 this becomes:

$$\iff -\sinh^2 d_H(A, B) = 1 - \frac{(a\alpha + b\beta - c\gamma)^2}{(a^2 + b^2 - c^2)(\alpha^2 + \beta^2 - \gamma^2)} = q(A, B) \tag{6.33}$$

Ratcliffe notes that $\eta(\mathbf{x},\mathbf{y}) = 0$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are positive scalar multiples of each other. This means that any two vectors that lie on the same ray emanating at the origin of $\mathbb{E}^3$ have a Hyperbolic Distance between them of zero. For distances between points inside the unit circle (the Poincaré Disk) of $\mathbb{H}^2$ the following *Identity* is now shown:

$$q(\mathcal{P}, \mathcal{Q}) = -\sinh^2 d_{hyp}(A, B) \tag{6.34}$$

which is equivalent to the form given in Section 2.2.6:

Definition 65. *The Hyperbolic Distance d_{hyp} between ψ and ϕ is:*

$$d_{hyp}(\psi, \phi) = arccosh\sqrt{\frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}} \tag{6.35}$$

where $|\psi\rangle$ is the Dirac Notation for the unit vector tied to the origin of a 3-dimensional Euclidean Space that parametrises an Imaginary unit sphere [19] and ψ is a Point on the surface.

Chapter 7

Conclusions and Further Research

The original aim of the Thesis was to question the extent to which Quantum 2-States can be realised by Wildberger's Universal Hyperbolic Geometry and highlight the advantages or disadvantages of so doing. We have seen in Chapter 3 that the standard definition of a 2-D Cubit Space is:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \tag{7.1}$$

Where $|\psi\rangle$ describes a Point on the 2-dimensional Elliptic Surface $\mathbb{S}^2$ and can only be used to represent the Pure State. Further to this, a General Quantum State may be represented as a Density Operator:

$$\rho = \frac{1}{2}\begin{bmatrix} 1+\gamma & \alpha - i\beta \\ \alpha + i\beta & 1-\gamma \end{bmatrix} \tag{7.2}$$

Where both $|\psi\rangle$ and ρ are derived from the Bloch Vector:

$$\mathbf{v} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z} \tag{7.3}$$

7.1 2-Dimensional Universal Qubit Space

What justification is there for employing NJ Wildberger's Universal Geometry for Quantum Computation? In QC in general, the geometric concept of a vector-length is considered analogous to a *probability.* We have seen that for the Unit Vector, the Universal Elliptic distance function is appropriate and generates a model equivalent to the Bloch Sphere from an underlying Real 3-Space. Whereas, where the underlying 3-Space is Imaginary we would turn to Universal Hyperbolic distance. It is worth putting these distance functions side by side:

$$\begin{aligned} Q(\rho_1, \rho_2) &= 1 - \frac{(a\alpha + b\beta + c\gamma)^2}{(\alpha^2+\beta^2+\gamma^2)(a^2+b^2+c^2)} \\ q(\rho_1, \rho_2) &= 1 - \frac{(a\alpha + b\beta - c\gamma)^2}{(a^2+b^2-c^2)(\alpha^2+\beta^2-\gamma^2)} \end{aligned}$$

It is really rather remarkable that the Stereographic Projection of a Real Space as compared to the Gnomonic Projection of a Complex Space can produce expressions defining respective distances in such a concise and similar fashion. We have seen that this is indeed a key feature of Wildberger's Universal approach [33, 34] from which the author has gained a level of familiarity with the relative Spaces encountered that would not have been possible via the traditional algebraic route. We submit that for the purposes of introducing quantum spaces to new students, Wildberger's model and method of derivation has merit.

7.2 Universal Elliptic Qubit Space

We have seen that by employing Universal Elliptic Geometry we can generate a vector of unit length from any Point *[**X, Y**]* in the Real 2-dimensional Euclidean Plane $\mathbb{E}^2$ and that the set of all such vectors generates the unit Sphere, across the surface of which we have the following distance function:

$$Q(V, V') = 1 - \frac{(XX' + YY' + 1)^2}{(X^2 + Y^2 + 1)(X'^2 + Y'^2 + 1)} \tag{7.4}$$

Where [X,Y] and [X',Y'] are Points in the Equatorial Plane corresponding to V and V' on the Sphere. See Definition 47 for the proof of the equivalence of this distance function with both the Fubini-Study Metric ξ and the traditional Elliptic distance function d_{ell}

