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Raviolo Vertex Algebras, Cochains, and Conformal Blocks

Luigi Alfonsi, Hyungrok Kim, and Charles A. S. Young[*]

Department of Physics, Astronomy, and Mathematics, University of Hertfordshire, College Lane, Hatfield AL10 9AB, UK
[*]Correspondence to be sent to: e-mail: c.young8@herts.ac.uk
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Raviolo vertex algebras were introduced recently by Garner and Williams in [25]. Working at the level of cochain complexes, in the present paper we construct spaces of conformal blocks, or more precisely their duals, coinvariants, in the raviolo setting. We prove that the raviolo state-field map correctly captures the limiting behaviour of coinvariants as marked points collide.

1 Introduction

Vertex algebras capture the physicists' notion of operator product expansions and the state-field correspondence in chiral conformal field theory in one complex dimension. Since their introduction [6] they have become a powerful and ubiquitous tool in mathematical physics and representation theory. Textbook references include [18, 31, 33].

The definition of vertex algebras appears to be closely tied to the special properties of complex dimension one, and specifically of the formal disc D and the formal punctured disc $D^\times = D \setminus \{0\}$,

$$D = \mathsf{Spec}\,\mathbb{C}[[z]], \qquad D^\times = \mathsf{Spec}\,\mathbb{C}((z)).$$

Roughly speaking, the formal punctured disc $D^\times$ describes the possible collision geometries of two marked points in the complex plane (one fixed at the origin, the other movable). See Figure 1.

What happens in higher dimensions? It has long been expected by experts that the language of factorization algebras (as developed in the algebro-geometric setting in [20] following [4]—cf. also [3, 5, 13]—and in the smooth setting in [8, 9]) in principle allows vertex algebras to generalize to higher dimensions. In this direction, see especially [26], [42, §4], and [24], and also [37], all broadly in the smooth setting and using (pre)factorization algebras constructed using the Dolbeault resolution of the holomorphic structure sheaf; and in the complex-algebraic setting see [21], [29], [32]. Writing down explicit axioms in closed form for higher-dimensional vertex algebras remains a challenge, however.

Recently though, Garner and Williams ([25] and see also [23]) have considered a particularly tractable instance of this general problem, namely the case of theories with one real topological dimension and one complex holomorphic dimension, that is, theories on spacetimes modelled on $\mathbb{R} \times \mathbb{C}$, such as twists of three-dimensional supersymmetric Yang–Mills theory [17]. Such a spacetime structure can be neatly

captured by a transversely holomorphic foliation [1, 14, 35]: that is, a foliation of the spacetime three-manifold by curves such that the leaf space has the structure of a Riemann surface.

In that topological-holomorphic setting, marked points are allowed to collide in the complex plane, but only if, when they do, they are separated in the topological direction. The upshot is that pairwise collisions are no longer described by the formal punctured disc $D^\times$, but rather by the *formal raviolo*

$$\mathrm{Rav} := D \sqcup_{D^\times} D,$$

the scheme obtained by gluing together two copies of the formal disc along their common copy of the formal punctured disc. Again, see Figure 1, and §2.1 below.

What's so nice about this setting is that, on the one hand, it is a sufficiently mild generalization that it is still possible to write down explicit axioms for the resulting *raviolo vertex algebras* in a form closely parallel to the usual case—see [25]—while on the other hand it is sufficiently different that it exhibits many of the features expected in the higher setting. Indeed, the formal raviolo Rav is no longer an affine scheme, unlike the punctured disc $D^\times$. Its structure sheaf has higher cohomology, and one consequence is that in the raviolo vertex algebras of [25] the usual lowering operators/negative modes disappear from degree zero, and reappear in cohomological degree one.

Now let us describe the contents of the present paper. We do essentially two things: first, we work at the level of cochain complexes rather than their cohomologies, and second, we introduce notions of configuration space and rational conformal blocks in the raviolo setting.

In [25], the sheaf cohomology $H^\bullet(\mathrm{Rav}, \mathcal{O})$ is regarded as a graded commutative algebra. It plays the role, there, that the commutative algebra of functions on the formal punctured disc, $\Gamma(D^\times, \mathcal{O}) = H^0(D^\times, \mathcal{O}) \cong \mathbb{C}((z))$, plays in the case of standard vertex algebras—that is, roughly speaking, it is what organizes the positive and negative modes in the state-field correspondence. However, the cohomology $H^\bullet(\mathrm{Rav}, \mathcal{O})$ comes endowed with additional higher structure, which is lost in this picture (as the authors of [25] remark). One way to keep track of that higher structure is to work, instead, with the *derived global sections* $R\Gamma^\bullet(\mathrm{Rav}, \mathcal{O})$ of the structure sheaf. This latter comes with the structure of a differential graded (dg) commutative algebra, which, via homotopy transfer, encodes all the higher products on its cohomology,

$$H^\bullet(\mathrm{Rav}, \mathcal{O}) \equiv H^\bullet(R\Gamma(\mathrm{Rav}, \mathcal{O})).$$

Thus, for us it will be (a certain model of) $R\Gamma^\bullet(\mathrm{Rav}, \mathcal{O})$, which plays the role of $\Gamma(D^\times, \mathcal{O}) = \mathbb{C}((z))$ in the usual case. For simplicity we focus exclusively on the raviolo analogues of the Kac–Moody vertex algebras at level zero. We write down in Section 2 the definition of the raviolo vacuum module and its state-field map at the level of cochain complexes (i.e., dg vector spaces) rather than graded vector spaces. One immediate consequence is that for us there are lowering operators *both* in degree one (representing cohomology classes) *and* in degree zero; see the discussion in §2.6.

Then our second and main goal is to introduce rational conformal blocks (or more precisely, their duals, rational coinvariants) in the raviolo setting. To do that, after reviewing the standard definition of rational coinvariants in Section 3, we define in Section 4 a notion of *ravioli configuration space*, $\mathrm{RavConf}_N$, which plays the role of the usual configuration space

$$\mathrm{Conf}_N = \mathbb{A}_\mathbb{C}^N \setminus \text{diagonals}$$

for standard vertex algebras. Mirroring the passage from the punctured disc $D^\times$ to the formal raviolo Rav, the ravioli configuration space will be defined by gluing together $N!$ copies of $\mathbb{A}_\mathbb{C}^N$ along the complements of their diagonals. The resulting (non-separated) scheme looks locally like $\mathbb{A}_\mathbb{C}^N$ everywhere except on diagonals, just like Conf_N. But the diagonals themselves, rather than being removed, instead appear with multiplicity > 1 (as one expects since, whenever marked points coincide, one has to keep track of their ordering in the topological/leaf direction). We introduce a model of the derived space of sections

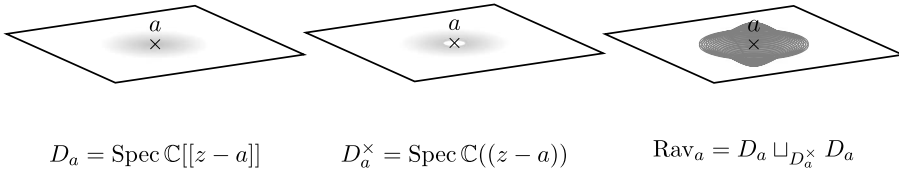$$\mathcal{A}_N \simeq R\Gamma(\mathrm{RavConf}_N, \mathcal{O})$$

$$D_a = \operatorname{Spec} \mathbb{C}[[z-a]] \qquad D_a^\times = \operatorname{Spec} \mathbb{C}((z-a)) \qquad \operatorname{Rav}_a = D_a \sqcup_{D_a^\times} D_a$$

Fig. 1. Sketch of copies of the formal disc D, the formal punctured disc $D^\times$, and the formal raviolo Rav associated to a point a in the complex plane.

of the structure sheaf on this configuration space. It is a dg commutative algebra, and it plays the role of the commutative algebra of functions $\mathcal{B}_N := \Gamma(\operatorname{Conf}_N, \mathcal{O}) = \mathbb{C}[z_i, (z_i - z_j)^{-1}]_{i,j=1; i \neq j}^N$ on configuration space in the usual case.

Our model $\mathcal{A}_N$ is so chosen that it is possible to write down (see Section 5) explicit raviolo analogues of all the constructions reviewed in Section 3 for the usual case. We arrive at the definition of the space—more precisely, the dg $\mathcal{A}_N$-module—of *ravioli coinvariants*

$$\mathrm{F}(\mathfrak{g}; \mathcal{A}_N; M_1, \ldots, M_N).$$

(See §5.3.)

The main result of the paper is then Theorem 20 in Section 6, which shows that the state-field map for the raviolo vacuum module, defined in Section 2, correctly captures the limiting behaviour of ravioli coinvariants as two marked points, each with copies of the vacuum module attached, are brought close together.

As we discuss at greater length in §2.6 below, the limiting behaviour of conformal blocks as two or three marked points collide is arguably what motivates the usual vertex algebra axioms (notably, Borcherds identity), and at any rate those axioms can certainly be reconstructed by considering such limits. Theorem 20 establishes an analogous setup in the raviolo case.

The proof of the main theorem, Theorem 20, is given in Section 7. Finally, in an appendix, we recall some background material on semisimplicial sets and the Thom–Sullivan functor.
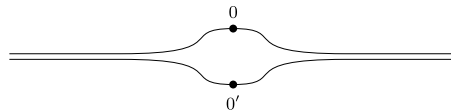
2 The Raviolo Vacuum Module in Cochain Complexes
2.1 The formal raviolo

The *formal raviolo*,

$$\operatorname{Rav} := D \sqcup_{D^\times} D,$$

is the $\mathbb{C}$-scheme obtained by gluing two copies of the formal disc $D = \operatorname{Spec} \mathbb{C}[[z]]$ along their common copy of the formal punctured disc $D^\times = \operatorname{Spec} \mathbb{C}((z))$ ($\operatorname{Rav} = D \sqcup_{D^\times} D$ is an infinitesimal analogue of the affine-line-with-a-doubled-origin, $\mathbb{A}_{\mathbb{C}}^1 \sqcup_{\operatorname{Spec} \mathbb{C}[z,z^{-1}]} \mathbb{A}_{\mathbb{C}}^1$, which is usually pictured like this

Both are prototypical examples of *nonseparated* schemes obtained by gluing; see for example [16, I-44]. Note that $D^\times$ is open in D.).

For our purposes, it is best to visualize the affine line $\mathbb{A}_{\mathbb{C}}^1 = \operatorname{Spec} \mathbb{C}[z]$ as a copy of the complex *plane*. The formal disc, formal punctured disc, and formal raviolo at some given closed point $a \in \mathbb{C}$ may be pictured as in Figure 1.

2.2 Functions on the formal raviolo

By definition, then, the formal raviolo Rav is the pushout in the category of $\mathbb{C}$-schemes

$$
\begin{array}{ccc}
D^\times & \longrightarrow & D \\
\downarrow & & \downarrow \\
D & \longrightarrow & \mathrm{Rav}
\end{array}
$$

or, equivalently, the coequalizer in $\mathbb{C}$-schemes

$$
\mathrm{Rav} = \mathrm{colim}\left(D^\times \rightrightarrows D \sqcup D \right).
$$

This latter is a useful way to think of Rav because it presents it explicitly as the colimit of a diagram corresponding to a semisimplicial object in affine schemes. (The notion of a semisimplicial object is recalled in Appendix A.) Namely it is the colimit of the Čech nerve $\check{C}(\mathscr{U}) = \left(U_1 \cap_{\mathrm{Rav}} U_2 \rightrightarrows U_1 \sqcup U_2 \right)$ of the open cover $\mathscr{U} = \{U_1, U_2\}$ of Rav by two copies of the formal disc $U_1 \cong U_2 \cong D$ whose intersection in Rav is, by definition, a copy of the punctured disc, $U_1 \cap_{\mathrm{Rav}} U_2 \cong D^\times$. On applying the global sections functor $\Gamma(-, \mathcal{O})$, one obtains the semicosimplicial object $\Gamma(\check{C}(\mathscr{U}), \mathcal{O})$ in commutative algebras. Commutative algebras embed in differential graded (dg) commutative algebras. The derived global sections of the structure sheaf $\mathcal{O}$ on Rav are then given, by definition, by taking homotopy limit in dg commutative algebras,

$$
R\Gamma(\mathrm{Rav}, \mathcal{O}) := \mathrm{holim}_{\mathbf{dgCAlg}_{\mathbb{C}}} \Gamma(\check{C}(\mathscr{U}), \mathcal{O})
$$
$$
= \mathrm{holim}_{\mathbf{dgCAlg}_{\mathbb{C}}} \left(\mathbb{C}((z)) \leftleftarrows \mathbb{C}[[z]] \times \mathbb{C}[[z]] \right).
$$

As we shall recall in more detail in §4.2 and Appendix A, the Thom–Sullivan construction provides a means of computing such homotopy limits. Namely, we let $\mathbb{C}\{\{z\}\}$ denote the dg commutative algebra

$$
\mathbb{C}\{\{z\}\} := \left\{ \omega \in \mathbb{C}((z)) \otimes \mathbb{C}[\upsilon, d\upsilon] : \omega|_{\upsilon=0} \in \mathbb{C}[[z]] \text{ and } \omega|_{\upsilon=1} \in \mathbb{C}[[z]] \right\}.
$$

and we then have

$$
\mathbb{C}\{\{z\}\} \simeq R\Gamma(\mathrm{Rav}, \mathcal{O}).
$$

This will be our model, in dg commutative algebras, of the derived global sections of the structure sheaf of Rav (Any other model of $R\Gamma(\mathrm{Rav}, \mathcal{O})$ will be related to this one by a zig-zag of quasi-isomorphisms. In particular this should be true of (the local analogue of) the dg commutative algebra $\mathcal{A}$ of [25,§1.2].).

Informally one should think of $\mathbb{C}\{\{z\}\}$ as the "functions on the formal raviolo Rav", just as the Laurent series $\mathbb{C}((z))$ are the "functions on the formal punctured disc $D^\times$". The fact that $\mathbb{C}\{\{z\}\}$ is nontrivially differential graded is a reflection of the fact that Rav is not an affine scheme. In particular the cohomology of $\mathbb{C}\{\{z\}\}$ computes the sheaf cohomology of the structure sheaf $\mathcal{O}$ of Rav. As a graded vector space, this cohomology is given by

$$
H^\bullet(\mathbb{C}\{\{z\}\}) \cong_{\mathbf{grVect}_{\mathbb{C}}} \begin{cases} \mathbb{C}[[z]] & \bullet = 0 \\ z^{-1}\mathbb{C}[z^{-1}] & \bullet = 1 \\ 0 & \text{otherwise.} \end{cases}
$$

(The classes in degree one have representatives in $z^{-1}\mathbb{C}[z^{-1}]d\upsilon$. Indeed, such one-forms are closed, obviously, but fail to be exact in $\mathbb{C}\{\{z\}\}$ because of the boundary conditions. For example the would-be primitive $z^{-1}\upsilon$ is not in $\mathbb{C}\{\{z\}\}$ since it is not regular in z when pulled back to $\upsilon = 1$.)

The cohomology $H^\bullet(\mathbb{C}\{\{z\}\})$ comes with the structure of a graded commutative algebra. It is isomorphic, as a graded commutative algebra, to the algebra called $\mathcal{K}$, or $\mathbb{C}\langle\langle z \rangle\rangle$, in [25]:

$$
H^\bullet(\mathbb{C}\{\{z\}\}) \cong_{\mathbf{grAlg}_{\mathbb{C}}} \mathcal{K} \equiv \mathbb{C}\langle\langle z \rangle\rangle.
$$

The cohomology has, however, *more* structure than that of a graded commutative algebra. Indeed, $H^\bullet(\mathbb{C}\{\!\{z\}\!\})$ gets the structure of a *minimal C_∞-algebra*, coming from homotopy transfer of the dg commutative algebra structure on $\mathbb{C}\{\!\{z\}\!\} \simeq R\Gamma(\mathrm{Rav}, \mathcal{O})$ itself. (See e.g., [34] for a discussion of homotopy transfer, and specifically [7] and [34,§13.1.9] for the dg commutative case.) In practice, what that means is that $H^\bullet(\mathbb{C}\{\!\{z\}\!\})$ is endowed with a family $(c_k)_{k\geq 2}$ of *higher products*, the first of which, c_2 is the binary product of the graded commutative algebra structure. One way to keep track of this extra structure is to work in dg commutative algebras rather than passing to their cohomologies.

2.3 Splitting

Let us define the dg commutative algebras

$$\mathbb{C}\{\!\{z\}\!\}_- := \big\{\omega \in z^{-1}\mathbb{C}[z^{-1}] \otimes \mathbb{C}[v, \mathrm{d}v] : \omega|_{v=0} = 0 \text{ and } \omega|_{v=1} = 0\big\}$$

$$\mathbb{C}\{\!\{z\}\!\}_+ := \mathbb{C}[[z]] \otimes \mathbb{C}[v, \mathrm{d}v].$$

Then evidently there are maps of dg commutative algebras

$$\mathbb{C}\{\!\{z\}\!\}_- \hookrightarrow \mathbb{C}\{\!\{z\}\!\} \hookleftarrow \mathbb{C}\{\!\{z\}\!\}_+$$

such that at the level of dg vector spaces

$$\mathbb{C}\{\!\{z\}\!\} = \mathbb{C}\{\!\{z\}\!\}_- \oplus \mathbb{C}\{\!\{z\}\!\}_+.$$

Moreover there are strong deformation retracts of dg vector spaces (The map ι is given by the embedding of $\mathbb{C}[[z]] = \mathbb{C}[[z]] \otimes \mathbb{C} \xrightarrow{\mathrm{id}\otimes 1} \mathbb{C}[[z]] \otimes \mathbb{C}[v, \mathrm{d}v] = \mathbb{C}\{\!\{z\}\!\}_+$, i.e., as "constant 0-forms in the v direction". The map $(-)|_{\frac{1}{2}}$ is given by pulling back forms to (say) $v = \frac{1}{2}$. A suitable homotopy h is given by "$h(\omega)(v) = \int_{\frac{1}{2}}^{v} \omega$", by which we mean, more precisely, the following: we have $\omega = f(v) + F(v)\mathrm{d}v$ for some $f(v), F(v) \in \mathbb{C}[[z]] \otimes \mathbb{C}[v]$, and we define $h(f(v) + F(v)\mathrm{d}v) := \int_0^v F(v')\mathrm{d}v'$.)

$$h \circlearrowright \mathbb{C}\{\!\{z\}\!\}_+ \underset{\iota}{\overset{(-)|_{\frac{1}{2}}}{\rightleftarrows}} \mathbb{C}[[z]] \cong H^0(\mathbb{C}\{\!\{z\}\!\})$$

and (A suitable homotopy k is given by is given by $k(f(v) + F(v)\mathrm{d}v) := \int_0^v F(v')\mathrm{d}v' - v\int_0^1 F(v')\mathrm{d}v'$. Cf. e.g., [2, Prop. 16].)

$$k \circlearrowright \mathbb{C}\{\!\{z\}\!\}_- \underset{(-)\mathrm{d}v}{\overset{\int_0^1}{\rightleftarrows}} z^{-1}\mathbb{C}[z^{-1}][-1] \cong H^1(\mathbb{C}\{\!\{z\}\!\}).$$

2.4 Residue pairing

Let us remark in passing that there is a map

$$\mathrm{Res} : \mathbb{C}\{\!\{z\}\!\} \to \mathbb{C}[-1]$$

given by

$$\mathrm{Res}\,\omega := \int_{v=0}^{1} \mathrm{res}_z\,\omega$$

where $\mathrm{res}_z : \mathbb{C}((z)) \to \mathbb{C}$ is the usual formal residue map picking out the coefficient of z^{-1}. In cohomology, this map reproduces the residue pairing from [25]. For example one sees that it correctly pairs z^k with $\mathrm{d}v/z^{k+1}$, since $\mathrm{Res}\,\mathrm{d}v/z = 1$.