7.3 Universal Hyperbolic Qubit Space

Turning now to the main topic under investigation; 2-D Cubit Space and the possible Universal Hyperbolic representation thereof. We have:

$$\rho = \frac{1}{2}\begin{bmatrix} 1+\gamma & \alpha - i\beta \\ \alpha + i\beta & 1-\gamma \end{bmatrix} \longleftrightarrow \mathbf{v} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z} \tag{7.5}$$

From Chapter 6 we have a Hyperbolic Point $\mathcal{P}$ defined as the proportion $[x : y : z]$ where $x + y = z$ is a Line through the Origin of the 3-dimensional Subspace $\mathbb{E}_3$:

$$\mathcal{P} \equiv [X, Y] = \left[\frac{x}{z}, \frac{y}{z}\right] \tag{7.6}$$

We can say therefore that we have the following map:

$$\rho = \frac{1}{2}\begin{bmatrix} 1+\gamma & \alpha - i\beta \\ \alpha + i\beta & 1-\gamma \end{bmatrix} \longleftrightarrow \mathbf{v} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z} \longleftrightarrow \mathcal{P} = \left[\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}\right] \in \mathbb{H}^2 \tag{7.7}$$

Apart from the ability to picture the system, what can we say about the "Universal" analysis of the system. If we take 2 arbitrary Points $\mathcal{P}$ and $\mathcal{P}'$ then the Universal Hyperbolic distance between them is:

$$q(\mathcal{P}, \mathcal{P}') = 1 - \frac{(\mathbf{x} \circ \mathbf{y})^2}{|\mathbf{x}|^2 |\mathbf{y}|^2} = 1 - \frac{(a\alpha + b\beta - c\gamma)^2}{(a^2 + b^2 - c^2)(\alpha^2 + \beta^2 - \gamma^2)} \tag{7.8}$$

Here, $\mathbf{x}$ and $\mathbf{y}$ are the vectors of $\mathbb{E}_3$ for $\mathcal{P}$ and $\mathcal{P}'$ respectively and $\mathbf{x} \circ \mathbf{y}$ is the Lorentzian Inner Product; similarly $|\mathbf{x}|$ is the Lorentzian Magnitude so that we have an Inner Product Space.

7.4 The Universal Hyperbolic Plane

Wildberger's observation that Universal Hyperbolic Geometry creates a more complete picture of the Hyperbolic Plane than that seen in the Poincaré Disk model is also worthy of further research. What, for example is the significance of a Point $\mathcal{P}$ lying outside of the unit circle of $\mathbb{H}^2$, since any vector $\mathbf{v}$ in $\mathbb{E}^3$ will by necessity project to a point inside it? On page 126 of *Exploring the Infinite* [6] C Esher puts this question rather well, with reference to Circle Limit III:

Figure 7.1: MC Esher's Circle Limit III

"...All these series arise at a ninety-degree angle from infinitely far away... and again loose themselves there. Not a single component of these series ever reaches the boundary line. Outside it, however, is absolute nothingness. Yet the round world cannot exist without that emptiness around it... because in the nothingness are located the immaterial and rigidly geometrically ordered centers of the circle arcs out of which the skeleton is made up..."

To elaborate, we refer to the diagram of the Universal Hyperbolic Plane where the black circular line is Esher's bounding line and the Line at Infinity of the Poincaré Disk.

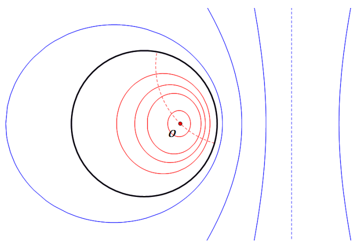

Figure 7.2: The Universal Hyperbolic Plane

Hyperbolic Distances to Points on this line are infinitely large and are generated by projecting vectors in the xy-Plane of the underlying 3-Space; in other words, they would have no Imaginary element. The black line represents the outer boundary for all traditional views of the Hyperbolic Plane. Wildberger's approach allows us to consider $\mathbb{H}^2$ in its entirety. Every solid line (red, blue and black) represents Points equidistant from the Point marked $\boldsymbol{O}$; hence every solid line is a Hyperbolic Circle centred at $\boldsymbol{O}$. The red dotted line crosses every circle orthogonally. It is a Hyperbolic *straight* Line as it represents the shortest distance between Points along it. The red dotted line is the circle arc whose centre is in Esher's "nothingness outside". The blue dotted line is the second Line at Infinity and we draw attention to the fact that this Line and the Point $\boldsymbol{O}$ are in the dual relationship described by Apollonius.