2.5 Vacuum module

Let us now pick a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, for example $\mathfrak{sl}_2$. We get the dg Lie algebra, that is, the Lie algebra in dg vector spaces over $\mathbb{C}$, given by

$$\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}.$$

It is the *raviolo loop algebra*, the raviolo analogue of the usual loop algebra $\mathfrak{g} \otimes \mathbb{C}((z))$. It has dg Lie subalgebras $\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_-$ and $\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_+$. We shall think of elements of $\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_-$ as *lowering operators* or *negative modes*, and of elements of $\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_+$ as *raising operators* or *positive modes*. The PBW theorem holds for dg Lie algebras and hence, in view of the splitting above, we have that

$$U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}) \cong U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_-) \otimes U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_+)$$

as dg vector spaces and moreover as $U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_-), U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_+)$-bimodules. In particular $U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\})$ is free as a $U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_-), U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_+)$-bimodule.

Let $\mathscr{V}$ denote the module over $\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}$ induced (One can check that $\mathscr{V}$ models the derived tensor product $U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}) \otimes^{\mathbb{L}}_{U(\mathfrak{g}\otimes\mathbb{C}\{\!\{z\}\!\}_+)} \mathbb{C}|0\rangle$ where

$$\otimes^{\mathbb{L}} : D(_{U(\mathfrak{g}\otimes\mathbb{C}\{\!\{z\}\!\})}\mathbf{Mod}_{U(\mathfrak{g}\otimes\mathbb{C}\{\!\{z\}\!\}_+)}) \times D(_{U(\mathfrak{g}\otimes\mathbb{C}\{\!\{z\}\!\}_+)}\mathbf{Mod}) \to D(_{U(\mathfrak{g}\otimes\mathbb{C}\{\!\{z\}\!\})}\mathbf{Mod}).)$$

from the trivial one-dimensional module $\mathbb{C}|0\rangle$ over $\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_+$:

$$\mathscr{V} := U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}) \otimes_{U(\mathfrak{g}\otimes\mathbb{C}\{\!\{z\}\!\}_+)} \mathbb{C}|0\rangle.$$

Following the usual convention for vertex algebras, we call vectors in $\mathscr{V}$ *states*. The representation $\mathscr{V}$ is the *raviolo vacuum Verma module (at level zero)*.

By the PBW decomposition above, there is an isomorphism

$$\mathscr{V} \cong U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_-) \otimes_{\mathbb{C}} \mathbb{C}|0\rangle$$

of left $U(\mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_-)$-modules in dg vector spaces.

2.6 State-field map

Let $\mathscr{V}((x))$ denote the dg vector space of formal Laurent series in x with coefficients in $\mathscr{V}$, which one thinks of as a completion of the tensor product $\mathscr{V} \otimes \mathbb{C}((x))$. Let $\mathscr{V}\{\!\{x\}\!\}$ denote the following dg vector space,

$$\mathscr{V}\{\!\{x\}\!\} := \big\{\omega \in \mathscr{V}((x)) \otimes \mathbb{C}[u, \mathrm{d}u] : \omega|_{u=0} \in \mathscr{V}[[x]] \text{ and } \omega|_{u=1} \in \mathscr{V}[[x]]\big\},$$

which we similarly think as a completion of the tensor product $\mathscr{V} \otimes \mathbb{C}\{\!\{x\}\!\}$. Now we define a *state-field map*, namely a (degree-zero) map of dg vector spaces (Here by $\mathrm{Hom}_{\mathbf{dgVect}_\mathbb{C}}$ we mean the internal Hom in dg vector spaces, just as in the usual definition of the state-field map—cf. §3.4—$\mathrm{Hom}_{\mathbf{Vect}_\mathbb{C}}$ really means the internal Hom in vector spaces.)

$$Y(-;x) : \mathscr{V} \to \mathrm{Hom}_{\mathbf{dgVect}_\mathbb{C}}(\mathscr{V}, \mathscr{V}\{\!\{x\}\!\}).$$

We do so recursively, as follows. First, we set

$$Y(|0\rangle; x) := \mathrm{id}_{\mathscr{V}}.$$

Then for all homogeneous $X \in \mathfrak{g} \otimes \mathbb{C}\{\!\{x\}\!\}_-$ and all homogeneous states $B \in \mathscr{V}$ we define

$$Y(XB; x) := X_+(x)Y(B; x) + (-1)^{|X||B|}Y(B; x) X_-(x),$$

where it remains to say what $X_+(x)$ and $X_-(x)$ are. To do so, it is enough to define, for all $a \in \mathfrak{g}$ and all $p(v, dv) \in \mathbb{C}[v, dv]$ such that $p|_{v=0} = p|_{v=1} = 0$, first

$$\left(a \otimes \frac{p(v, dv)}{z}\right)_+ (x) := \sum_{k=0}^{\infty} \left(a \otimes \frac{p(v, dv)}{z^{k+1}}\right) x^k$$

$$\left(a \otimes \frac{p(v, dv)}{z}\right)_- (x) := \sum_{k=0}^{\infty} \left(a \otimes z^k\right) \frac{p(1 - u, -du)}{x^{k+1}}$$

and then, for any $f(z) \in \mathbb{C}((z))$,

$$\left(a \otimes \frac{\partial}{\partial z} f(z) p(v, dv)\right)_{\pm} (x) := \frac{\partial}{\partial x} \left(a \otimes f(z) p(v, dv)\right)_{\pm} (x).$$

More explicitly, one has the following, by induction.

> **Lemma 1** (Explicit formula for the state-field map). Given a collection of homogeneous lowering operators $X^i \in \mathfrak{g} \otimes \mathbb{C}\{\!\{z\}\!\}_-$, $i = 1, \ldots, n$, we have
>
> $$Y_{\mathrm{Rav}}(X^1 \ldots X^n |0\rangle, x) =$$
>
> $$\sum_{m=0}^{n} \sum_{(\mu, \nu) \in \mathrm{Unshf}_m^n} (-1)^{n-m+\chi(|X^1|, \ldots, |X^n|, \mu, \nu)} X_+^{\mu_1}(x) \ldots X_+^{\mu_m}(x) X_-^{\nu_{n-m}}(x) \ldots X_-^{\nu_1}(x).$$
>
> The sum is over unshuffles, that is, permutations $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_{n-m})$ of $(1, \ldots, n)$ such that $\mu_1 < \cdots < \mu_m$ and $\nu_1 < \cdots < \nu_{n-m}$, and $(-1)^{\chi(|X^1|, \ldots, |X^n|, \mu, \nu)}$ is the Koszul sign of an unshuffle of the X^i, defined such that $(-1)^{\chi(|X^1|, \ldots, |X^n|, \mu, \nu)} X^1 \ldots X^n = X^{\mu_1} \ldots X^{\mu_m} X^{\nu_{n-m}} \ldots X^{\nu_1}$ in the symmetric algebra $\mathrm{Sym}(\mathfrak{g} \otimes \mathbb{C}\{\!\{w - z_N\}\!\}_-)$.

Some remarks are called for about this definition.

First, one should compare it to that of the usual state-field map for the vacuum Verma module $\mathbb{V}$ over the loop algebra $\mathfrak{g} \otimes \mathbb{C}((x))$. (Cf. §3.4 below.) In some informal but intuitively helpful sense, the former collapses to the latter if one ignores all of the factors $p(v, dv)$.

To understand the role of the polynomial differential form $p(v, dv)$, let us consider in turn the examples

$$p(v, dv) = dv \qquad \text{and} \qquad p(v, dv) = v(1 - v).$$

The lowering operators

$$a \otimes \frac{dv}{z^k}, \qquad a \in \mathfrak{g}, \quad k \geq 1,$$

represent non-trivial cohomology classes in degree one. What we call dv/z^k corresponds to what [25] call $\Omega^{k-1} \propto \lambda^{k-1}\omega$. On such cohomology classes, our definition here coincides with that of [25, §2.3]. (We wrote only a special case of the normal-ordered product above. Cf. [25, Defn. 2.1.3].) In addition to those lowering operators, there are also lowering operators in degree zero, for example,

$$a \otimes \frac{v(1 - v)}{z^k}, \qquad a \in \mathfrak{g}, \quad k \geq 1,$$

which are neither closed nor exact. This is the first instance of what we meant above by working in dg vector spaces rather than their cohomologies.

In view of [25], we expect that this definition of the state-field map will make the raviolo vacuum module $\mathscr{V}$ above into an example of what should probably be called a raviolo vertex algebra in dg vector spaces. It should be possible to spell out suitable axioms for such a structure, following [25] in the case

of raviolo vertex algebras in graded vector spaces and standard references for vertex algebras, [18, 31, 33] (Note also that (standard) vertex algebras internal to the category of dg vector spaces have been studied in [10, 11], with rather different motivations. Vertex algebras internal to the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, i.e., super vertex algebras, are ubiquitous in the literature; see (but also contrast, because the reference deals with supersymmetric vertex algebras in different sense)[28].).

However, in the present paper we want to do something slightly different. Let us adopt the standard perspective that the reason the usual vertex algebra axioms (locality, Borcherds identity, etc.) are the way they are is that they formalize the behaviour of what physicists would call operator product expansions (OPEs) in chiral conformal field theories (CFTs). More precisely, they capture the limiting behaviour of conformal blocks as two or more insertion points, associated to copies of the vacuum module, are brought close together. That relationship between vertex algebras and conformal blocks is known to hold for algebraic curves in very great generality—see [18] and references therein, following especially [38]. But in particular it holds in the prototypical setting of conformal blocks in genus zero, that is, on the complex plane or the Riemann sphere.

The crucial point, for us, is that in that latter genus zero setting it is well known how to define conformal blocks *without reference* to vertex algebras. Namely, conformal blocks are defined as the duals of rational coinvariants, as we are about to recall in detail, following [19] and [18, §13.3] (This is what we mean by "conformal blocks". The term has a number of closely related meanings in the mathematics and theoretical physics literature. See e.g., the discussion in [12, §8].).

Consequently, if the axioms of vertex algebras were mysteriously lost, one principled way to *recover* them would be by studying the limiting behaviour of such rational coinvariants.

The goal of the present paper is to establish the analogous relationship between vertex algebras and coinvariants in the raviolo case. Namely, we shall define a notion of coinvariants in the raviolo case, and then we shall show that the state-field map we defined above for the raviolo vacuum module emerges naturally from the behaviour of these raviolo coinvariants in the limit in which marked points collide.

3 Rational Coinvariants and Conformal Blocks

In this section we review the standard definition of rational coinvariants/conformal blocks on the complex plane with punctures. We focus exclusively on conformal blocks associated to an untwisted affine Kac–Moody algebra; that is, in physics language, the chiral sector of a WZW model. Moreover for simplicity we consider only the case of level zero, that is, we shall work with loop algebras and not their central extensions (It is worth recalling that level zero is in certain important senses a generic value. The non-generic value is the *critical level*, $k = -h^\vee$ in standard normalizations, at which for example the usual Sugawara conformal vector of the vacuum Verma module becomes singular.).

3.1 Fixed punctures

In this subsection we work over the complex numbers $\mathbb{C}$.

Let $a_1, \ldots, a_N \in \mathbb{C}$ be pairwise distinct complex numbers. We think of them as *marked points* or *punctures* in the complex plane.

Let $\mathbb{C}[w, (w - a_i)^{-1}]'_{1 \le i \le N}$ denote the (non-unital) commutative $\mathbb{C}$-algebra of rational expressions in w singular at most at the points $a_1, \ldots, a_N$ and vanishing as $w \to \infty$. It is a subalgebra of the unital commutative $\mathbb{C}$-algebra

$$\Gamma(\mathbb{A}^1_{\mathbb{C}} \setminus \{a_1, \ldots, a_N\}, \mathcal{O}) = \mathbb{C}[w, (w - a_i)^{-1}]_{1 \le i \le N}$$

of sections of the structure sheaf $\mathcal{O}$ of the affine line $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[w]$ over the Zariski open subset $\mathbb{A}^1_{\mathbb{C}} \setminus \{a_1, \ldots, a_N\}$.

For each puncture a_i, $1 \le i \le N$, we have the algebra of formal series and of formal Laurent series,

$$\mathbb{C}[[w - a_i]] = \Gamma(\operatorname{Disc}_1(a_i), \mathcal{O}) \qquad \text{and} \quad \mathbb{C}((w - a_i)) = \Gamma(\operatorname{Disc}_1^\times(a_i), \mathcal{O}), \qquad (1)$$

which are to be thought of as the algebras of regular functions on, respectively, the formal disc $\operatorname{Disc}_1(a_i) = \operatorname{Spec} \mathbb{C}[[w - a_i]]$ and the formal punctured disc $\operatorname{Disc}_1^\times(a_i) = \operatorname{Spec} \mathbb{C}((w - a_i)) = \operatorname{Disc}_1(a_i) \setminus \overline{a_i}$ at the closed point a_i of $\mathbb{A}^1_{\mathbb{C}}$.

There are embeddings of commutative $\mathbb{C}$-algebras

$$\mathbb{C}[w, (w - a_i)^{-1}]'_{1 \leq i \leq N} \hookrightarrow \bigoplus_{i=1}^{N} \mathbb{C}((w - a_i)) \hookleftarrow \bigoplus_{i=1}^{N} \mathbb{C}[[w - a_i]] \tag{2a}$$

—on the left, by Laurent-expanding at each of the marked points; on the right, by the canonical embedding summand by summand—such that, at the level of vector spaces, there is an isomorphism

$$\bigoplus_{i=1}^{N} \mathbb{C}((w - a_i)) \cong_{\mathbb{C}} \mathbb{C}[w, (w - a_i)^{-1}]'_{1 \leq i \leq N} \oplus \bigoplus_{i=1}^{N} \mathbb{C}[[w - a_i]]. \tag{2b}$$

Let us now pick a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. We get Lie algebras over $\mathbb{C}$

$$\mathfrak{b} := \bigoplus_{i=1}^{N} \mathfrak{g} \otimes \mathbb{C}((w - a_i))$$

$$\mathfrak{b}_+ := \bigoplus_{i=1}^{N} \mathfrak{g} \otimes \mathbb{C}[[w - a_i]], \qquad \mathfrak{b}_- := \mathfrak{g} \otimes \mathbb{C}[w, (w - a_i)^{-1}]'_{1 \leq i \leq N}$$

and embeddings of Lie algebras over $\mathbb{C}$

$$\mathfrak{b}_- \hookrightarrow \mathfrak{b} \hookleftarrow \mathfrak{b}_+, \tag{3a}$$

which again give rise to an isomorphism of the underlying vector spaces,

$$\mathfrak{b} \cong_{\mathbb{C}} \mathfrak{b}_- \oplus \mathfrak{b}_+. \tag{3b}$$

Let M_i, $1 \leq i \leq N$ be $\mathfrak{g}$-modules in the category of $\mathbb{C}$-vector spaces. We make each M_i into a module over the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[[w - a_i]]$ by declaring $X \otimes 1$ acts as X and $X \otimes (w - a_i)^k$ acts as 0 for all $k \geq 1$ and all $X \in \mathfrak{g}$. In other words, we pull back M_i along the map of Lie algebras $\mathfrak{g} \otimes \mathbb{C}[[w - a_i]] \to \mathfrak{g} \otimes \mathbb{C}[[w - a_i]]/\mathfrak{g} \otimes (w - a_i)\mathbb{C}[[w - a_i]] \cong \mathfrak{g}$.

> **Remark 2.** More generally, one may start with, for each $1 \leq i \leq N$, *any* module M_i over the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[[w - a_i]]$, which is *smooth*. Recall, from [18,§5.1.5], that a $\mathfrak{g} \otimes \mathbb{C}[[x]]$-module M is said to be *smooth* if, for all $m \in M$, the Lie ideal $\mathfrak{g} \otimes x^k\mathbb{C}[[x]]$ acts as zero on m for all sufficiently large k.

We may then construct the induced $\mathfrak{g} \otimes \mathbb{C}((w - a_i))$ module

$$\mathbb{M}_i := \mathrm{Ind}_{\mathfrak{g} \otimes \mathbb{C}[[w-a_i]]}^{\mathfrak{g} \otimes \mathbb{C}((w-a_i))} M_i := U(\mathfrak{g} \otimes \mathbb{C}((w - a_i))) \otimes_{U(\mathfrak{g} \otimes \mathbb{C}[[w-a_i]])} M_i. \tag{4}$$

Equivalently, one sees that

$$\mathbb{M} := \bigotimes_{i=1}^{N} \mathbb{M}_i = \mathrm{Ind}_{\mathfrak{b}_+}^{\mathfrak{b}} M, \quad \text{where} \quad M := \bigotimes_{i=1}^{N} M_i$$

On pulling back by the embedding $\mathfrak{b}_- \hookrightarrow \mathfrak{b}$, $\mathbb{M}$ is, in particular, a module over $\mathfrak{b}_-$.

The space of *rational coinvariants (at level zero)* associated to these data $a_1, \ldots, a_N$; $\mathfrak{g}$; $M_1, \ldots, M_N$ is then by definition

$$F(\mathfrak{g}; a_1, \ldots, a_N; M_1, \ldots, M_N) := \mathbb{M}/\mathfrak{b}_- := \mathbb{M}/(\mathfrak{b}_-.\mathbb{M}) \cong_{\mathbb{C}} \mathbb{C} \otimes_{U(\mathfrak{b}_-)} \mathbb{M}$$

and the space of *rational conformal blocks (at level zero)* is by definition the dual space

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{M}/\mathfrak{b}_-, \mathbb{C}) \cong_{\mathbb{C}} \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{C}}(\mathfrak{b}_-)}(\mathbb{M}, \mathbb{C}).$$

Here we make $\mathbb{C}$ into the trivial $\mathfrak{b}_-$ module. Equivalently it is the $U(\mathfrak{b}_-)$-module obtained by pulling back by the counit map $U(\mathfrak{b}_-) \to \mathbb{C}$.

It follows from (3) together with the PBW theorem that there is an isomorphism $U(\mathfrak{b}) \cong U(\mathfrak{b}_-) \otimes_{\mathbb{C}} U(\mathfrak{b}_+)$ of $(U(\mathfrak{b}_-), U(\mathfrak{b}_+))$-bimodules. Consequently the space of coinvariants is canonically isomorphic to underlying vector space of the $\mathfrak{g}$-module $M = M_1 \otimes \ldots \otimes M_N$ from which we began:

$$\mathbb{M}/\mathfrak{b}_- = \mathbb{C} \otimes_{U(\mathfrak{b}_-)} \mathbb{M} \cong_{\mathbb{C}} \mathbb{C} \otimes_{U(\mathfrak{b}_-)} U(\mathfrak{b}) \otimes_{U(\mathfrak{b}_+)} M$$

$$\cong_{\mathbb{C}} \mathbb{C} \otimes_{U(\mathfrak{b}_-)} U(\mathfrak{b}_-) \otimes_{\mathbb{C}} U(\mathfrak{b}_+) \otimes_{U(\mathfrak{b}_+)} M$$

$$\cong_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}} M = M.$$

The richness of the space of coinvariants really emerges when one allows the marked points $a_1, \ldots, a_N$ to vary (One sense in which this is true is that the rational coinvariants/conformal blocks obey the celebrated KZ equations. It would be interesting to investigate whether the coinvariants we introduce below in the ravioli setting obey some analogous equations. (One possible obstruction is that, in the raviolo case, every level is "critical" in the sense that the Sugawara construction fails [25,§4.4]; we thank an anonymous referee for pointing this out.)). We turn to this now.