Appendix A

Euclidean Geometry

This Thesis investigates various forms of Non-Euclidean Geometry; which as the name suggests are geometric systems defined primarily by what they are *not*. Euclid formulated his geometry in approximately 300BCE in one of the most influential books in the whole of Mathematical history; *The Elements* is an encyclopaedia of the Mathematical knowledge of the day. Euclid's geometry has one very useful property in that it is the geometry of the Universe as we experience it in our day-to-day lives; for this reason it is the geometry of Newtonian Mechanics which we now know to be an *approximation* of universal Mechanics. The geometry most appropriate to Einstein's Relativity for example is of a quite different nature. As one would expect, the rules defining Euclidean Geometry have been much improved in the 2300 years since *The Elements* and we present now a breakdown of Hilbert's Axioms as a more modern and rigorous set of rules and definitions. The complete picture of Euclidean Geometry is a summation of the various sub-geometries which will now be defined. Main sources: [5][8][13]

A.0.1 Incidence Geometry

Definition 66. *The following three axioms formulate a geometry consisting of points and lines plus a relationship called Incidence:*

1^{st} *Axiom of Incidence:*	*For every point P and for every point Q not P there exists a unique line l that is incident with P and Q hence P and Q are called Collinear*
2^{nd} *Axiom of Incidence:*	*For every line l there exists at least two distinct points incident with l*
3^{rd} *Axiom of Incidence:*	*There exist three distinct points with the property that no line is incident with all of them*

Although this relationship is undefined it may be interpreted as *"lies on"* or *"passes through"*.

A.0.2 Betweenness

Definition 67. *The notation $A * B * C$ is interpreted to mean that point B is between points A and C and all three are Collinear*

1^{st} *Axiom of Betweenness:*	$A * B * C \iff C * B * A$ *and points* A*,* B *and* C *are collinear*
2^{nd} *Axiom of Betweenness:*	*Given any two distinct points* B *and* D *on* l *there exist points* A*,* C *and* E *where* A, B, C, D, E *are collinear such that* $A * B * D$*,* $B * C * D$ *and* $B * D * E$
3^{rd} *Axiom of Betweenness:*	*If* A*,* B *and* C *are three distinct collinear points then one and only one of the points is between the other two*
4^{th} *Axiom of Betweenness:*	*For every line* l *and for any* A*,* B *and* C *not on* l*, then:*

4.1 *If* A *and* B *are on the same side of* l *and if* B *and* C *are on the same side of* l
$\implies$ A *and* C *are on the same side of* l

4.2 *If* A *and* B *are on opposite sides of* l *and if* B *and* C *are on opposite sides of* l
$\implies$ A *and* C *are on the same side of* l

4.3 *If* A *and* B *are on opposite sides of* l *and if* B *and* C *are on the same side of* l
$\implies$ A *and* C *are on opposite sides of* l

A.0.3 Congruence

Definition 68. *The relation known as Congruence* ($\cong$)*, between line segments and angles has the following properties:*

1^{st} *Axiom of Congruence:*	*If* A *and* B *are distinct points, and if* A' *is any point, then for each Ray* r *emanating from* A' *there is a unique point* B' *on* r *such that* $B' \neq A'$ *and* $AB \cong A'B'$
2^{nd} *Axiom of Congruence:*	*If* $AB \cong CD$ *and* $AB \cong EF$ *then* $CD \cong EF$
3^{rd} *Axiom of Congruence:*	*If* $A * B * C$*,* $A' * B' * C'$*,* $AB \cong A'B'$*,* $BC \cong B'C'$ *then* $AC \cong A'C'$
4^{th} *Axiom of Congruence:*	*Given any angle between Rays* $\overline{AB}$ *and* $\overline{BC}$ *and given a Ray* $\overline{A'B'}$ *then there exists a unique Ray* $\overline{B'C'}$ *such that* $\angle ABC \cong \angle A'B'C'$
5^{th} *Axiom of Congruence:*	*If* $\angle A \cong \angle B$ *and* $\angle A \cong \angle C$ *then* $\angle B \cong \angle C$
6^{th} *Axiom of Congruence:*	*If 2 sides and the included angle of* $\triangle A$ *are congruent respectively to* $\triangle B$ *then* $\triangle A \cong \triangle B$