3.2 Movable punctures and configuration space

Our fixed set of distinct marked points $a_1, \ldots, a_N \in \mathbb{C}$ from the previous subsection is now to be thought of just one choice of closed point of the *configuration space*

$$\mathrm{Conf}_N := \mathbb{A}_{\mathbb{C}}^N \setminus \bigcup_{1 \le i < j \le N} \overline{(z_i = z_j)} \tag{5}$$

obtained by starting with the affine scheme $\mathbb{A}_{\mathbb{C}}^N = \mathbf{Spec}\, \mathbb{C}[z_1, \ldots, z_N]$ and removing (the Zariski closures $\overline{(z_i = z_j)}$ of the generalized points $(z_i = z_j)$ of) all the diagonal hyperplanes.

Following the approach of [18,§13], one can think that going from fixed to movable marked points is a matter of changing the ground ring from $\mathbb{C}$ to the $\mathbb{C}$-algebra $\mathcal{B}_N := \Gamma(\mathrm{Conf}_N, \mathcal{O})$ of regular functions on configuration space. Since Conf_N is the complement in $\mathbb{A}_{\mathbb{C}}^N$ of the zero locus of the function

$$\Delta_N := \prod_{1 \le i < j \le N} (z_i - z_j)$$

we have by definition—see for example, [16,§I]—that

$$\mathcal{B}_N = \mathbb{C}[z_1, \ldots, z_N][\Delta_N^{-1}] \tag{6a}$$

is the localization of $\mathbb{C}[z_1, \ldots, z_N] = \Gamma(\mathbb{A}_{\mathbb{C}}^N, \mathcal{O})$ obtained by adjoining an inverse to Δ_N, and

$$\mathrm{Conf}_N = \mathbf{Spec}\, \mathcal{B}_N$$

is again an affine scheme. Of course, once we can invert Δ_N, we can invert any $(z_i - z_j)$, so that, in more suggestive notation,

$$\mathcal{B}_N = \mathbb{C}[z_i, (z_i - z_j)^{-1}]_{\substack{1 \le i, j \le N \\ i \ne j}}. \tag{6b}$$

Now we should ask what, in this setting, the analogues of the embeddings of $\mathbb{C}$-algebras in (2) should be. First, in place of $\mathbb{C}[w, (w - a_i)^{-1}]_{1 \le i \le N} = \Gamma(\mathbb{A}_{\mathbb{C}}^1 \setminus \{a_1, \ldots, a_N\}, \mathcal{O})$ we should consider

$$\mathcal{B}_{N+1} = \mathbb{C}[z_1, \ldots, z_N, w]\left[\left(\Delta_N \prod_{i=1}^N (w - z_i)\right)^{-1}\right] = \Gamma(\mathrm{Conf}_{N+1}, \mathcal{O}). \tag{7}$$

This is a $\mathcal{B}_N$-algebra in the obvious way. Let $\mathcal{B}'_{N+1}$ denote the (non-unital) subalgebra consisting of those functions that vanish as $w \to \infty$.

Next, we want the analogue of the disc $\mathrm{Disc}_1(a_i) = \mathrm{Spec}\,\mathbb{C}[[w-a_i]]$ and the punctured disc $\mathrm{Disc}_1^\times(a_i) = \mathrm{Spec}\,\mathbb{C}((w-a_i))$ near a closed point $a_i \in \mathbb{A}^1_\mathbb{C}$. Recall that $\mathbb{C}[[w-a_i]]$ is the completion of $\mathbb{C}[w]$ with respect to the ideal $(w-a_i)\mathbb{C}[w]$, and $\mathbb{C}((w-a_i))$ is then the localization of $\mathbb{C}[[w-a_i]]$ obtained by adjoining an inverse to $w-a_i$. Here, since the ground ring is now $\mathcal{B}_N$, we consider the completion of $\mathcal{B}_N[w]$ with respect to $(w-z_i)\mathcal{B}_N[w]$, that is, the ring $\mathcal{B}_N[[w-z_i]]$, and then its localization $\mathcal{B}_N((w-z_i))$.

We arrive at the following analogue of (2): there are embeddings of commutative algebras in $\mathcal{B}_N$-modules

$$\mathcal{B}'_{N+1} \hookrightarrow \bigoplus_{i=1}^N \mathcal{B}_N((w-z_i)) \hookleftarrow \bigoplus_{i=1}^N \mathcal{B}_N[[w-z_i]] \tag{8}$$

such that, at the level of $\mathcal{B}_N$-modules, there is an isomorphism

$$\bigoplus_{i=1}^N \mathcal{B}_N((w-z_i)) \cong_{\mathcal{B}_N} \mathcal{B}'_{N+1} \oplus \bigoplus_{i=1}^N \mathcal{B}_N[[w-z_i]]. \tag{9}$$

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ as before. By extension of scalars we obtain the Lie algebra over $\mathcal{B}_N$, that is, the Lie algebra in $\mathcal{B}_N$-modules, given by

$$_\mathcal{B}\mathfrak{g} := \mathcal{B}_N \otimes \mathfrak{g}.$$

We have also Lie algebras in $\mathcal{B}_N$-modules given by

$$_\mathcal{B}\mathfrak{b} := \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathcal{B}_N((w-z_i))$$

$$_\mathcal{B}\mathfrak{b}_+ := \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathcal{B}_N[[w-z_i]], \qquad _\mathcal{B}\mathfrak{b}_- := \mathfrak{g} \otimes \mathcal{B}'_{N+1}$$

and embeddings of Lie algebras in $\mathcal{B}_N$-modules

$$_\mathcal{B}\mathfrak{b}_- \hookrightarrow {}_\mathcal{B}\mathfrak{b} \hookleftarrow {}_\mathcal{B}\mathfrak{b}_+, \tag{10a}$$

which give rise to an isomorphism of the underlying $\mathcal{B}_N$-modules,

$$_\mathcal{B}\mathfrak{b} \cong {}_\mathcal{B}\mathfrak{b}_- \oplus {}_\mathcal{B}\mathfrak{b}_+. \tag{10b}$$

Let M_i, $1 \le i \le N$, be $\mathfrak{g}$-modules as before. By extension of scalars we get $_\mathcal{B}\mathfrak{g}$-modules $_\mathcal{B}M_i := \mathcal{B}_N \otimes M_i$, that is, $_\mathcal{B}\mathfrak{g}$-module objects in the category of $\mathcal{B}_N$-modules. We make each $_\mathcal{B}M_i$ into a module over the Lie algebra $\mathfrak{g} \otimes \mathcal{B}_N[[w-z_i]]$ by pulling back along the map of Lie algebras $\mathfrak{g} \otimes \mathcal{B}_N[[w-z_i]] \to \mathfrak{g} \otimes \mathcal{B}_N[[w-z_i]]/\mathfrak{g} \otimes (w-z_i)\mathcal{B}_N[[w-z_i]] \cong {}_\mathcal{B}\mathfrak{g}$.

(Again, as in Remark 2, there is actually no obstruction to taking $_\mathcal{B}M_i$ to be any smooth module over the Lie algebra $\mathfrak{g} \otimes \mathcal{B}_N[[w-z_i]]$; we focus on the above class of such modules purely for concreteness.)

We may then construct the induced $\mathfrak{g} \otimes \mathcal{B}_N((w-z_i))$-module

$$_\mathcal{B}\mathbb{M}_i := \mathrm{Ind}_{\mathfrak{g} \otimes \mathcal{B}_N[[w-z_i]]}^{\mathfrak{g} \otimes \mathcal{B}_N((w-z_i))} {}_\mathcal{B}M_i$$

$$:= U_{\mathcal{B}_N}(\mathfrak{g} \otimes \mathcal{B}_N((w-z_i))) \otimes_{U_{\mathcal{B}_N}(\mathfrak{g} \otimes \mathcal{B}_N[[w-z_i]])} {}_\mathcal{B}M_i. \tag{11}$$

(Here we write $U_{\mathcal{B}_N}(-) : \mathbf{Alg}^{\mathcal{L}\mathrm{ie}}(\mathbf{Mod}_{\mathcal{B}_N}) \to \mathbf{Alg}^{\mathcal{A}\mathrm{ss}}(\mathbf{Mod}_{\mathcal{B}_N})$ for the functor whose action on objects is to take a Lie algebra over $\mathcal{B}_N$ to its universal envelope, an associative algebra over $\mathcal{B}_N$.)

The tensor product of these modules,

$$\mathcal{_B}\mathbb{M} := \bigotimes_{i=1}^{N} \mathcal{_{B_N}} \mathcal{_B}\mathbb{M}_i,$$

is equivalently the induced module

$$\mathcal{_B}\mathbb{M} = \operatorname{Ind}_{\mathcal{_B}\mathfrak{b}_+}^{\mathcal{_B}\mathfrak{b}} \mathcal{_B}M := U_{\mathcal{B_N}}(\mathfrak{b}) \otimes_{U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_+)} \mathcal{_B}M, \quad \text{where} \quad \mathcal{_B}M := \bigotimes_{i=1}^{N} \mathcal{_{B_N}} \mathcal{_B}M_i.$$

On pulling back by the embedding $\mathcal{_B}\mathfrak{b}_- \hookrightarrow \mathcal{_B}\mathfrak{b}$, $\mathcal{_B}\mathbb{M}$ is, in particular, a module over $\mathcal{_B}\mathfrak{b}_-$.

The space of *rational coinvariants (at level zero)* associated to these data $\mathfrak{g}; M_1, \dots, M_N$ is then by definition the $\mathcal{B_N}$-module

$$F(\mathfrak{g}; \mathcal{B_N}; M_1, \dots, M_N) := \mathcal{_B}\mathbb{M}\big/ \mathcal{_B}\mathfrak{b}_- := \mathcal{_B}\mathbb{M}/(\mathcal{_B}\mathfrak{b}_- \cdot \mathcal{_B}\mathbb{M})$$

$$\cong_{\mathcal{B_N}} \mathcal{B_N} \otimes_{U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_-)} \mathcal{_B}\mathbb{M} \tag{12}$$

and the space of *rational conformal blocks (at level zero)* is by definition the dual

$$\operatorname{Hom}_{\mathcal{B_N}}(\mathcal{_B}\mathbb{M}\big/ \mathcal{_B}\mathfrak{b}_-, \mathcal{B_N}) \cong_{\mathcal{B_N}} \operatorname{Hom}_{\mathcal{_B}\mathfrak{b}_-}(\mathcal{_B}\mathbb{M}, \mathcal{B_N}).$$

Here we make $\mathcal{B_N}$ into the trivial $\mathcal{_B}\mathfrak{b}_-$-module. Equivalently it is the $U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_-)$-module obtained by pulling back by the counit map $U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_-) \to \mathcal{B_N}$.

Once more, the data of the triple of Lie algebras (10) together with the PBW theorem imply that there is an isomorphism, $U_{\mathcal{B_N}}(\mathfrak{b}) \cong U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_-) \otimes_{\mathcal{B_N}} U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_+)$, now of $(U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_-), U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_+))$-bimodules in $\mathcal{B_N}$-modules, and hence that the space of coinvariants is canonically isomorphic to $\mathcal{_B}M = \mathcal{B_N} \otimes M$:

$$\mathcal{_B}\mathbb{M}\big/ \mathcal{_B}\mathfrak{b}_- = \mathcal{B_N} \otimes_{U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_-)} \mathcal{_B}\mathbb{M}$$

$$\cong_{\mathcal{B_N}} \mathcal{B_N} \otimes_{U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_-)} U_{\mathcal{B_N}}(\mathfrak{b}) \otimes_{U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_+)} \mathcal{_B}M$$

$$\cong_{\mathcal{B_N}} \mathcal{B_N} \otimes_{U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_-)} U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_-) \otimes_{\mathcal{B_N}} U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_+) \otimes_{U_{\mathcal{B_N}}(\mathcal{_B}\mathfrak{b}_+)} \mathcal{_B}M$$

$$\cong \mathcal{B_N} \otimes_{\mathcal{B_N}} \mathcal{_B}M = \mathcal{_B}M = \mathcal{B_N} \otimes M. \tag{13}$$

3.3 Taking coinvariants

For any $\mathbb{C}$-point $(a_1, \dots, a_N) \in \operatorname{Conf}_N$ we can apply the evaluation map

$$\operatorname{ev}_{a_1,\dots,a_N} : \mathcal{B_N} \to \mathbb{C}$$

to recover the space of coinvariants for this particular choice of fixed marked points, as we had it in §3.1. That is, the constructions above are all suitably functorial, so that there is a map

$$\operatorname{ev}_{a_1,\dots,a_N} : F(\mathfrak{g}; \mathcal{B_N}; M_1, \dots, M_N) \to F(\mathfrak{g}; a_1, \dots, a_N; M_1, \dots, M_N).$$

In checking this, one notes in particular the following lemma.

Lemma 3. For each i, $\mathcal{_B}\mathbb{M}_i \cong \mathcal{B_N} \otimes_{\mathbb{C}} \mathbb{M}_i$. Hence $\mathcal{_B}\mathbb{M} \cong \mathcal{B_N} \otimes_{\mathbb{C}} \mathbb{M}$.

Proof. Let us define

$$\mathfrak{g}_- := (w - z_N)^{-1}\mathbb{C}[(w - z_N)^{-1}] \quad \text{and} \quad \mathcal{_B}\mathfrak{g}_- := (w - z_N)^{-1}\mathcal{B_N}[(w - z_N)^{-1}].$$

These are Lie algebras over $\mathbb{C}$ and $\mathcal{B}_N$ respectively. Certainly we have

$$_{\mathcal{B}}\mathfrak{g}_- \cong \mathcal{B}_N \otimes_{\mathbb{C}} \mathfrak{g}_- \tag{14}$$

(since these are just Lie algebras of polynomials) and hence that $U_{\mathcal{B}_N}(_{\mathcal{B}}\mathfrak{g}_-) \cong \mathcal{B}_N \otimes_{\mathbb{C}} U(\mathfrak{g}_-)$. We have the "local" Lie algebra splitting, that is, the embeddings of Lie algebras in $\mathcal{B}_N$-modules

$$_{\mathcal{B}}\mathfrak{g}_- \hookrightarrow \mathfrak{g} \otimes \mathcal{B}_N((w - z_N)) \hookleftarrow \mathfrak{g} \otimes \mathcal{B}_N[[w - z_N]]$$

such that as $\mathcal{B}_N$-modules

$$_{\mathcal{B}}\mathfrak{g}_- \oplus \mathfrak{g} \otimes \mathcal{B}_N[[w - z_N]] \cong \mathfrak{g} \otimes \mathcal{B}_N((w - z_N)).$$

Therefore $_{\mathcal{B}}\mathbb{M}_i$ is free as a module over $U_{\mathcal{B}_N}(\mathfrak{g}_-)$:

$$_{\mathcal{B}}\mathbb{M}_i \cong U_{\mathcal{B}_N}(_{\mathcal{B}}\mathfrak{g}_-) \otimes_{\mathcal{B}_N} {}_{\mathcal{B}}\mathbb{M}_i$$

The analogous statements hold for $\mathfrak{g}_-$ and $\mathbb{M}_i$. The result follows. ∎

On the other hand, we can now consider fixing vectors $m_i \in \mathbb{M}_i$, for $1 \leq i \leq N$, while letting the marked points vary. More precisely, the unit map $1 : \mathbb{C} \to \mathcal{B}_N$ induces embeddings (of vector spaces)

$$\mathbb{M}_i \hookrightarrow \mathcal{B}_N \otimes \mathbb{M}_i \cong {}_{\mathcal{B}}\mathbb{M}_i; \quad m \mapsto 1 \otimes m \tag{15}$$

for each i, and hence $\mathbb{M} \hookrightarrow \mathcal{B}_N \otimes \mathbb{M} \cong {}_{\mathcal{B}}\mathbb{M}$. We may take the class of the vector $m_1 \otimes \dots \otimes m_N \in \mathbb{M} \cong 1 \otimes \mathbb{M} \hookrightarrow \mathcal{B}_N \otimes \mathbb{M} \cong {}_{\mathcal{B}}\mathbb{M}$ in the space of coinvariants in (13). We shall write this class as

$$\left[m_1 \underset{\overset{\uparrow}{z_1}}{\otimes} \cdots \otimes m_N \underset{\overset{\uparrow}{z_N}}{} \right].$$

It is an element of $\mathcal{B}_N \otimes M$, that is, an $M = \bigotimes_{i=1}^N M_i$-valued rational function of $z_1, \dots, z_N$ singular at most on the diagonals $z_i - z_j$, $1 \leq i < j \leq N$.

We call applying this map of $\mathcal{B}_N$-modules

$$\mathbb{M} \to \mathcal{B}_N \otimes M; \quad m_1 \otimes \cdots \otimes m_N \mapsto \left[m_1 \underset{\overset{\uparrow}{z_1}}{\otimes} \cdots \otimes m_N \underset{\overset{\uparrow}{z_N}}{} \right], \tag{16}$$

the operation of *taking coinvariants*.

3.4 The usual vacuum Verma module and state-field map

Let now

$$\mathbb{V} := \mathrm{Ind}_{\mathfrak{g} \otimes \mathbb{C}[[s]]}^{\mathfrak{g} \otimes \mathbb{C}((s))} \mathbb{C}|0\rangle$$

denote the module over the loop algebra $\mathfrak{g} \otimes \mathbb{C}((s))$ induced from the trivial one-dimensional module $\mathbb{C}|0\rangle$ over $\mathfrak{g} \otimes \mathbb{C}[[s]]$ generated a vector $|0\rangle$. This module $\mathbb{V}$ is called the *vacuum Verma module (at level zero)*.

The vacuum Verma module $\mathbb{V}$ comes equipped with a linear map

$$Y(-, s) : \mathbb{V} \to \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{C}}}(\mathbb{V}, \mathbb{V}((s))), \qquad A \mapsto Y(A, s) = \sum_{n \in \mathbb{Z}} A_{(n)} s^{-n-1}, \tag{17}$$

called the *state-field map*. Vectors in $\mathbb{V}$ are called *states*, and one can think that $Y(-, s)$ sends each state $A \in \mathbb{V}$ to the formal sum of its *modes* $A_{(n)} \in \mathbf{End}_{\mathbf{Vect}_{\mathbb{C}}}(\mathbb{V})$. The state-field map satisfies certain axioms (notably Borcherds identity), which can be found in standard references including [18, 31, 33] and which make

$\mathbb{V}$ into a *vertex algebra*. From our present perspective the important point is that one way to motivate these axioms is by studying the limit of rational coinvariants as points collide, as we now describe.