A.0.4 Continuity

Definition 69. *The notion of Continuity requires the assumption of the following Principles of Continuity*

Lemma 1. *The Circular Continuity Principle: If a circle has one point inside and one point outside another circle, then the two circles intersect at two points.*

Lemma 2. *The Elementary Continuity Principle: If one endpoint of a segment is inside a circle and one end is outside the same circle, then the segment intersects the circle.*

Archimedes' Axiom:	*Given any segment* $\overline{AB}$ *then there exists a segment* $\overline{CD}$ *such that* $n \cdot \overline{AB} \cong \overline{CD}$
Aristotle's Axiom:	*Given any Right Triangle ABC with Right angle at B, then as* $\overline{BC}$ *is extended* $\overline{AC}$ *extends indefinitely.*
Dedekind's Axiom:	*Is the geometric analogue of the continuous nature of the Real Number Line.*

Archimedes' Axiom is not purely a geometric axiom as it asserts the existence of *number*. For a fuller description of Continuity see [8]

A.0.5 Neutral Geometry

The Axioms defined so far formulate a geometry that was dubbed *Absolute Geometry* by Janus Bolyai and later in 1965 the term *Neutral Geometry* was applied. This geometry is neutral in the sense that it leaves out the final and most controversial of all the Axioms [8]

A.0.6 Axioms of Parallelism

Hilbert's Parallel Axiom for Euclidean Geometry

The Axiom of Parallelism completes the list of Axioms that define Euclidean Geometry [8]. It is exactly this Axiom that differentiates Euclidean, Spherical (Elliptic) and Hyperbolic geometries. For centuries it was believed that this Axiom, known as Euclid's Fifth Postulate was redundant in that it could be deduced from Euclid's first four postulates:

Euclid's First Postulate: That there exists a straight line segment from any point to any point.
Euclid's Second Postulate: That any straight line segment can be extended indefinitely in either direction.
Euclid's Third Postulate: That it is possible to describe a circle with any centre and any radius.
Euclid's Fourth Postulate: That all right-angles are equal.

Compared to these four seemingly self-evident postulates, the fifth has proved to be a rather slippery subject and has had a profound influence on the development of Mathematics.

Definition 70. *For every line l and every point P not on l there exists at most one line m through P such that m is parallel to l*

Throughout the 19th Century, variations on Euclid's postulates lead to the independent discovery of alternative geometric realities; most notably by Carl Friedrich Gauss, János Bolyai, Nikolai Ivanovich Lobachevsky and later developed by Bernhard Riemann. In each case, it was the negating of the need for the Parallel Postulate that gave rise to the new insights.

For the Euclidean case as stated in Definition 70, also known in this form as *Playfair's Axiom,* there is one and only one line m through P so that the notion of *Parallelism* is well defined and seemingly obvious; although all attempts to deduce the Parallel Postulate from the others given here have failed!

For the Elliptic case, where every point on the surface of a sphere is distinct there are no lines m; as each Great Circle intersects twice at antipodal points. For the Hyperbolic case, there are infinitely many lines m; as hyperbolic straight lines intersect only at the limiting boundary which is infinitely far away. It is, in the author's opinion a rather wonderful fact that the Hyperbolic reality of Parallel lines meeting at infinity chimes exactly with the *observed* Euclidean reality of perspective.

Appendix B

The Geometry of Group Structures

In Chapters 5 and 6 we present our findings on the work of Norman J Wildberger as it might apply to representations of a single Quantum State. Since, as we have seen, Group Structures play an important role in the formalism we present here a fully constructable, geometric understanding of an Abelian Group purely for the reader's interest.