Let us now specialize our discussion of rational coinvariants above to the case in which $M_{N-1} = \mathbb{C}|0\rangle$ and $M_N = \mathbb{C}|0\rangle$. In that case

$$M = \bigotimes_{i=1}^{N-2} M_i \otimes \mathbb{C} \otimes \mathbb{C} = \bigotimes_{i=1}^{N-2} M_i \quad \text{and} \quad \mathbb{M} = \bigotimes_{i=1}^{N-2} \mathbb{M}_i \otimes \mathbb{V}_{N-1} \otimes \mathbb{V}_N.$$

Now, by identifying local coordinates $w - z_i$, we may identify each of the Lie algebras $\mathfrak{g} \otimes \mathbb{C}((w - z_i))$ with a single copy $\mathfrak{g} \otimes \mathbb{C}((s))$ (This identification is sometimes left implicit, but is an important assumption. Other ways of picking preferred local coordinates and hence identifying copies of $\mathbb{V}$ at different points are possible and can be important in applications: e.g., when one wants to go to what physicists would call the cylinder geometry.), and thereby identify their vacuum Verma modules with a single abstract copy of $\mathbb{V}$:

$$\mathbb{V} \cong \mathbb{V}_i := \operatorname{Ind}_{\mathfrak{g} \otimes \mathbb{C}[[w-z_i]]}^{\mathfrak{g} \otimes \mathbb{C}((w-z_i))} \mathbb{C}|0\rangle.$$

Pick vectors $m_i \in \mathbb{M}_i$ for $1 \leq i \leq N-2$, and states $A, B \in \mathbb{V}$ in this abstract copy of $\mathbb{V}$. On taking coinvariants, we get the M-valued rational function

$$\left[\underset{z_1}{m_1} \otimes \cdots \otimes \underset{z_{N-2}}{m_{N-2}} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{A} \right] \in \mathcal{B}_N \otimes M. \tag{18}$$

The usual state-field map captures the behaviour of this function in the limit in which the marked point z_N becomes close to marked point z_{N-1}, while the points $z_1, \ldots, z_{N-1}$, the vectors $m_1, \ldots, m_{N-2}$, and the states A, B are all held fixed.

> **Theorem 4** (Relation of the state-field map Y to rational coinvariants). For all $A, B \in \mathbb{V}$ and $m_i \in \mathbb{M}_i$, $1 \leq i \leq N-2$, we have
>
> $$\iota_{z_N \to z_{N-1}} \left[\underset{z_1}{m_1} \otimes \cdots \otimes \underset{z_{N-2}}{m_{N-2}} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{A} \right]$$
> $$= \left[\underset{z_1}{m_1} \otimes \cdots \otimes \underset{z_{N-2}}{m_{N-2}} \otimes \underset{z_{N-1}}{Y(A; z_N - z_{N-1}) B} \right].$$

We give a proof of this very standard fact below, in §7.1. Note that while the left-hand side here is manifestly in $\mathcal{B}_{N-1}((z_N - z_{N-1})) \otimes M$, since it is the expansion of an element of $\mathcal{B}_N \otimes M$, a priori the right-hand side is merely an element of $(\mathcal{B}_{N-1} \otimes M)((z_N - z_{N-1}))$: indeed what is meant by the right-hand side is the series obtained by computing the coinvariant in $F(\mathfrak{g}; \mathcal{B}_{N-1}; M_1, \ldots, M_{N-2}, \mathbb{C}) \cong \mathcal{B}_{N-1} \otimes M$ order by order in $z_N - z_{N-1}$.

Now let us close this digression on the usual rational conformal blocks associated to Kac–Moody algebras at level zero, and return to the raviolo case.

4 Ravioli Configuration Space

As we just saw, the usual rational coinvariants/conformal blocks form a module over the commutative algebra

$$\mathcal{B}_N := \Gamma(\operatorname{Conf}_N, \mathcal{O}) = \mathbb{C}[z_1, \ldots, z_N][\operatorname{Discr}_N^{-1}]$$

$$= \mathbb{C}[z_i, (z_i - z_j)^{-1}]_{\substack{1 \leq i, j \leq N \\ i \neq j}}$$

of global sections of the structure sheaf on the configuration space of N distinguishable pairwise distinct marked points in the complex plane,

$$\mathrm{Conf}_N := \mathbb{A}_{\mathbb{C}}^N \setminus \bigcup_{1 \le i < j \le N} \overline{(z_i = z_j)}.$$

Remark 5. In more geometrical language, the module of conformal blocks is the $\Gamma(\mathrm{Conf}_N, \mathcal{O})$-module of global sections of a trivial vector bundle over configuration space, whose typical fibre we described in §3.1; see [18,§13.3]. Cf. also for example, [39], [15].

In this section our goal is to introduce a suitable notion of *configuration space in the ravioli setting*,

$$\mathrm{RavConf}_N,$$

and then to construct a model

$$\mathcal{A}_N \simeq R\Gamma(\mathrm{RavConf}_N, \mathcal{O}),$$

in dg commutative algebras, of the derived global sections of its structure sheaf. This is a prelude to defining ravioli analogues of rational coinvariants/conformal blocks, which we shall do in the next section.

In the ravioli setting, we again want to describe configurations of N distinguishable marked points in the complex plane. However, we now want to allow them to coincide, but with the stipulation that, whenever two points *do* coincide, we want to keep track of which point is "on top" of which.

Formally then, what we shall do is to glue together $N!$ copies of the affine scheme $\mathbb{A}_{\mathbb{C}}^N$ along the complements of the diagonal hyperplanes, as follows.

Let S_N denote the group of permutations of the set $[1, N] = \{1, \dots, N\}$. We shall identify S_N, as a set, with the set of *total orders* on $[1, N]$, by associating $\sigma \in S_N$ with the total order $\prec_\sigma$ on $[1, N]$ defined by

$$\sigma(1) \prec_\sigma \sigma(2) \prec_\sigma \cdots \prec_\sigma \sigma(N).$$

Let $\mathrm{Partial}_{[1,N]}$ denote the set of all *partial orders* $\prec$ on the set $[1, N] = \{1, \dots, N\}$. We make $\mathrm{Partial}_{[1,N]}$ itself into a partially ordered set (or, equivalently, a skeletal preorder) in which there is an arrow $\prec \to \prec'$ if and only if $\prec'$ refines $\prec$ in the obvious sense. Given a partial order $\prec \in \mathrm{Partial}_{[1,N]}$, let $\mathcal{O}(U_\prec)$ denote the commutative algebra

$$\mathcal{O}(U_\prec) := \mathbb{C}[z_1, \dots, z_N]\left[\prod_{\substack{i,j \in [1,N] \\ i \ne j, i \not\prec j, j \not\prec i}} \frac{1}{z_i - z_j} \right] \in \mathbf{Alg}^{Com}(\mathbf{Vect}_{\mathbb{C}}).$$

That is, $\mathcal{O}(U_\prec)$ is the localization of the polynomial algebra $\mathbb{C}[z_1, \dots, z_N]$ in which $z_i - z_j$ is invertible precisely for those distinct i and j that are $\prec$-incomparable. If $\prec'$ refines $\prec$ then there is a canonical inclusion of $\mathbb{C}$-algebras $\mathcal{O}(U_{\prec'}) \to \mathcal{O}(U_\prec)$, and hence a map of affine $\mathbb{C}$-schemes $U_\prec \to U_{\prec'}$. This defines a functor (and hence a diagram in $\mathbb{C}$-schemes. In fact $\mathrm{RavConf}_N$ as we are about to define it is the colimit in $\mathbb{C}$-schemes of this diagram, $\mathrm{RavConf}_N = \mathrm{colim}_{\prec \in \mathrm{Partial}_{[1,N]}} U_\prec$.)

$$U : \mathrm{Partial}_{[1,N]} \to \mathbf{Alg}^{Com}(\mathbf{Vect}_{\mathbb{C}})^{\mathrm{op}} \equiv \mathbf{AffSch}_{\mathbb{C}} \hookrightarrow \mathbf{Sch}_{\mathbb{C}}.$$

4.1 Definition of RavConf$_N$ by gluing Čech data

Recall that if we are *given* a $\mathbb{C}$-scheme X, we can by definition always cover it with a collection $\mathscr{U} = \{U_i\}_{i \in I}$ of affine patches $U_i \in \mathbf{AffSch}_{\mathbb{C}}$ indexed by some totally ordered index set $(I, <)$. The Čech nerve of this cover is the semisimplicial object in $\mathbb{C}$-schemes given by the diagram

$$\check{C}(\mathscr{U}) = \left(\cdots \ \Longrightarrow \ \bigsqcup_{\substack{i,j \in I \\ i < j}} U_{i,j} \ \Longrightarrow \ \bigsqcup_{i \in I} U_i \right)$$

where

$$U_{i,j} := U_i \cap_X U_j$$

is the intersection in X of the affine patches U_i and U_j. (We recall the meaning of semisimplicial objects in a category in Appendix A.) This is a diagram in $\mathbb{C}$-schemes whose colimit is the original scheme X:

$$X = \operatorname{colim} \check{C}(\mathscr{U}).$$

Moreover we may always find a Leray cover, and the Čech cohomology of $\mathcal{O}$ with respect to such a cover computes the sheaf cohomology of $\mathcal{O}$. This is the case in particular if all the intersections $U_{i,j,\dots,k}$ are themselves affine.

With this in mind, let us define the *ravioli configuration space* $RavConf_N$ as follows. It is covered by the collection of $N!$ affine schemes

$$\mathscr{U} := \left\{ U_{\prec_\sigma} \cong \mathbb{A}_{\mathbb{C}}^N : \sigma \in S_N \right\}.$$

We glue these affine patches together as follows. Given a collection of partial orders $\prec_1, \dots, \prec_k \in$ $\mathrm{Partial}_{[1,N]}$, let

$$\prec_1 \wedge \cdots \wedge \prec_k$$

denote their finest common coarsening, or in other words their meet, or categorical product, in $\mathrm{Partial}_{[1,N]}$. All meets exist in $\mathrm{Partial}_{[1,N]}$; at worst the meet may be the initial object, namely the partial order in which no two elements are comparable. Given any distinct $\sigma_1, \sigma_2 \in S_N$ we *define* the intersection in $RavConf_N$ of the affine patches $U_{\prec_{\sigma_1}}$ and $U_{\prec_{\sigma_2}}$ to be $U_{\prec_{\sigma_1} \wedge \prec_{\sigma_2}}$:

$$U_{\prec_{\sigma_1}} \cap_{RavConf_N} U_{\prec_{\sigma_2}} := U_{\prec_{\sigma_1} \wedge \prec_{\sigma_2}},$$

with the inclusions $U_{\prec_{\sigma_1} \wedge \prec_{\sigma_2}} \hookrightarrow U_{\prec_{\sigma_1}}$ and $U_{\prec_{\sigma_1} \wedge \prec_{\sigma_2}} \hookrightarrow U_{\prec_{\sigma_2}}$ being the canonical inclusions we noted in the definition of the functor U above. This gluing data satisfies the triple overlap condition, and we define $RavConf_N$ to be the resulting $\mathbb{C}$-scheme. (See e.g., [16,§I.2.4 and Corollary I-14] for a discussion of the gluing construction.) Indeed, given any $\sigma_1, \dots, \sigma_k \in S_N$, the intersection of $U_{\prec_{\sigma_1}}, \dots, U_{\prec_{\sigma_k}}$ in $RavConf_N$ is then a copy of the affine scheme $U_{\prec_{\sigma_1} \wedge \cdots \wedge \prec_{\sigma_k}}$:

$$U_{\prec_{\sigma_1}} \cap_{RavConf_N} \cdots \cap_{RavConf_N} U_{\prec_{\sigma_k}} := U_{\prec_{\sigma_1} \wedge \cdots \wedge \prec_{\sigma_k}}.$$

Thus, we pick and fix arbitrarily any total order $<$ on S_N, and define the semisimplicial object in affine $\mathbb{C}$-schemes given by the diagram

$$\check{C}(\mathscr{U}) = \left(\cdots \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} \bigsqcup_{\substack{\sigma_1, \sigma_2 \in S_N \\ \sigma_1 < \sigma_2}} U_{\prec_{\sigma_1} \wedge \prec_{\sigma_2}} \mathrel{\substack{\longrightarrow \\ \longrightarrow}} \bigsqcup_{\sigma \in S_N} U_{\prec_\sigma} \right);$$

this diagram has a colimit in the category of $\mathbb{C}$-schemes, and we define $RavConf_N$ to be that colimit:

$$RavConf_N := \operatorname{colim}_{\mathbf{Sch}_{\mathbb{C}}} \check{C}(\mathscr{U}).$$

Informally then, $RavConf_2$ is a copy of $\mathbb{A}_{\mathbb{C}}^2$ but with a doubled diagonal; $RavConf_3$ is a copy of $\mathbb{A}_{\mathbb{C}}^3$ but with all pairwise diagonals doubled and the main diagonal $z_1 = z_2 = z_3$ having multiplicity $3! = 6$; and so on (We emphasize that, like the formal raviolo Rav itself, this configuration space $RavConf_N$ is not separated as a scheme for any $N \geq 2$; the underlying topological space of the corresponding analytic space is not Hausdorff.).

4.2 Derived sections and the Thom–Sullivan functor

Our goal is now to give a model in dg commutative algebras of the derived global sections $R\Gamma(\mathcal{O}, RavConf_N)$ of the structure sheaf on $RavConf_N$.

Given any finite Leray cover $\mathcal{U} = \{U_i\}_{i \in I}$ of a $\mathbb{C}$-scheme X, the derived global sections $R\Gamma(X, \mathcal{O})$ of its structure sheaf $\mathcal{O}$ is the dg commutative algebra defined, up to zig-zags of quasi-isomorphisms, as the homotopy limit in dg commutative algebras

$$R\Gamma(X, \mathcal{O}) = \operatorname{holim} \Gamma(\check{C}(\mathcal{U}), \mathcal{O}).$$

The *Thom–Sullivan functor* $\mathrm{Th}^\bullet$ provides one way of computing any such homotopy limit, that is, the homotopy limit of any diagram given by a semicosimplicial object in dg commutative algebras. We recall the definition of this functor in Appendix A and refer the reader to [30], [32, Appendix A], or for example [2] for further discussion.

The model of the homotopy limit that the Thom–Sullivan functor produces can be understood as consisting of polynomial differential forms on a single $(|I| - 1)$-simplex, valued in $\mathcal{O}(\bigcap_{i \in I} U_i)$, together with polynomial differential forms on every face of that simplex. Each face is labelled by some subset $S \subset I$, and the form on that face is valued in $\mathcal{O}(\bigcap_{i \in S} U_i)$. (These intersections are taken in X.) These forms are required to satisfy the natural compatibility conditions under pullbacks.

Whenever, as is true in our case, the maps $\mathcal{O}(\bigcap_{i \in S} U_i) \to \mathcal{O}(\bigcap_{i \in I} U_i)$, $S \subset I$, are all embeddings of commutative algebras, then these compatibility conditions mean that the forms on the faces of the $(|I| - 1)$-simplex are actually determined by the form in the bulk of the simplex itself. The model of the homotopy limit is then a dg commutative algebra of polynomial differential forms on the $(|I| - 1)$-simplex, valued in $\mathcal{O}(\bigcap_{i \in I} U_i)$, subject to boundary conditions:

$$\mathrm{Th}^\bullet(\Gamma(\check{C}(\mathcal{U}), \mathcal{O})) = \Big\{ \omega \in \mathcal{O}(\bigcap_{i \in I} U_i) \otimes \mathbb{C}[u_i, \mathrm{d}u_i]_{i \in I} / \langle \sum_{i \in I} u_i - 1, \sum_{i \in I} \mathrm{d}u_i \rangle$$

$$: \omega|_{\{u_i = 0 \, \forall i \in I \setminus S\}} \in \mathcal{O}(\bigcap_{i \in S} U_i) \otimes \mathbb{C}[u_i, \mathrm{d}u_i]_{i \in S} / \langle \sum_{i \in S} u_i - 1, \sum_{i \in S} \mathrm{d}u_i \rangle$$

$$\text{for all nonempty subsets } S \subset I \Big\}.$$

(Here $\omega|_{\{u_i = 0 \, \forall i \in I \setminus S\}}$ denotes the pullback.)

Thus, in our case, we arrive at the following. Define the dg commutative algebra $\mathcal{A}_N$ by

$$\mathcal{A}_N := \Big\{ \omega \in \mathcal{B}_N \otimes \mathbb{C}[u_\sigma, \mathrm{d}u_\sigma]_{\sigma \in S_N} / \langle \sum_{\sigma \in S_N} u_\sigma - 1, \sum_{\sigma \in S_N} \mathrm{d}u_\sigma \rangle$$

$$: \omega|_{\{u_\sigma = 0 \, \forall \sigma \in S_N \setminus S\}} \in \mathcal{O}(U_{\bigwedge_{\sigma \in S} \prec_\sigma}) \otimes \mathbb{C}[u_i, \mathrm{d}u_i]_{i \in S} / \langle \sum_{i \in S} u_i - 1, \sum_{i \in S} \mathrm{d}u_i \rangle$$

$$\text{for all nonempty subsets } S \subset S_N \Big\}.$$

Theorem 6. This $\mathcal{A}_N$ is a model, in dg commutative algebras, of the derived global sections of the structure sheaf on the ravioli configuration space $\mathrm{RavConf}_N$:

$$\mathcal{A}_N \simeq R\Gamma(\mathrm{RavConf}_N, \mathcal{O}).$$

Because this algebra $\mathcal{A}_N$ will play a central role for us, it is worth noting the following equivalent description. For any distinct $i, j \in [1, N]$, let $S_N^{ij} \subset S_N$ denote the set of total orders on $[1, N]$ in which i precedes j:

$$S_N^{ij} := \{\sigma \in S_N : i \prec_\sigma j\}. \tag{19}$$

Lemma 7. The definition of $\mathcal{A}_N$ above is equivalent to

$$\mathcal{A}_N = \Big\{ \omega \in \mathcal{B}_N \otimes \mathbb{C}[u_\sigma, \mathrm{d}u_\sigma]_{\sigma \in S_N} / \langle \sum_{\sigma \in S_N} u_\sigma - 1, \sum_{\sigma \in S_N} \mathrm{d}u_\sigma \rangle$$

$$: \text{for all distinct } i, j \in [1, N] \text{ the pullback } \omega|_{\{u_\sigma = 0 \, \forall \sigma \in S_N^{ij}\}}$$

$$\text{is regular in } z_i - z_j \Big\}.$$

Proof. The idea is that by imposing the boundary conditions at these particular faces of the simplex, we are in fact imposing all the boundary conditions in the definition of $\mathcal{A}_N$, because all other faces on which the boundary conditions are non-empty arise as intersections of these faces.

To see this in detail, it is enough to check that, for every nonempty subset $S \subset S_N$ of the set of total orders on $[1, N]$, if ω obeys the boundary conditions given in the lemma, then it obeys the boundary condition given in the definition of $\mathcal{A}_N$ at the face $\{u_\sigma = 0 \forall \sigma \in S_N \setminus S\}$ corresponding to S. To that end, pick any such S and let $\prec_S$ denote the finest common coarsening,

$$\prec_S := \bigwedge_{\sigma \in S} \prec_\sigma,$$

or in other words, the partial order in which $i \prec_S j$ if and only if $i \prec_\sigma j$ for all $\sigma \in S$. The set S is then the intersection (in S_N) of the subsets S_N^{ij} as i, j range over all pairs of distinct elements of $[1, N]$ such that $i \prec_S j$:

$$S = \bigcap_{i \prec_S j} S_N^{ij}.$$

It follows that our hyperplane $\{u_\sigma = 0 \forall \sigma \in S_N \setminus S\}$ is the intersection (in $\mathbb{A}_{\mathbb{C}}^{N!}$) of the hyperplanes $\{u_\sigma = 0 \forall \sigma \in S_N \setminus S_N^{ij}\}$ as i, j range over all pairs of distinct elements of $[1, N]$ that are comparable with respect to $\prec_S$:

$$\{u_\sigma = 0 \forall \sigma \in S_N \setminus S\} = \bigcap_{i \prec_S j} \{u_\sigma = 0 \forall \sigma \in S_N \setminus S_N^{ij}\}.$$

Now, of course, $S_N \setminus S_N^{ij} = S_N^{ji}$, and $z_i - z_j = -(z_j - z_i)$. So we see that by imposing the boundary conditions in the statement of the lemma, we are thereby imposing the condition that the pullback of ω to the face $\{u_\sigma = 0 \forall \sigma \in S_N \setminus S\}$ is regular in $z_i - z_j$ for all $i \prec_S j$. This is the boundary condition on that face in the definition of $\mathcal{A}_N$, as required. ∎

5 Rational Ravioli Coinvariants

Having defined the ravioli configuration space $\mathrm{RavConf}_N$ and a model $\mathcal{A}_N$ of the derived global sections of its structure sheaf, we now define the ravioli analogues of rational coinvariants/conformal blocks from Section 3. As far as possible, we shall follow the same construction of spaces of coinvariants with movable marked points we reviewed starting in §3.2, with $\mathrm{RavConf}_N$ playing the role of Conf_N and $\mathcal{A}_N$ playing the role of $\mathcal{B}_N$.

In what follows, the dg commutative algebra

$$\mathcal{A}_N \simeq R\Gamma(\mathrm{RavConf}_N, \mathcal{O})$$

will play the role of the base ring (i.e., we shall work in dg $\mathcal{A}_N$-modules) in the same way that the commutative algebra

$$\mathcal{B}_N = \Gamma(\mathrm{Conf}_N, \mathcal{O})$$

was the base ring (i.e., we worked in $\mathcal{B}_N$-modules) in the usual setting in §3.2 of rational coinvariants with N movable marked points.

Now, the commutative algebra $\mathcal{B}_{N+1} \cong \mathcal{B}_N[w, (w - z_i)^{-1}]_{i=1}^N$ was naturally a module over $\mathcal{B}_N$. We want something similar in the ravioli setting. Namely we would like to show that $\mathcal{A}_{N+1}$ is a commutative algebra in dg $\mathcal{A}_N$-modules in some natural way.

To that end, let

$$\pi_N : S_{N+1} \to S_N; \quad \sigma \mapsto (\sigma(1), \sigma(2), \ldots, \widehat{N+1}, \ldots, \sigma(N+1)) \tag{20}$$

denote the surjective map of sets that sends total orders on $[1, N+1]$ to total orders on $[1, N]$ by forgetting about the position of $N + 1$. We define a map of dg commutative algebras

$$\iota_N : \mathbb{C}[u_\sigma, \mathrm{d}u_\sigma]_{\sigma \in S_N} \to \mathbb{C}[u_\sigma, \mathrm{d}u_\sigma]_{\sigma \in S_{N+1}}$$

by setting

$$\iota_N(u_\sigma) := \sum_{\tau \in \pi_N^{-1}(\sigma)} u_\tau. \tag{21}$$

For example, when $N = 2$, we have

$$\iota_2(u_{(12)}) = u_{(123)} + u_{(132)} + u_{(312)}, \qquad \iota_2(u_{(21)}) = u_{(213)} + u_{(231)} + u_{(321)}.$$

Lemma 8. There is an injective map of dg commutative algebras

$$\iota_N : \mathcal{A}_N \hookrightarrow \mathcal{A}_{N+1},$$

(overloading notation somewhat) given by the tensor product of ι_N above with the obvious embedding $\mathcal{B}_N \hookrightarrow \mathcal{B}_{N+1}$.

Proof. First observe that ι_N maps the dg ideal $\langle \sum_{\sigma \in S_N} u_\sigma - 1, \sum_{\sigma \in S_N} \mathrm{d}u_\sigma \rangle$ to the dg ideal $\langle \sum_{\tau \in S_{N+1}} u_\tau - 1, \sum_{\tau \in S_{N+1}} \mathrm{d}u_\tau \rangle$, so it defines a map between the polynomial differential forms on the $(N! - 1)$-simplex and those on the $((N + 1)! - 1)$-simplex. Now we need to check that this map respects the defining boundary conditions of $\mathcal{A}_{N+1}$. It is enough to consider elements $\omega \in \mathcal{A}_N$ of the form

$$\omega = p \otimes \lambda, \quad p \in \mathcal{B}_N, \quad \lambda \in \mathbb{C}[u_\sigma, \mathrm{d}u_\sigma]_{\sigma \in S_N}.$$

Obviously, p is not singular in $z_i - z_{N+1}$ for any i. Pick any distinct $i, j \in [1, N]$ and suppose that p is singular in $z_i - z_j$. We must check that $\iota_N(\lambda)$ vanishes on pullback to the face of the $(N+1)! - 1$-simplex given by $\{u_\tau = 0 \forall \tau \in S_{N+1}^{ij}\}$. On that zero locus, we have that $\iota_N(u_\sigma) = 0$ for every $\sigma \in S_N^{ij}$. That is, the operation of pulling back to the zero locus of all the u_τ with $\tau \in S_{N+1}^{ij}$ factors through the operation of pulling back first to the zero locus of the images $\iota_N(u_\sigma)$ with $\sigma \in S_N^{ij}$. That latter pullback commutes with the dg algebra map ι_N, that is, $\iota_N(\lambda)|_{\{\iota_N(u_\sigma) = 0 \forall \sigma \in S_N^{ij}\}} = \iota_N(\lambda|_{\{u_\sigma = 0 \forall \sigma \in S_N^{ij}\}})$. And finally, the pullback $\lambda|_{\{u_\sigma = 0 \forall \sigma \in S_N^{ij}\}}$ vanishes, since by assumption ω obeys the defining boundary conditions of $\mathcal{A}_N$. ■

For example, when $N = 2$, we have the following well-defined element of $\mathcal{A}_2$:

$$\frac{u_{(12)} u_{(21)}}{z_1 - z_2} \in \mathcal{A}_2,$$

and its image is a well-defined element of $\mathcal{A}_3$:

$$\frac{(u_{(123)} + u_{(132)} + u_{(312)})(u_{(213)} + u_{(231)} + u_{(321)})}{z_1 - z_2} \in \mathcal{A}_3.$$

In this way, $\mathcal{A}_{N+1}$ has the structure of a commutative algebra not just in dg vector spaces but in dg $\mathcal{A}_N$-modules.

In passing, let us note that the map $\iota_N : \mathcal{A}_N \hookrightarrow \mathcal{A}_{N+1}$ of Lemma 8 has a natural family of generalizations. Given any subset $J \subset [1, N]$ we have a corresponding surjection

$$\pi_{J \subset [1,N]} : S_N \twoheadrightarrow S_{|J|}$$

between the sets of total orders (given by forgetting about the positions of elements of $[1, N] \setminus J$) and hence a map of dg commutative algebras

$$\iota_{J \subset [1,N]} : \mathbb{C}[u_\sigma, \mathrm{d}u_\sigma]_{\sigma \in S_{|J|}} \to \mathbb{C}[u_\sigma, \mathrm{d}u_\sigma]_{\sigma \in S_N}. \tag{22}$$

On tensoring this with the obvious embedding of commutative algebras

$$\mathcal{B}_{|J|} \xrightarrow{\cong} \mathbb{C}[z_i, (z_i - z_j)^{-1}]_{i,j \in J, i \neq j} \hookrightarrow \mathcal{B}_N$$

we obtain an embedding of dg commutative algebras

$$\iota_{J \subset [1,N]} : \mathcal{A}_{|J|} \hookrightarrow \mathcal{A}_N. \tag{23}$$

For example we have the three maps

$$\iota_{\{1,2\} \subset [1,3]}, \iota_{\{1,3\} \subset [1,3]}, \iota_{\{2,3\} \subset [1,3]} : \mathcal{A}_2 \hookrightarrow \mathcal{A}_3.$$

Example 9. Such maps are a rich source of interesting elements of $\mathcal{A}_N$. Let $i, j \in [1, N]$ be distinct. Then the elements

$$\frac{\mathrm{d}\iota_{\{i,j\}}(u_{(12)})}{z_i - z_j}$$

belong to $\mathcal{A}_N$. Compare Example 12 below.

5.1 Expansion maps

Recall that we write

$$\iota_{w \to z_s} : \mathcal{B}_{N+1} \to \mathcal{B}_N((w - z_s))$$

for the map of commutative algebras in $\mathcal{B}_N$-modules given by taking the Laurent expansion in small $w - z_s = z_{N+1} - z_s$ with $z_1, \ldots, z_N$ held fixed. (To keep track of the distinguished role of the last coordinate, we shall continue to write $w := z_{N+1}$.) One thinks of $\mathcal{B}_N((w - z_s))$ as a certain completion of the tensor product

$$\mathcal{B}_N \otimes_{\mathbb{C}} \mathbb{C}((w - z_s)) \cong \Gamma(\mathrm{Conf}_N, \mathcal{O}) \otimes_{\mathbb{C}} \Gamma(D^\times, \mathcal{O}).$$

In a similar spirit, let us now introduce a commutative algebra in dg $\mathcal{A}_N$-modules $\mathcal{A}_N\{\!\{w - z_s\}\!\}$, which we think of as a certain completion of the tensor product

$$\mathcal{A}_N \otimes_{\mathbb{C}} \mathbb{C}\{\!\{w - z_s\}\!\} \simeq R\Gamma(\mathrm{RavConf}_N, \mathcal{O}) \otimes_{\mathbb{C}} R\Gamma(\mathrm{Rav}, \mathcal{O}).$$

Namely, we define

$$\mathcal{A}_N\{\!\{w - z_s\}\!\} :=$$

$$\Big\{\omega \in \mathcal{B}_N((w - z_s)) \otimes \mathbb{C}[v, dv] \otimes \mathbb{C}[u_\sigma, du_\sigma]_{\sigma \in S_N} / \langle \sum_{\sigma \in S_N} u_\sigma - 1, \sum_{\sigma \in S_N} du_\sigma \rangle$$

$$: \text{for all distinct } i, j \in [1, N] \text{ the pullback } \omega|_{\{u_\sigma = 0 \forall \sigma \in S_N^{ij}\}}$$

$$\text{is regular in } z_i - z_j,$$

$$\text{and both } \omega|_{v=0} \text{ and } \omega|_{v=1} \text{ are regular in } w - z_s \Big\}.$$

We think of $\mathcal{A}_N\{\!\{w - z_s\}\!\}$ as the dg commutative algebra of functions on the formal raviolo near the sth of N distinguishable movable marked points, in the same way that $\mathcal{B}_N((w - z_s))$ is the commutative algebra of functions on the formal punctured disc near the sth of N distinguishable movable marked points, cf. [18,§13.2].

Now we shall define a map of commutative algebras in dg $\mathcal{A}_N$-modules

$$\mathcal{A}_{N+1} \to \mathcal{A}_N\{\!\{w - z_s\}\!\}$$

given by Laurent expanding in $w - z_s$ and simultaneously pulling back to a certain judiciously chosen curved copy of $\Delta^1_{\mathbb{C}} \times \Delta^{N!-1}_{\mathbb{C}}$ inside the algebro-geometric $((N + 1)! - 1)$-simplex

$$\Delta^{(N+1)!-1}_{\mathbb{C}} := \text{Spec} \left(\mathbb{C}[u_\sigma]_{\sigma \in S_{N+1}} / \langle \sum_{\sigma \in S_{N+1}} u_\sigma - 1 \rangle \right) \hookrightarrow \mathbb{A}^{(N+1)!}_{\mathbb{C}}.$$

Namely, we first define a map of affine schemes

$$\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^{N!}_{\mathbb{C}} = \text{Spec} \left(\mathbb{C}[v] \otimes \mathbb{C}[u_\sigma]_{\sigma \in S_N} \right) \to \mathbb{A}^{(N+1)!}_{\mathbb{C}} = \text{Spec}\, \mathbb{C}[u_\sigma]_{\sigma \in S_{N+1}}$$

or equivalently a map of commutative algebras

$$\mathbb{C}[u_\sigma]_{\sigma \in S_{N+1}} \to \mathbb{C}[v] \otimes \mathbb{C}[u_\sigma]_{\sigma \in S_N}$$

by sending

$$u_\sigma \mapsto \begin{cases} (1 - v)u_{(\dots, s, \dots)} & \text{if } \sigma = (\dots, N + 1, s, \dots) \\ v u_{(\dots, s, \dots)} & \text{if } \sigma = (\dots, s, N + 1, \dots) \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Since

$$\sum_{\sigma \in S_{N+1}} u_\sigma \mapsto \sum_{\sigma \in S_N} (v u_\sigma + (1 - v) u_\sigma) = \sum_{\sigma \in S_N} u_\sigma,$$

this map induces a map of affine schemes

$$p_{N+1 \to s} : \Delta^1_{\mathbb{C}} \times \Delta^{N!-1}_{\mathbb{C}} \to \Delta^{(N+1)!-1}_{\mathbb{C}}.$$

One should think of this as embedding a curved copy of $\Delta^1_{\mathbb{C}} \times \Delta^{N!-1}_{\mathbb{C}}$ into $\Delta^{(N+1)!-1}_{\mathbb{C}}$. We get also the map $p^*_{N+1 \to s}$ of dg commutative algebras from the polynomial differential forms on $\Delta^{(N+1)!-1}_{\mathbb{C}}$ to the polynomial

differential forms on $\Delta_{\mathbb{C}}^1 \times \Delta_{\mathbb{C}}^{N!-1}$,

$$p_{N+1 \to s}^* : \mathbb{C}[u_\sigma, du_\sigma]_{\sigma \in S_{N+1}} / \langle \sum_{\sigma \in S_{N+1}} u_\sigma - 1, \sum_{\sigma \in S_{N+1}} du_\sigma \rangle$$

$$\to \mathbb{C}[v, dv] \otimes \mathbb{C}[u_\sigma, du_\sigma]_{\sigma \in S_N} / \langle \sum_{\sigma \in S_N} u_\sigma - 1, \sum_{\sigma \in S_N} du_\sigma \rangle,$$

given again by (25).

We can now define an analogue of the Laurent-expansion map $\iota_{w \to z_s}$ for $\mathcal{A}_N$. (Here we shall overload the notation $\iota_{w \to z_s}$ somewhat.)

Definition-Proposition 10. There is a map of commutative algebras in dg $\mathcal{A}_N$-modules

$$\iota_{w \to z_s} : \mathcal{A}_{N+1} \to \mathcal{A}_N\{\!\{w - z_s\}\!\}$$

given by $\iota_{w \to z_s} \otimes p_{N+1 \to s}^*$. That is, we take the formal Laurent expansion in small $w - z_s$ with $z_1, \ldots, z_N$ held fixed, and we pull back to the copy of $\Delta_{\mathbb{C}}^1 \times \Delta_{\mathbb{C}}^{N!-1}$ in $\Delta_{\mathbb{C}}^{(N+1)!-1}$ defined by the map $p_{N+1 \to s}$ above.

Proof. We certainly have a map of commutative algebras in dg vector spaces

$$\mathcal{A}_{N+1} \to \mathcal{B}_N((w - z_s)) \otimes \mathbb{C}[v, dv] \otimes \mathbb{C}[u_\sigma, du_\sigma]_{\sigma \in S_N} / \langle \sum_{\sigma \in S_N} u_\sigma - 1, \sum_{\sigma \in S_N} du_\sigma \rangle.$$

What has to be checked is first that this map respects the defining boundary conditions of $\mathcal{A}_N\{\!\{w - z_s\}\!\}$ and second that it is a map of $\mathcal{A}_N$-modules.

Let $\omega \in \mathcal{A}_{N+1}$ be any element. We must show that $\iota_{w \to z_s}(\omega)$ obeys the defining boundary conditions of $\mathcal{A}_N\{\!\{w - z_s\}\!\}$.

Consider first $(\iota_{w \to z_s}(\omega))|_{v=0}$. On the preimage of the zero locus of v under the map $p_{N+1 \to s}^*$, we have that u_τ vanishes for all $\tau \in S_{N+1}$ except for those of the form $(\ldots, N+1, s, \ldots)$. Thus, in particular, on that preimage we have that $u_\tau = 0$ for all $\tau \in S_{N+1}^{s,N+1}$. Therefore the pullback of ω to that preimage is regular in $w - z_s$ by virtue of the defining boundary conditions of $\mathcal{A}_{N+1}$, and hence $(\iota_{w \to z_s}(\omega))|_{v=0}$ is regular in $w - z_s$, as required.

The argument for $(\iota_{w \to z_s}(\omega))|_{v=1}$ is similar.

Next consider $(\iota_{w \to z_s}(\omega))|_{u_\sigma = 0 \forall \sigma \in S_N^{ij}}$ for distinct $i, j \in [1, N] \setminus \{s\}$. We must show that this is regular in $z_i - z_j$. When we set to zero u_σ for all $\sigma \in S_N^{ij}$, we are thereby setting to zero the images $p_{N+1 \to s}^*(u_\tau)$ of u_τ for all $\tau \in S_{N+1}^{ij}$. Therefore the pullback $\omega|_{(p_{N+1 \to s}^*)^{-1}(\{u_\sigma = 0 \forall \sigma \in S_N^{ij}\})}$ of ω to this preimage is regular in $z_i - z_j$, by virtue of the defining boundary conditions of $\mathcal{A}_{N+1}$. Hence $(\iota_{w \to z_s}(\omega))|_{u_\sigma = 0 \forall \sigma \in S_N^{ij}}$ is regular in $z_i - z_j$, again as required.

Finally we consider $(\iota_{w \to z_s}(\omega))|_{u_\sigma = 0 \forall \sigma \in S_N^{is}}$ for $i \in [1, N] \setminus \{s\}$. When we set to zero u_σ for all $\sigma \in S_N^{is}$, we are thereby setting to zero the images $p_{N+1 \to s}^*(u_\tau)$ of u_τ *both* for all $\tau \in S_{N+1}^{is}$ *and* crucially also for all $\tau \in S_{N+1}^{i,N+1}$. Therefore the defining boundary conditions of $\mathcal{A}_{N+1}$ guarantee that the pullback $\omega|_{(p_{N+1 \to s}^*)^{-1}(\{u_\sigma = 0 \forall \sigma \in S_N^{is}\})}$ of ω to this preimage is regular in both $z_i - z_s$ and crucially also in $z_i - w$. Hence $(\iota_{w \to z_s}(\omega))|_{u_\sigma = 0 \forall \sigma \in S_N^{is}}$ is regular in $z_i - z_s$, again as required. (One should keep in mind that the process of taking Laurent expansions introduces additional singularities. For example

$$\iota_{w \to z_s} \frac{1}{w - z_i} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(z_s - z_i)^{k+1}} (w - z_s)^k.$$

These are dealt with by the "and crucially" part of the argument above.)

The argument for $(\iota_{w \to z_s}(\omega))|_{u_\sigma = 0 \forall \sigma \in S_N^{si}}$ is similar.