B.1 Group Structure on the Circle

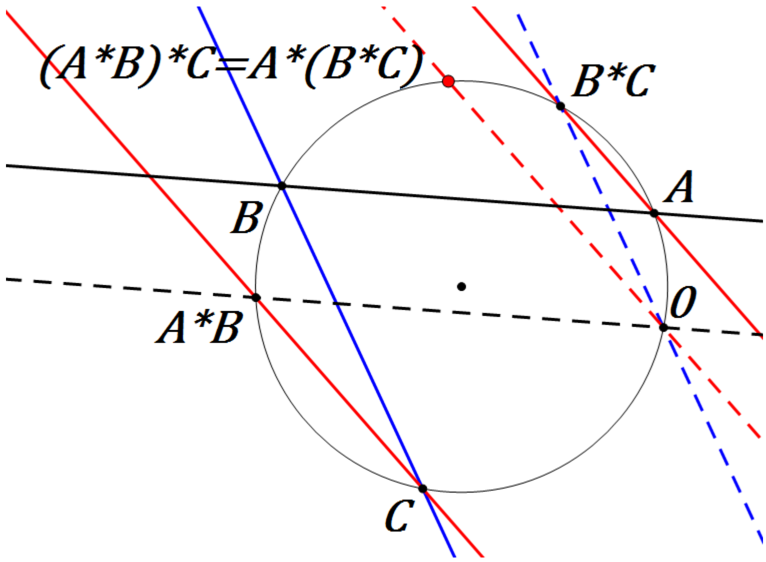

Figure B.1: The Group Structure on the Circle

In order to demonstrate that the Abelian Group Axioms given in Section 2.3 exist in a completely geometric form we consider an arbitrary binary operation denoted $(*)$ for Points A and B on a general circle. Refer to Figure B.1

We define $A * B$ as being the intersection resulting from taking a line parallel to AB through an arbitrary but fixed point called O also on the circle. This is referenced to the the work of Franz Lemmermeyer [15] and S. Shirali [7]. We see that if A and B are on the circle then so is $A * B$ so that we have Closure and that:

(G1) $\quad A * (B * C) = (A * B) * C$
(G4) $\quad A * B = B * A$

The special case $A * A$ is the intersection of the line parallel to the *Tangent Line* at A taken through O so that $A * A \sim A^2$ has an interpretation. See Figure B.2

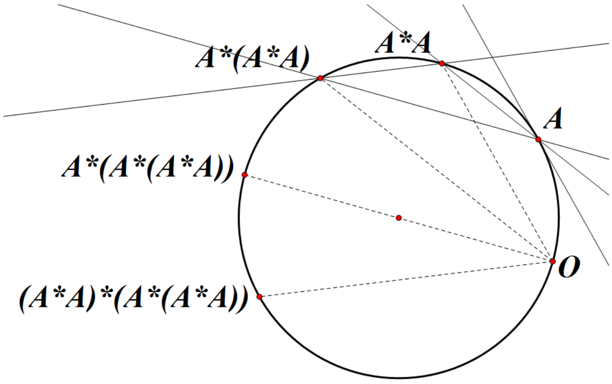

Figure B.2: The Special Case $A * A * A * ... \sim A^n$

Staying with Figure B.2, if after n-repetitions the point A^n is Incident (see Section A.0.1) with the point O, then the points $\{A, A^1, ...A^n\}$ form a Regular n-sided Polygon.

Proof. Let $A = z \in \mathbb{C} : |z| = 1$ where $z = e^{i2\pi}$ then $z^{\frac{1}{n}} = e^{\frac{i2\pi}{n}}$ is called the Primary Root of z which has exactly n-roots in total, equally spaced around the unit circle of the Argand Diagram. □

Referring now to Figure B.3 we see that the point O acts as an Identity:

(G2) $\quad A * O = O * A = A$

since the line through O parallel to AO is the line AO. If we then define A^{-1} as the intersection resulting from taking a line through A parallel to the Tangent Line at O then we also have the existence of an Inverse:

(G3) $\quad A * A^{-1} = A^{-1} * A = O$

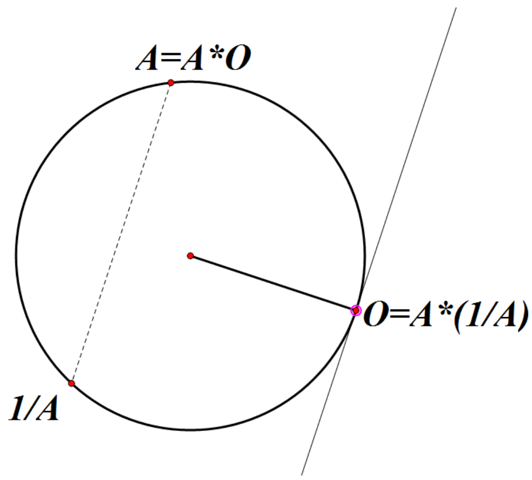

Figure B.3: Existence of Inverse and Identity for the Group Structure

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