It remains to check that the map is a map of dg $\mathcal{A}_N$-modules, cf. Lemma 8. But this follows from the observation that, for every $\sigma \in S_N$, we have

$$p^*_{N+1 \to s}(\iota_N(u_\sigma)) = \sum_{\tau \in \pi_N^{-1}(\sigma)} p^*_{N+1 \to s}(u_\tau) = v u_\sigma + (1 - v) u_\sigma = u_\sigma.$$

∎

5.2 Cospan of dg Lie algebras

Let us define $\mathcal{A}'_{N+1}$ to be the nonunital dg subalgebra of $\mathcal{A}_{N+1}$ consisting of those elements vanishing as $w \to \infty$. At this stage we have maps of commutative algebras in dg $\mathcal{A}_N$-modules

$$\mathcal{A}'_{N+1} \to \bigoplus_{k=1}^N \mathcal{A}_N\{\{w - z_k\}\} \leftarrow \bigoplus_{k=1}^N \mathcal{A}_N\{\{w - z_k\}\}_+.$$

(Here $\mathcal{A}_N\{\{w - z_k\}\}_+ := \mathcal{A}_N[[w - z_k]] \otimes \mathbb{C}[v, \mathrm{d}v]$, as in §2.3.)

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, as earlier. On tensoring with $\mathfrak{g}$ we get maps of Lie algebras in dg $\mathcal{A}_N$-modules. Namely, let us define

$$\mathfrak{a} := \mathfrak{g} \otimes \bigoplus_{k=1}^N \mathcal{A}_N\{\{w - z_k\}\}, \qquad \mathfrak{a}_+ := \mathfrak{g} \otimes \bigoplus_{k=1}^N \mathcal{A}_N\{\{w - z_k\}\}_+, \qquad \mathfrak{a}_- := \mathfrak{g} \otimes \mathcal{A}'_{N+1}.$$

Then we have the cospan of Lie algebras in dg $\mathcal{A}_N$-modules

$$\mathfrak{a}_- \xrightarrow{I_{\text{Global}}} \mathfrak{a} \xleftarrow{I_{\text{Ravioli}}} \mathfrak{a}_+.$$

This is analogous to the cospan of Lie algebras in $\mathcal{B}_N$-modules we had in Section 3 above. (In what follows we shall often omit the map I_{Ravioli} and simply identify elements of $\mathfrak{a}_+$ with their embedded images in $\mathfrak{a}$.)

Moreover, we still have the following.

Proposition 11. As a dg $\mathcal{A}_N$-module, $\mathfrak{a}$ is the sum (although not, as we shall see, the direct sum) of the images of $\mathfrak{a}_+$ and $\mathfrak{a}_-$:

$$\mathfrak{a} = \mathfrak{a}_+ + I_{\text{Global}}(\mathfrak{a}_-).$$

Proof. We must show that every element of $\mathfrak{a}$ can be written as a sum of an element of $\mathfrak{a}_+$ and an element of $I_{\text{Global}}(\mathfrak{a}_-)$. Let us define

$$\mathfrak{a}^\pm := \mathfrak{g} \otimes \bigoplus_{i=1}^N \mathcal{A}_N\{\{w - z_i\}\}_\pm$$

where on the right $\pm$ means restricting to non-negative (respectively, strictly negative) powers of $w - z_i$. Thus, $\mathfrak{a}^+ \equiv \mathfrak{a}_+$, but of course $\mathfrak{a}^- \neq I_{\text{Global}}(\mathfrak{a}_-)$. At the level of dg $\mathcal{A}_N$-modules we evidently have $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$. Given any element $X \in \mathfrak{a}$, let $X = X^+ + X^-$ be its corresponding decomposition. It is enough to show that X^- is in the image of $I_{\text{Global}}(\mathfrak{a}_-)$ modulo terms in $\mathfrak{a}^+$. In other words, it is enough to construct a map of dg $\mathcal{A}_N$-modules

$$g : \mathfrak{a}^- \to \mathfrak{a}_-,$$

the "building global objects" map, with the property that $(I_{\text{Global}}(g(X^-)))^- = X^-$ for all $X \in \mathfrak{a}$. To that end, we first note that the map

$$p_{N+1 \to s} : \Delta_{\mathbb{C}}^1 \times \Delta_{\mathbb{C}}^{N!-1} \to \Delta_{\mathbb{C}}^{(N+1)!-1}$$

we defined above has a left inverse

$$q_s : \Delta_{\mathbb{C}}^{(N+1)!-1} \to \Delta_{\mathbb{C}}^1 \times \Delta_{\mathbb{C}}^{N!-1} \tag{225}$$

given by (here, recall (22))

$$q_s^* := \iota_{\{s,N+1\} \subset [1,N+1]} \otimes \iota_{[1,N] \subset [1,N+1]}.$$

That is, explicitly, we have

$$q_s^*(v) = \sum_{\tau \in S_{N+1}^{s,N+1}} u_\tau, \qquad q_s^*(u_\sigma) = \iota_N(u_\sigma), \ \sigma \in S_N,$$

with ι_N as in (21) and $S_{N+1}^{s,N+1}$ as in (19). To see that q_s is left inverse to $p_{N+1 \to s}$ we note first that $p_{N+1 \to s}^* q_s^*(u_\sigma) = p_{N+1 \to s}^*(\iota_N(u_\sigma)) = u_\sigma$ for every $\sigma \in S_N$, as we checked in the proof of Definition-Proposition 10 above; and second that

$$p_{N+1 \to s}^* q_s^*(v) = p_{N+1 \to s}^* \left(\sum_{\tau \in S_{N+1}^{s,N+1}} u_\tau \right) = v \sum_{\sigma \in S_N} u_\sigma = v.$$

where in the last equality we have used the defining relation $\sum_{\sigma \in S_N} u_\sigma = 1$ of the algebro-geometric simplex $\Delta_{\mathbb{C}}^{N!-1}$.

Now, suppose we are given an element $\omega \in \mathcal{A}_N\{\!\{w - z_k\}\!\}_-$. Such an ω is in particular a polynomial differential form valued in $(w - z_k)^{-1} \mathcal{B}_N[(w - z_k)^{-1}]$. We apply to it the map of commutative algebras in $\mathcal{B}_N$-modules

$$(w - z_k)^{-1} \mathcal{B}_N[(w - z_k)^{-1}] \hookrightarrow \mathcal{B}'_{N+1}.$$

The result is an element of the dg commutative algebra

$$\mathcal{B}'_{N+1} \otimes \mathbb{C}[v, dv] \otimes \mathbb{C}[u_\sigma, du_\sigma]_{\sigma \in S_N} / \langle \sum_{\sigma \in S_N} u_\sigma - 1, \sum_{\sigma \in S_N} du_\sigma \rangle$$

obeying certain boundary conditions. We may map it to the dg commutative algebra

$$\mathcal{B}'_{N+1} \otimes \mathbb{C}[u_\sigma, du_\sigma]_{\sigma \in S_{N+1}} / \langle \sum_{\sigma \in S_{N+1}} u_\sigma - 1, \sum_{\sigma \in S_{N+1}} du_\sigma \rangle$$

via the map $q_{s,N}^*$. Let us check that the resulting form, call it $\tilde{\omega}$, obeys the defining boundary conditions of $\mathcal{A}_{N+1}$. First, consider singularities in $z_i - z_j$ for any $i, j \in [1, N]$ with $i \neq j$. We must consider the pullback of $\tilde{\omega}$ to $\{u_\sigma = 0 \forall \sigma \in S_{N+1}^{ij}\}$. On the latter zero locus, we have $q_{s,N}^*(u_\tau) = 0$ for every $\tau \in S_N^{ij}$. Therefore this pullback of $\tilde{\omega}$ is regular in $z_i - z_j$, since ω obeyed the boundary conditions of $\mathcal{A}_N$. Next, consider singularities in $w - z_s$. We are to consider the pullback of $\tilde{\omega}$ to $\{u_\sigma = 0 \forall \sigma \in S_{N+1}^{s,N+1}\}$ (or the same with $S_{N+1}^{N+1,s}$, for which the argument is similar). On that zero locus, $q_s^*(v)$ vanishes. Therefore $\tilde{\omega}$ is regular in $w - z_s$ there, since ω obeyed the boundary conditions of $\mathcal{A}_N\{\!\{w - z_s\}\!\}$. Finally note that $\tilde{\omega}$ is obviously regular everywhere in $z_i - w$ for $i \neq s$.

In this way we obtain a map of dg $\mathcal{A}_N$-modules

$$g_k : \mathcal{A}_N\{\!\{w - z_k\}\!\}_- \to \mathcal{A}'_{N+1} \tag{26}$$

for each k. For future use, let us remark that for each individual k, the map g_k is even a map of commutative algebras in dg $\mathcal{A}_N$-modules. (It is the analogue in our raviolo context of the map $(w - z_k)^{-1} \mathcal{B}_N[(w - z_k)^{-1}] \to \mathcal{B}'_{N+1}$ above.)

Hence we obtain a map of dg $\mathcal{A}_N$-modules

$$g : \mathfrak{g} \otimes \bigoplus_{k=1}^{N} \mathcal{A}_N \{\!\{ w - z_k \}\!\}_- \to \mathfrak{g} \otimes \mathcal{A}'_{N+1}.$$

By construction, we have that $(I_{\text{Global}}(g(X)))^- = X^-$ for all $X \in \mathfrak{a}$. Thus, finally,

$$X = I_{\text{Global}}(g(X)) + (X - I_{\text{Global}}(g(X))) \in I_{\text{Global}}(\mathfrak{a}_-) + \mathfrak{a}^+ \tag{27}$$

for every $X \in \mathfrak{a}$, as required. ∎

The map I_{Global} has a nontrivial kernel (For example, consider the case $N = 2, N + 1 = 3$. The element

$$\frac{u_{(123)} u_{(312)}}{w - z_2}$$

belongs to $\mathcal{A}'_3$, and it is in the kernel of the expansion map $\mathcal{A}_3 \to \mathcal{A}_2 \{\!\{ w - z_1 \}\!\} \oplus \mathcal{A}_2 \{\!\{ w - z_2 \}\!\}$. Indeed, $p^*_{3 \to 1}(u_{(123)}) = 0$ and $p^*_{3 \to 2}(u_{(312)}) = 0$.

(Observe, in passing, that for example neither $\frac{u_{(123)}}{w - z_2}$ nor $\frac{d u_{(123)}}{w - z_2}$ belong to $\mathcal{A}_3$: they are not regular in $w - z_2$ on pullback to $u_{(321)} = u_{(312)} = u_{(132)} = 0$.)). Of course, one could always simply *define* the subalgebra $\mathcal{A}'_{N+1}$ to be the quotient of $\mathcal{A}_{N+1}$ by the kernel of I_{Global} (a dg ideal). However, the following example illustrates a more profound disanalogy between the usual case and the ravioli case.

Example 12. Consider the case $N = 2, N + 1 = 3$. Let us write

$$v_{12} := \iota_{\{1,2\}}(u_{(12)}) = u_{(123)} + u_{(132)} + u_{(312)},$$

$$v_{13} := \iota_{\{1,3\}}(u_{(12)}) = u_{(123)} + u_{(213)} + u_{(132)},$$

$$v_{23} := \iota_{\{2,3\}}(u_{(12)}) = u_{(123)} + u_{(213)} + u_{(231)}$$

(and $v_{32} = 1 - v_{23}$ etc.). Consider the element

$$\Omega_{12} := \frac{d v_{31}}{w - z_1} \wedge \frac{d v_{32}}{w - z_2} - \frac{d v_{21}}{z_2 - z_1} \wedge \frac{d v_{32}}{w - z_2} - \frac{d v_{31}}{w - z_1} \wedge \frac{d v_{12}}{z_1 - z_2} \in \mathcal{A}'_3.$$

It is nonzero. The singular part of its expansion in small $w - z_1$ is

$$d v_{31} \wedge (d v_{32} - d v_{12}) \frac{1}{w - z_1} \frac{1}{z_1 - z_2}.$$

Consider the pullback of this to the copy of $\Delta^1 \times \Delta^1$ in Δ^3 defined by the map $p_{3 \to 1}$ above. We have, on this copy of $\Delta^1 \times \Delta^1$,

$$p^*_{3 \to 1}(v_{12}) = 0 + (1 - v) u_{(12)} + v u_{(12)} = u_{(12)}$$

$$p^*_{3 \to 1}(v_{13}) = 0 + v u_{(21)} + v u_{(12)} = v(u_{(21)} + u_{(12)}) = v$$

$$p^*_{3 \to 1}(v_{23}) = 0 + v u_{(21)} + (1 - v) u_{(21)} = u_{(21)}.$$

Therefore this pullback vanishes:

$$- d v \wedge d(u_{(12)} - u_{(12)}) \frac{1}{w - z_1} \frac{1}{z_1 - z_2} = 0.$$

A similar story holds for the expansion in small $w - z_2$.

We conclude that the image of this element $\Omega_{12} \in \mathcal{A}'_3$ in $\mathcal{A}_2 \{\!\{ w - z_1 \}\!\} \oplus \mathcal{A}_2 \{\!\{ w - z_2 \}\!\}$, while nonzero, is nonsingular in the local variables $w - z_i$ in each summand, that is, it lies in the subalgebra $\mathcal{A}_2 \{\!\{ w - z_1 \}\!\}_+ \oplus \mathcal{A}_2 \{\!\{ w - z_2 \}\!\}_+$.

Remark 13. The element Ω_{12} is closed and we expect that it represents a nontrivial cohomology class in $\mathcal{A}_3$. However we expect that its image in $\mathcal{A}_2\{\!\{w-z_1\}\!\} \oplus \mathcal{A}_2\{\!\{w-z_2\}\!\}$ is exact. That is, we expect that $H(\Omega_{12})$ is in the kernel of the induced map of cohomologies.

Remark 14. It is interesting to note the superficial similarity of the elements of which Ω_{12} is an example, namely

$$\frac{dv_{N+1,j}}{w-z_j} \wedge \frac{dv_{jk}}{z_j-z_k} + \frac{dv_{jk}}{z_j-z_k} \wedge \frac{dv_{k,N+1}}{z_k-w} + \frac{dv_{k,N+1}}{z_k-w} \wedge \frac{dv_{N+1,j}}{w-z_j},$$

with the relations

$$\frac{dz_i-dz_j}{z_i-z_j} \wedge \frac{dz_j-dz_k}{z_j-z_k} + \frac{dz_j-dz_k}{z_j-z_k} \wedge \frac{dz_k-dz_i}{z_k-z_i} + \frac{dz_k-dz_i}{z_k-z_i} \wedge \frac{dz_i-dz_j}{z_i-z_j} = 0,$$

which hold in the de Rham complex of holomorphic forms on the usual configuration space $\mathrm{Conf}_N = \mathbb{A}_\mathbb{C}^N \setminus \{\text{diagonals}\} = \mathrm{Spec}\,\mathcal{B}_N$, and which are examples of *Orlik–Solomon* relations associated to a hyperplane arrangement. See [36] and for example [40,§2].

As Example 12 shows, the image of $\mathfrak{a}_-$ in $\mathfrak{a}$ has nontrivial intersection with $\mathfrak{a}_+$. Let us define $\mathfrak{a}_0$ to be this intersection, a Lie subalgebra in dg $\mathcal{A}_N$-modules of $\mathfrak{a}$:

$$\mathfrak{a}_0 := I_{\mathrm{Global}}(\mathfrak{a}_-) \cap \mathfrak{a}_+ \subset \mathfrak{a}.$$

Remark 15. Recall that in the usual case reviewed in Section 3, we were able to arrange that $\mathfrak{b}_- \cap \mathfrak{b}_+ = 0$ in $\mathfrak{b}$, and hence that $\mathfrak{b} = \mathfrak{b}_- \oplus \mathfrak{b}_+$ was the direct sum of vector spaces, or of $\mathcal{B}_N$-modules. We did that by defining $\mathfrak{b}_- = \mathfrak{g} \otimes \mathcal{B}'_{N+1}$, where $\mathcal{B}'_{N+1}$ was the nonunital subalgebra of $\mathcal{B}_{N+1}$ consisting of those rational expressions in w vanishing as $w \to \infty$; that ensured, in (8), that the image of $\mathcal{B}'_{N+1}$ had trivial intersection with the image of $\bigoplus_{i=1}^N \mathcal{B}_N[[w-z_i]]$. In the construction of rational conformal blocks, [19], it is common to define $\mathcal{B}'_{N+1}$ that way, for precisely this reason. Doing so yields what is called the *modified space of conformal blocks* in [18,§13]. If one weakens that restriction on $\mathfrak{b}_-$ then one also has a non-trivial intersection $\mathfrak{b}_- \cap \mathfrak{b}_+ =: \mathfrak{b}_0$ in the usual setting, and the space of coinvariants has a residual quotient,

$$\mathrm{F}(\mathfrak{g};\mathcal{B}_N;M_1,\ldots,M_N) \cong (\mathcal{B}_N \otimes M)\big/\mathfrak{b}_0.$$

For example, if one insists only that elements of $\mathfrak{b}_-$ be *regular* as $w \to \infty$, then one finds $\mathfrak{b}_0 = \mathcal{B}_N \otimes \mathfrak{g}$, the Lie algebra of zero modes. In that case, the space of rational coinvariants is isomorphic as a $\mathcal{B}_N$-module to the quotient by the diagonal action of $\mathfrak{g}$ on $M = \bigotimes_{i=1}^N M_i$:

$$\mathrm{F}(\mathfrak{g};\mathcal{B}_N;M_1,\ldots,M_N) \cong \mathcal{B}_N \otimes \big(M/\mathfrak{g}\big).$$

In our present setting, there seems to be no obvious way to avoid this non-trivial intersection $\mathfrak{a}_0$. By the PBW theorem (which holds since $\mathcal{A}_N \supset \mathbb{Q}$) we get that

$$U_{\mathcal{A}_N}(\mathfrak{a}) \cong U_{\mathcal{A}_N}(I_{\mathrm{Global}}(\mathfrak{a}_-)) \otimes_{U_{\mathcal{A}_N}(\mathfrak{a}_0)} U_{\mathcal{A}_N}(\mathfrak{a}_+) \tag{28}$$

as $(U_{\mathcal{A}_N}(I_{\mathrm{Global}}(\mathfrak{a}_-)), U_{\mathcal{A}_N}(\mathfrak{a}_+))$-bimodules in dg $\mathcal{A}_N$-modules.

That has the following somewhat awkward consequence: $U_{\mathcal{A}_N}(\mathfrak{a})$ is not free as a $(U_{\mathcal{A}_N}(\mathfrak{a}_-), U_{\mathcal{A}_N}(\mathfrak{a}_+))$-bimodule in dg $\mathcal{A}_N$-modules, and therefore we have no reason to expect $U_{\mathcal{A}_N}(\mathfrak{a})$ to be cofibrant in that category with respect to its projective model structure, cf. Remark 17 below.

5.3 Ravioli coinvariants

Let now M_i, $1 \leq i \leq N$, be $\mathfrak{g}$-modules, just as in §3.1 and §3.2.

As in §3.1 we make M_i into a module over $\mathfrak{g}\otimes\mathbb{C}[[w-z_i]]$ by declaring that $\big(\mathfrak{g}\otimes(w-z_i)\mathbb{C}[[w-z_i]]\big).M_i = 0$, that is, that strictly positive modes act as zero.

We then further make M_i into a module over $\mathfrak{g} \otimes \mathbb{C}\{\{w - z_i\}\}_+$ (cf. §2.3) by declaring that $\mathfrak{g} \otimes \mathbb{C}[[w - z_i]] \otimes I$ acts as zero, where I is the dg ideal in $\mathbb{C}[\upsilon, \mathrm{d}\upsilon]$ generated by υ. That is, the action of an element of $\mathfrak{g} \otimes \{\{w - z_i\}\}_+$ on M_i factors through taking the zero mode of the pullback of that element to $\upsilon = 0$.

The module M_i is then smooth as a $\mathfrak{g} \otimes \mathbb{C}\{\{w - z_i\}\}_+$-module. By *smooth* we mean, following [18,§5.1.5] and [25, Def. 5.1.1], that for all $m \in M_i$ the Lie ideal $\mathfrak{g} \otimes (w - z_i)^k \mathbb{C}\{\{w - z_i\}\}_+$ acts as zero on m, for all sufficiently large k. Since M_i is smooth over $\mathfrak{g} \otimes \mathbb{C}\{\{w - z_i\}\}_+$, we can extend scalars in an obvious way to obtain a module $_{\mathcal{A}}M_i := \mathcal{A}_N \otimes M_i$ over $\mathfrak{g} \otimes \mathcal{A}_N\{\{w - z_i\}\}_+$, which is a smooth module over $\mathfrak{g} \otimes \mathcal{A}_N\{\{w - z_i\}\}_+$, for each i, and

$$\mathcal{A}_N \otimes M := \mathcal{A}_N \otimes \bigotimes_{i=1}^{N} M_i$$

is a left $U_{\mathcal{A}_N}(\mathfrak{a}_+)$-module.

> **Remark 16.** Much as in Remark 2, all we really require here is that, for each $1 \le i \le N$, $_{\mathcal{A}}M_i$ be a smooth module over the Lie algebra $\mathfrak{g} \otimes \mathcal{A}_N\{\{w - z_i\}\}_+$. We specialize to the above class of such modules for concreteness. (In particular, pulling back to $\upsilon = 0$, rather than say $\upsilon = 1$ or $\upsilon = \frac{1}{2}$, was a choice; and cf. also the discussion in Remark 18.)

We have the induced module

$$\mathscr{M} := U_{\mathcal{A}_N}(\mathfrak{a}) \otimes_{U_{\mathcal{A}_N}(\mathfrak{a}_+)} (\mathcal{A}_N \otimes M),$$

a left $U_{\mathcal{A}_N}(\mathfrak{a})$-module. It is equivalently, the tensor product (over $\mathcal{A}_N$) of the induced modules

$$\mathscr{M}_i := U_{\mathcal{A}_N}(\mathfrak{g} \otimes \mathcal{A}_N\{\{w - z_i\}\}) \otimes_{U_{\mathcal{A}_N}(\mathfrak{g} \otimes \mathcal{A}_N\{\{w - z_i\}\}_+)} (\mathcal{A}_N \otimes M_i)$$

at the marked points.

We may then define the space (more precisely, the dg $\mathcal{A}_N$-module) of *ravioli coinvariants* of $\mathfrak{g}$ acting on $M_1, \ldots, M_N$ to be

$$\mathrm{F}(\mathfrak{g}; \mathcal{A}_N; M_1, \ldots, M_N) := \mathcal{A}_N \otimes_{U_{\mathcal{A}_N}(\mathfrak{a}_-)} U_{\mathcal{A}_N}(\mathfrak{a}) \otimes_{U_{\mathcal{A}_N}(\mathfrak{a}_+)} (\mathcal{A}_N \otimes M).$$

By the PBW theorem, (28), we have that

$$\mathrm{F}(\mathfrak{g}; \mathcal{A}_N; M_1, \ldots, M_N) \cong \mathcal{A}_N \otimes_{U_{\mathcal{A}_N}(\mathfrak{a}_0)} (\mathcal{A}_N \otimes M) =: (\mathcal{A}_N \otimes M)/\mathfrak{a}_0 \tag{29}$$

as left $\mathcal{A}_N$-modules.

> **Remark 17.** This definition has the merit of being relatively concrete (although we don't have an explicit description of $\mathfrak{a}_0$). In principle however, one should really consider the derived tensor product
>
> $$\mathcal{A}_N \otimes^{\mathbb{L}}_{U_{\mathcal{A}_N}(\mathfrak{a}_-)} U_{\mathcal{A}_N}(\mathfrak{a}) \otimes^{\mathbb{L}}_{U_{\mathcal{A}_N}(\mathfrak{a}_+)} M, \tag{30}$$
>
> which will be modelled by the tensor product
>
> $$\widetilde{\mathrm{F}} := \mathcal{A}_N \otimes_{U_{\mathcal{A}_N}(\mathfrak{a}_-)} QU_{\mathcal{A}_N}(\mathfrak{a}) \otimes_{U_{\mathcal{A}_N}(\mathfrak{a}_+)} M$$
>
> where $QU_{\mathcal{A}}(\mathfrak{a})$ is a cofibrant replacement of $U_{\mathcal{A}}(\mathfrak{a})$ in the category of $(U_{\mathcal{A}_N}(\mathfrak{a}_-), U_{\mathcal{A}_N}(\mathfrak{a}_+))$-bimodules in dg $\mathcal{A}_N$-modules equipped with its projective model structure [27] (That is, a $(U_{\mathcal{A}_N}(\mathfrak{a}_-), U_{\mathcal{A}_N}(\mathfrak{a}_+))$-bimodule in dg $\mathcal{A}_N$-modules is the same thing as a left dg $U_{\mathcal{A}_N}(\mathfrak{a}_-) \otimes_{\mathcal{A}_N} U_{\mathcal{A}_N}(\mathfrak{a}_+^{\mathrm{op}})$)-module, so this is a special case of the model category structure on the category of (left) dg-modules over a (not necessarily graded-commutative) dg-algebra as defined in [27].).

We expect Theorem 20 below to apply to (a suitable choice of model $\widetilde{F}$ of) this derived space of coinvariants as well, but we do not show that here.

5.4 Worked example

In this subsection we shall consider in detail a concrete example of raviolo coinvariants. We take $\mathfrak{g} = \mathfrak{sl}_2$, and $N = 2$ marked points. Let us further take

$$M_1 = \mathbb{C}^2, \qquad M_2 = \mathbb{C}$$

where $\mathfrak{sl}_2$ acts on $\mathbb{C}^2$ in the defining (i.e., fundamental) representation and acts trivially on $\mathbb{C}$. Let e, f, h denote the basis of $\mathfrak{sl}_2$ with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, and denote by $\{m, f.m\}$ a basis of $\mathbb{C}^2$, where m is the $\mathfrak{sl}_2$-highest weight vector, with $e.m = 0$, $h.m = m$.

As in the discussion above, we promote M_1 to a smooth module over $\mathfrak{sl}_2 \otimes \mathbb{C}\{\!\{w - z_1\}\!\}_+ = \mathfrak{sl}_2 \otimes \mathbb{C}[[w - z_1]] \otimes \mathbb{C}[v, dv]$ by declaring that the dg ideal generated by $(w - z_1)$ and v act trivially. We similarly promote M_2 to a smooth module over $\mathfrak{sl}_2 \otimes \mathbb{C}\{\!\{w - z_2\}\!\}_+$.

Then $\mathcal{A}_2 \otimes M_1$ and $\mathcal{A}_2 \otimes M_2$ are smooth modules over respectively $\mathfrak{sl}_2 \otimes \mathcal{A}_2\{\!\{w - z_1\}\!\}_+$ and $\mathfrak{sl}_2 \otimes \mathcal{A}_2\{\!\{w - z_2\}\!\}_+$. As above, on first inducing to obtain modules $\mathscr{M}_1$ and $\mathscr{M}_2$ and then taking coinvariants, we obtain the space of ravioli coinvariants

$$F(\mathfrak{sl}_2; \mathcal{A}_2; \mathbb{C}^2, \mathbb{C}).$$

Let compute an example of taking coinvariants in this space. Let $|0\rangle$ span $\mathbb{C}$. The induced module $\mathscr{M}_2$ is a copy of the raviolo vacuum module $\mathscr{V}$ as we defined it in §2.6. It contains for example the vector

$$\left(f \otimes \frac{dv}{w - z_2} \right) |0\rangle$$

Let us consider the class in $F(\mathfrak{sl}_2; \mathcal{A}_2; \mathbb{C}^2, \mathbb{C})$ of

$$\left[m \otimes \left(f \otimes \frac{dv}{w - z_2} \right) |0\rangle \right].$$

Now, by definition of coinvariants,

$$0 = \left[\left(f \otimes \frac{dv_{23}}{w - z_2} \right) . (m \otimes |0\rangle) \right]$$

where we write, as in Example 12,

$$v_{23} = \iota_{\{2,3\}}(u_{(12)}) = u_{(123)} + u_{(213)} + u_{(231)}.$$

We compute

$$p^*_{3 \to 1}(v_{23}) = 1 - u_{(12)}, \qquad p^*_{3 \to 2}(v_{23}) = v$$

and thereby the expansions of $\left(f \otimes \frac{dv_{23}}{w - z_2} \right)$ at both marked points, 1 and 2. (For more detail on this step, see §6.1 below.) We conclude that

$$\left[m \otimes \left(f \otimes \frac{dv}{w - z_2} \right) |0\rangle \right] = \frac{du_{(12)}}{z_1 - z_2} [(f.m) \otimes |0\rangle]$$

where we pull out the scalar prefactor $\frac{du_{(12)}}{z_1 - z_2} \in \mathcal{A}_2$. In this way, we have expressed our initial coinvariant as the class of an element in $\mathcal{A}_2 \otimes \mathbb{C}^2 \otimes \mathbb{C}$.

Remark 18. We say "the class of", because $F(\mathfrak{sl}_2; \mathcal{A}_2; \mathbb{C}^2, \mathbb{C}) \cong \left(\mathcal{A}_N \otimes \mathbb{C}^2 \otimes \mathbb{C}\right)/\mathfrak{a}_0$ and we must still quotient by $\mathfrak{a}_0$. The dg $\mathcal{A}_2$-module $F(\mathfrak{sl}_2; \mathcal{A}_2; \mathbb{C}^2, \mathbb{C})$, as we have defined it above, is certainly not free. Recall the notations and calculations in Example 12, and consider the element

$$X_{12} := \frac{v_{13}v_{31}}{w - z_1} \wedge \frac{v_{23}v_{32}}{w - z_2} - \frac{v_{12}v_{21}}{z_2 - z_1} \wedge \frac{v_{23}v_{32}}{w - z_2} - \frac{v_{13}v_{31}}{w - z_1} \wedge \frac{v_{21}v_{12}}{z_1 - z_2} \in \mathcal{A}_3'.$$

Since by definition $[(f \otimes X_{12}).(m \otimes |0\rangle)] = 0$ in coinvariants, we find that

$$\left(\frac{u_{(12)}(1 - u_{(12)})}{(z_1 - z_2)}\right)^2 [(f.m) \otimes |0\rangle] = 0.$$

On the other hand, by inspection one confirms that the class of $\frac{du_{(12)}}{z_1 - z_2}[(f.m) \otimes |0\rangle]$ is non-zero. It is also interesting to note the following. Recall from Remark 16 that all we actually require of the module M_1 is that it be smooth over $\mathfrak{sl}_2 \otimes \mathcal{A}_2\{\!\{w - z_1\}\!\}_+ = \mathfrak{sl}_2 \otimes \mathcal{A}_2[[w - z_1]] \otimes \mathbb{C}[v, dv]$. We do not need to assume, as we did in the main text, that the action of v is by 0. If we wish we can instead take a different definition of M_1 as smooth $\mathfrak{sl}_2 \otimes \mathcal{A}_2\{\!\{w - z_1\}\!\}_+$-module, by taking v to act as $u_{(12)} \in \mathcal{A}_2$. By inspection, one confirms that when one does that, then the elements X_{12} and Ω_{12}, cf. Example 12, act as zero on M_1, and more generally that in fact $\mathfrak{a}_0$ acts trivially on $M_1 \otimes M_2$. With those particular choices, then, we expect the dg $\mathcal{A}_2$-module of ravioli coinvariants to be free, $\cong \mathcal{A}_2 \otimes M_1 \otimes M_2$.

6 Main Result

We can now state the main result of the present paper, which says that the state-field map for the raviolo vacuum module $\mathscr{V}$, as we defined it in §2.6, emerges naturally when one considers appropriate limits of the spaces of coinvariants introduced in §5.3 above. That is, Theorem 4 above continues to hold in the raviolo case, mutatis mutandis.

Indeed, let us again specialize to the case in which we insert a copy of the vacuum module at the $N - 1$st and Nth marked points, $M_{N-1} = \mathbb{C}|0\rangle$ and $M_N = \mathbb{C}|0\rangle$. By identifying local coordinates $w - z_i$ in the complex-algebraic direction, we may identify each of the dg Lie algebras $\mathfrak{g} \otimes \mathbb{C}\{\!\{w - z_i\}\!\}$ with a single copy $\mathfrak{g} \otimes \mathbb{C}\{\!\{s\}\!\}$, and thereby identify their vacuum Verma modules with a single abstract copy of $\mathscr{V}$:

$$\mathscr{V} \cong \mathscr{V}_i := \mathrm{Ind}_{\mathfrak{g} \otimes \mathbb{C}[[w - z_i]]}^{\mathfrak{g} \otimes \mathbb{C}\{\!\{w - z_i\}\!\}} \mathbb{C}|0\rangle.$$

There is then an evident embedding map of dg vector spaces

$$\mathscr{V}_i \hookrightarrow U_{\mathcal{A}_N}(\mathfrak{g} \otimes \mathcal{A}_N\{\!\{w - z_i\}\!\}) \otimes_{U_{\mathcal{A}_N}(\mathfrak{g} \otimes \mathcal{A}_N\{\!\{w - z_i\}\!\}_+)} (\mathcal{A}_N|0\rangle)$$

coming from the unit map $1 : \mathbb{C} \hookrightarrow \mathcal{A}_N$.

At the other sites, we pick arbitrary smooth modules M_i as before. Pick vectors $m_i \in \mathscr{M}_i$ for $1 \le i \le N-2$, and states $A, B \in \mathscr{V}$ in this abstract copy of $\mathscr{V}$. On taking coinvariants, we get the class

$$\left[m_1 \otimes \cdots \otimes m_{N-2} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{A}\right] \in F(\mathfrak{g}; \mathcal{A}_N; M_1, \ldots, M_{N-2}, \mathbb{C}, \mathbb{C})$$

$$\underset{z_1}{\overset{\uparrow}{}} \qquad \underset{z_{N-2}}{\overset{\uparrow}{}} \qquad \underset{z_{N-1}}{\overset{\uparrow}{}} \quad \underset{z_N}{\overset{\uparrow}{}} \tag{31}$$

Remark 19. One should keep in mind that, despite our attempt to make the notation as similar as possible to the usual case in (18), this object encodes a lot of information. It is a polynomial differential form on a simplex of dimension $N! - 1$, whose pullbacks to certain faces of that simplex encode the behaviour in regimes in which some of the insertion points are ordered in particular ways in the topological direction whenever they collide in the complex plane. For example, the pullback to the vertex given by $u_\sigma = 1$ encodes the behaviour in the regime in which *all* the points are so ordered, in the particular total order $\sigma(1) < \cdots < \sigma(N)$.

The raviolo vacuum module state-field map captures the behaviour of this coinvariant in the limit in which the Nth marked point becomes close to the $(N - 1)$st marked point, while remaining marked points, the vectors $m_1, \ldots, m_{N-2}$, and the states A, B are all held fixed.

Indeed, recall that $\mathcal{A}_N \simeq R\Gamma(\mathrm{RavConf}_N, \mathcal{O})$ models the dg commutative algebra of derived sections of the structure sheaf on ravioli configuration space. We have the expansion map

$$\iota_{z_N \to z_{N-1}} : \mathcal{A}_N \to \mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}$$

defined in the same way as $\iota_{z_{N+1} \to z_N}$ in §5.1 above.

Theorem 20 (Relation of the raviolo state-field map Y_{Rav} to coinvariants). For all homogeneous states $A, B \in \mathcal{V}$ and vectors $m_i \in \mathcal{M}_i$, $1 \le i \le N-2$, we have

$$\iota_{z_N \to z_{N-1}}\left[m_1 \otimes \cdots \otimes m_{N-2} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{A} \right]$$
$$= (-1)^{|A||B|}\left[m_1 \otimes \cdots \otimes m_{N-2} \otimes Y_{\mathrm{Rav}}(A; z_N - z_{N-1}) \underset{z_{N-1}}{B} \right]$$

where

$$Y_{\mathrm{Rav}}(-; x) : \mathcal{V} \to \mathrm{Hom}_{\mathbf{dgVect}_{\mathbb{C}}}(\mathcal{V}, \mathcal{V}\{\{x\}\})$$

is the raviolo state-field map defined in §2.6.

Proof. The proof is given in Section 7 below. ∎

6.1 Worked example

Before we give the proof of Theorem 20 it is instructive to work through a simple example in detail. Let $a \in \mathfrak{g}$ and consider the state

$$\left(a \otimes \frac{\mathrm{d}v}{z} \right) |0\rangle \in \mathcal{V}$$

in the vacuum module $\mathcal{V}$. Everything below will hold, mutatis mutandis, with $\mathrm{d}v$ replaced by $v(1 - v)$; cf. the discussion about cochains versus cohomology in §2.6 above.

We are first to identify this abstract copy of $\mathcal{V}$ with the local copy associated to the Nth marked point, by identifying the coordinate z with the local coordinate $w - z_N$ at that point. Then we are to insert the state above into a generic coinvariant with some vectors $m_1, \ldots, m_{N-2}$ at (what are about to be) the far marked points, and some state $B \in \mathcal{V}$ at (what is about to be) the nearby $N-1$st marked point. We obtain a coinvariant we shall call f,

$$f := \left[m_1 \otimes \cdots \otimes m_{N-2} \otimes \underset{z_{N-1}}{B} \otimes \left(a \otimes \frac{\mathrm{d}v}{w - z_N} \right) \underset{z_N}{|0\rangle} \right] \in \mathcal{A}_N \otimes M. \tag{32}$$

Our aim is to now to "swap" the lowering operator onto the other sites. The first step is to identify the lowering operator $a \otimes \frac{\mathrm{d}v}{w - z_N}$ as the expansion at this Nth site of some element of the global dg Lie algebra. To do that, we apply the map g_N of (26) to the element

$$a \otimes \frac{\mathrm{d}v}{w - z_N} \in \mathfrak{g} \otimes \mathbb{C}\{\{w - z_N\}\}_- \hookrightarrow \mathfrak{g} \otimes \mathcal{A}_N\{\{w - z_N\}\}_-.$$

We have, cf. (25),

$$q_N^*(v) = \iota_{\{N, N+1\} \subset [1, N+1]}(v) = \sum_{\sigma \in S_{N+1}^{N, N+1}} u_\sigma$$

and so we find that

$$G := g_N(a \otimes \frac{\mathrm{d}v}{w - z_N}) = a \otimes \frac{\mathrm{d} \sum_{\sigma \in S_{N+1}^{N,N+1}} u_\sigma}{w - z_N} \in \mathfrak{g} \otimes \mathcal{A}'_{N+1}$$

where we have introduced a name, G, for this element. By definition of coinvariants, we have

$$-f = \sum_{i=1}^{N-2} (-)^* [m_1 \otimes \cdots \otimes \underset{z_i}{(\iota_{w \to z_i} G)} \underset{z_i}{m_i} \otimes \cdots \otimes \underset{z_{N-2}}{m_{N-2}} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{|0\rangle}]$$
$$+ (-)^{|B|} [\underset{z_1}{m_1} \otimes \cdots \otimes \underset{z_{N-2}}{m_{N-2}} \otimes (\iota_{w \to z_{N-1}} G) \underset{z_{N-1}}{B} \otimes \underset{z_N}{|0\rangle}]$$

Here $(-)^*$ are certain Koszul signs, which we do not write out in full.

We are to take the expansion $\iota_{z_N \to z_{N-1}} f \in \mathcal{A}_{N-1}\{\!\{z_N - z_{N-1}\}\!\} \otimes M$ of this element $f \in \mathcal{A}_N \otimes M$. As we shall discuss in (7.2.1) below, coinvariants are suitably functorial, so that

$$-\iota_{z_N \to z_{N-1}} f = \sum_{i=1}^{N-2} (-)^* [m_1 \otimes \cdots \otimes \underset{z_i}{(\iota_{z_N \to z_{N-1}} \iota_{w \to z_i} g)} \underset{z_i}{m_i} \otimes \cdots \otimes \underset{z_{N-2}}{m_{N-2}} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{|0\rangle}]$$
$$+ (-1)^{|B|} [\underset{z_1}{m_1} \otimes \cdots \otimes \underset{z_{N-2}}{m_{N-2}} \otimes \underset{z_{N-1}}{(\iota_{z_N \to z_{N-1}} \iota_{w \to z_{N-1}} g)} \underset{z_{N-1}}{B} \otimes \underset{z_N}{|0\rangle}] \tag{33}$$

Now let us actually compute these expansions of G at the other sites. Recall the definition of the expansion map, Definition-Proposition 10. For every $s \in \{1, \dots, N-1\}$ we have

$$p^*_{N+1 \to s} \left(\sum_{\sigma \in S_{N+1}^{N,N+1}} u_\sigma \right) = v \sum_{\sigma \in S_N^{N,s}} u_\sigma + (1-v) \sum_{\sigma \in S_N^{N,s}} u_\sigma = \sum_{\sigma \in S_N^{N,s}} u_\sigma \tag{34}$$

Thus, in particular, the expansion of G at the $(N-1)$st site is given by

$$\iota_{w \to z_{N-1}} G = -\sum_{k=0}^{\infty} \frac{\mathrm{d} \sum_{\sigma \in S_N^{N,N-1}} u_\sigma}{(z_N - z_{N-1})^{k+1}} \left(a \otimes (w - z_{N-1})^k \right).$$

We are working in dg $\mathcal{A}_N$-modules, so that the factor $\frac{\mathrm{d} \sum_{\sigma \in S_N^{N,N-1}} u_\sigma}{(z_N - z_{N-1})^{k+1}}$ is a scalar. Let us apply the expansion map $\iota_{z_N \to z_{N-1}}$ to this scalar prefactor. Note that

$$p^*_{N \to N-1} \sum_{\sigma \in S_N^{N,N-1}} u_\sigma = (1-u) \sum_{\sigma \in S_{N-1}} u_\sigma = (1-u).$$

Here we write u here rather than v for the coordinate of

$$\mathbb{C}\{\!\{z_N - z_{N-1}\}\!\} := \big\{ \omega \in \mathbb{C}((z_N - z_{N-1})) \otimes \mathbb{C}[u, \mathrm{d}u]$$
$$: \omega|_{u=0} \in \mathbb{C}[[z_N - z_{N-1}]] \text{ and } \omega|_{u=1} \in \mathbb{C}[[z_N - z_{N-1}]] \big\}$$

to avoid a clash with the coordinate v of for example, $\mathbb{C}\{\!\{w - z_s\}\!\}$. Note that the above is consistent with the fact that $\sum_{\sigma \in S_N^{N,N-1}} u_\sigma = \iota_{\{N-1,N\} \subset [1,N]} (1-u)$; see (22) for the definition of $\iota_{\{N-1,N\} \subset [1,N]}$. We obtain that

$$\iota_{z_N \to z_{N-1}} \frac{\mathrm{d} \sum_{\sigma \in S_N^{N,N-1}} u_\sigma}{(z_N - z_{N-1})^{k+1}} = 1 \frac{\mathrm{d}(1-u)}{(z_N - z_{N-1})^{k+1}} \in \mathcal{A}_{N-1}\{\!\{z_N - z_{N-1}\}\!\}.$$

Next let us consider the expansions of g at the far sites. Crucially, for each $i \leq N - 2$, the operations $\iota_{z_N - z_{N-1}}$ and $\iota_{w \to z_i}$ commute. This is true for the coefficients in $\mathbb{C}[z_i, w, (z_i - z_j)^{-1}, (w - z_i)^{-1}]_{1 \leq i,j \leq N; i \neq j} \equiv \mathcal{B}_{N+1}$ just as in the usual case; and for the forms on the simplex, one checks (similarly to (34)) that on the one hand

$$p^*_{N \to N-1} p^*_{N+1 \to i} \left(\sum_{\sigma \in S^{N,N+1}_{N+1}} u_\sigma \right) = \sum_{\sigma \in S^{N,i}_N} u_\sigma = \sum_{\sigma \in S^{N-1,i}_{N-1}} u_\sigma$$

while on the other hand

$$p^*_{N+1 \to i} p^*_{N \to N-1} \left(\sum_{\sigma \in S^{N,N+1}_{N+1}} u_\sigma \right) = \sum_{\sigma \in S^{N-1,N+1}_{\{1,\ldots,N-1,N+1\}}} u_\sigma = \sum_{\sigma \in S^{N-1,i}_{N-1}} u_\sigma.$$

(We don't get such agreement when $i = N - 1$, which is reassuringly consistent with the fact that the usual expansion maps $\iota_{w \to z_{N-1}}$ and $\iota_{z_N \to z_{N-1}}$ certainly do not commute—indeed this failure is in some sense at the heart of how vertex algebras work.)

Thus, for each $i \leq N - 2$, we have $\iota_{z_N \to z_{N-1}} \iota_{w \to z_i} G = \iota_{w \to z_i} \iota_{z_N \to z_{N-1}} G$ and here

$$\iota_{z_N - z_{N-1}} G = + \sum_{k=0}^{\infty} (z_N - z_{N-1})^k \otimes \left(a \otimes \frac{d \sum_{\sigma \in S^{N-1,N+1}_{\{1,\ldots,N-1,N+1\}}} u_\sigma}{(w - z_{N-1})^{k+1}} \right).$$

We recognize the terms in this expansion as elements of the global Lie algebra, and by definition of coinvariants we obtain that

$$\iota_{z_N \to z_{N-1}} f = \sum_{k=0}^{\infty} [m_1 \otimes \cdots \otimes m_{N-2} \otimes (-)^{|B|} (z_N - z_{N-1})^k \left(a \otimes \frac{dv}{(w - z_{N-1})^{k+1}} \right) \underset{z_{N-1}}{B}]$$
$$+ \sum_{k=0}^{\infty} [m_1 \otimes \cdots \otimes m_{N-2} \otimes (-)^{|B|} \frac{d(1-u)}{(z_N - z_{N-1})^{k+1}} \left(a \otimes (w - z_{N-1})^k \right) \underset{z_{N-1}}{B}]$$

(Here we dropped the vacuum state at the Nth site, cf. §7.3 below.)

We recognize the first and second lines here as, respectively, the $(-)_+$ and $(-)_-$ parts of the raviolo mode expansion of the state $(a \otimes \frac{dv}{w - z_N})|0\rangle$ with which we began, as we defined it in §2.6.

7   Proof of Theorem 20

In this section we prove Theorem 20, namely that the raviolo state-field map Y_{Rav} from §2.6 emerges naturally when one considers appropriate limits of the spaces of ravioli coinvariants introduced in §5.3.

To separate concerns, we shall first warm up by rehearsing a proof of the analogous statement in the usual case, Theorem 4. Our approach is similar to that of [41].

7.1 Proof of Theorem 4

We first need to recall a functoriality property of coinvariants and the property known as propagation of vacua.

7.1.1 *Functoriality of coinvariants*

Let us consider certain spaces of coinvariants with $N - 1$ movable marked points. The construction of coinvariants in Section 3 of course goes through with $N - 1$ in place of N, yielding the $\mathcal{B}_{N-1}$-module

$$F(\mathfrak{g}; \mathcal{B}_{N-1}; M_1, \ldots, M_{N-1}).$$

But we may also choose to work over $\mathcal{B}_N$, or over $\mathcal{B}_{N-1}((z_N - z_{N-1}))$, that is, to allow our coefficient functions to depend in some prescribed way on the formal variable z_N, even though there are now only

modules assigned to the points $z_1, \ldots, z_{N-1}$. More precisely, we may consider the following cospans of commutative algebras

$$\begin{array}{ccc}
\mathcal{B}_N[w, (w - z_i)^{-1}]'_{1 \le i \le N-1} & \longrightarrow & \mathcal{B}_{N-1}((z_N - z_{N-1}))[w, (w - z_i)^{-1}]'_{1 \le i \le N-1} \\
\uparrow & & \uparrow \\
\bigoplus_{j=1}^{N-1} \mathcal{B}_N((w - z_j)) & \longrightarrow & \bigoplus_{j=1}^{N-1} \mathcal{B}_{N-1}((z_N - z_{N-1}))((w - z_j)) \\
\uparrow & & \uparrow \\
\bigoplus_{j=1}^{N-1} \mathcal{B}_N[[w - z_j]] & \longrightarrow & \bigoplus_{j=1}^{N-1} \mathcal{B}_{N-1}((z_N - z_{N-1}))[[w - z_j]]
\end{array} \tag{35}$$

—in $\mathcal{B}_N$-modules on the left and in $\mathcal{B}_{N-1}((z_N - z_{N-1}))$-modules on the right. We obtain corresponding spaces of coinvariants, which we denote respectively as

$$\mathsf{F}(\mathfrak{g}; \mathcal{B}_N; M_1, \ldots, M_{N-1}) \quad \text{and} \quad \mathsf{F}(\mathfrak{g}; \mathcal{B}_{N-1}((z_N - z_{N-1})); M_1, \ldots, M_{N-1}).$$

Moreover the algebra map

$$\mathcal{B}_N \to \mathcal{B}_{N-1}((z_N - z_{N-1}))$$

given by expanding in small $z_N - z_{N-1}$ for fixed $z_1, \ldots, z_{N-1}$ allows us to change base ring, in the sense that it induces the embeddings of commutative algebras in $\mathcal{B}_{N-1}$-modules shown as horizontal arrows in the diagram above. Let us use $\iota_{\substack{\text{base} \\ \text{change}}}$ for that change-of-base map. The diagram above commutes. In this way, one has the following functoriality property of coinvariants.

Lemma 21 (Base change commutes with taking coinvariants). The following diagram of $\mathcal{B}_{N-1}$-modules commutes:

$$\begin{array}{ccc}
\bigotimes_{i=1}^{N-1} \mathbb{M}_i & \longrightarrow & \bigotimes_{i=1}^{N-1} \iota_{\substack{\text{base} \\ \text{change}}} \mathbb{M}_i \\
\downarrow & & \downarrow \\
\mathsf{F}(\mathfrak{g}; \mathcal{B}_N; M_1, \ldots, M_{N-1}) & & \mathsf{F}(\mathfrak{g}; \mathcal{B}_{N-1}((z_N - z_{N-1})); M_1, \ldots, M_{N-1}) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{B}_N \otimes M & \longrightarrow & \mathcal{B}_{N-1}((z_N - z_{N-1})) \otimes M
\end{array}$$

Let us stress that in the horizontal maps in (37) above, we are merely performing a change of base ring. By contrast, we reserve the notation $\iota_{z_N \to z_{N-1}}$ for the algebra map, which expands in small $z_N - z_{N-1}$ for fixed $z_1, \ldots, z_{N-1}$ *and* $w \equiv z_{N+1}$. Thus, for example,

$$\iota_{z_N \to z_{N-1}} : \mathcal{B}'_{N+1} \to \mathcal{B}_{N-1}[w, (w - z_j)^{-1}]_{1 \le j \le N-1}((z_N - z_{N-1}))$$

sends the element $1/(w - z_N)$ to its expansion $\sum_{k=0}^{\infty} \frac{(z_N - z_{N-1})^k}{(w - z_{N-1})^{k+1}}$.

One should keep in mind that the Laurent-expansion maps $\iota_{w \to z_{N-1}}$ and $\iota_{z_N \to z_{N-1}}$ do not commute. For example, they fail to commute when applied to $1/(w - z_N)$. (Indeed, they map from $\mathcal{B}_{N+1}$ into different rings. In some sense, this fact is central to the notion of vertex algebras: see e.g., the discussion in [18,§1].) On the other hand, for all $i \le N - 2$, the Laurent expansion maps $\iota_{w \to z_i}$ and $\iota_{z_N \to z_{N-1}}$ do commute.

7.1.2 *Propagation of vacua*

When $M_N = \mathbb{C}$, there is a canonical identification, of $\mathcal{B}_N$-modules, between our initial space of coinvariants with N marked points and one with only $N - 1$ marked points (The reader will notice that while the choice $M_N = \mathbb{C}$ is crucial here, the choice $M_{N-1} = \mathbb{C}$ actually plays no role. And indeed, the construction goes through more generally, and yields the structure of $\mathbb{M}_{N-1}$ as a *module over the vertex*

algebra $\mathbb{V}$.):

$$\mathsf{F}(\mathfrak{g}; \mathcal{B}_N; M_1, \ldots, M_{N-2}, \mathbb{C}, \mathbb{C}) \cong \mathcal{B}_N \otimes M \cong \mathsf{F}(\mathfrak{g}; \mathcal{B}_N; M_1, \ldots, M_{N-2}, \mathbb{C}).$$

One has, moreover, the following equality

$$\left[m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{|0\rangle} \right] = \left[m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \otimes \underset{z_{N-1}}{B} \right]. \tag{36}$$

This property is an example of what is sometimes called *propagation of vacua*.

7.1.3 Completion of the proof of Theorem 4

The equality in (36) establishes the statement of Theorem 4 in the special case that $A = |0\rangle$ is the vacuum state.

Next we shall show that for all states $A \in \mathbb{V}$, the class

$$\left[m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{A} \right] \in \mathsf{F}(\mathfrak{g}; \mathcal{B}_N; M_1, \ldots, M_{N-2}, \mathbb{C}, \mathbb{C}) \cong \mathcal{B}_N \otimes M$$

has a representative of the form

$$\sum_i \left[A_i^{(-)}.(m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2}) \otimes A_i^{(+)}.\underset{z_{N-1}}{B} \otimes \underset{z_N}{|0\rangle} \right]$$

for some finite sum over i and for certain $A_i^{(-)}$ and $A_i^{(+)}$ belonging to $U(\mathfrak{g} \otimes (w - z_N)^{-1}\mathbb{C}[(w - z_N)^{-1}])$. Here, when we write $A_i^{(-)}.(m_1 \otimes \ldots \otimes m_{N-2})$, the action is by definition via the embedding

$$\iota_{\text{far}} : \mathfrak{g} \otimes (w - z_N)^{-1}\mathbb{C}[(w - z_N)^{-1}] \hookrightarrow \mathfrak{g} \otimes (w - z_N)^{-1}\mathcal{B}_N[(w - z_N)^{-1}]$$

$$\hookrightarrow \bigoplus_{i=1}^{N-2} \mathfrak{g} \otimes \mathcal{B}_N[[w - z_i]].$$

Call this embedding ι_{far}. Similarly when we write $A_i^{(+)}.B$, the action is via the embedding into $\mathfrak{g} \otimes \mathcal{B}_N[[w - z_{N-1}]]$, which we shall call ι_{near}:

$$\iota_{\text{near}} : \mathfrak{g} \otimes (w - z_N)^{-1}\mathbb{C}[(w - z_N)^{-1}] \hookrightarrow \mathfrak{g} \otimes (w - z_N)^{-1}\mathcal{B}_N[(w - z_N)^{-1}]$$

$$\hookrightarrow \mathfrak{g} \otimes \mathcal{B}_N[[w - z_{N-1}]].$$

Indeed, we may suppose

$$A = X^1_{-k_1} \cdots X^n_{-k_n} |0\rangle$$

for some number $n \in \mathbb{Z}_{\geq 0}$ of elements $X^i \in \mathfrak{g}$ and mode numbers $-k_i \in \mathbb{Z}_{<0}$. Here $X_k := X \otimes (w - z_N)^k$. (Such states A span $\mathbb{V}$ as a $\mathbb{C}$-vector space.) By a straightforward induction on n, one checks that

$$\left[m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{A} \right]$$

$$= (-1)^n \sum_{m=0}^{n} \sum_{(\mu,\nu) \in \text{Unshf}_m^n} \left[(\iota_{\text{far}} \overleftarrow{A}_\mu)(m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2}) \otimes (\iota_{\text{near}} \overleftarrow{A}_\nu)\underset{z_{N-1}}{B} \otimes \underset{z_N}{|0\rangle} \right] \tag{37}$$

where the inner sum is over unshuffles, that is, permutations $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_{n-m})$ of $(1, \ldots, n)$ such that $\mu_1 < \cdots < \mu_m$ and $\nu_1 < \cdots < \nu_{n-m}$, and where we write

$$\overleftarrow{A}_\mu := X^{\mu_m}_{-k_{\mu_m}} \cdots X^{\mu_1}_{-k_{\mu_1}}, \qquad \overleftarrow{A}_\nu := X^{\nu_{n-m}}_{-k_{\nu_{n-m}}} \cdots X^{\nu_1}_{-k_{\nu_1}}.$$

Then by propagation of vacua as in (36), we may regard the right-hand side in (37) as an element of the space of coinvariants $F(\mathfrak{g}; \mathcal{B}_N; M_1, \ldots, M_{N-2}, \mathbb{C})$. The equality in (37) is in $\mathcal{B}_N \otimes M$. We may apply the change-of-base map $\iota_{\substack{\text{base} \\ \text{change}}}$ to both sides to obtain the equality

$$\iota_{\substack{\text{base} \\ \text{change}}} \left[m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \underset{z_{N-2}}{\otimes} \underset{z_{N-1}}{B} \underset{z_N}{\otimes} A \right]$$

$$= (-1)^n \sum_{m=0}^n \sum_{(\mu, \nu) \in \text{Unshf}^n_m} \iota_{\substack{\text{base} \\ \text{change}}} \left[(\iota_{\text{far}} \overleftarrow{A}_\mu)(m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \underset{z_{N-2}}{\otimes}) \otimes (\iota_{\text{near}} \overleftarrow{A}_\nu) \underset{z_{N-1}}{B} \right]$$

$$= (-1)^n \sum_{m=0}^n \sum_{(\mu, \nu) \in \text{Unshf}^n_m} \left[\iota_{\substack{\text{base} \\ \text{change}}} (\iota_{\text{far}} \overleftarrow{A}_\mu)(m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \underset{z_{N-2}}{\otimes}) \otimes \iota_{\substack{\text{base} \\ \text{change}}} (\iota_{\text{near}} \overleftarrow{A}_\nu) \underset{z_{N-1}}{B} \right]$$

$$= (-1)^n \sum_{m=0}^n \sum_{(\mu, \nu) \in \text{Unshf}^n_m} \left[(\iota_{\substack{\text{base} \\ \text{change}}} \iota_{\text{far}} \overleftarrow{A}_\mu) \iota_{\substack{\text{base} \\ \text{change}}} (m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \underset{z_{N-2}}{\otimes}) \otimes \iota_{\substack{\text{base} \\ \text{change}}} (\iota_{\text{near}} \overleftarrow{A}_\nu) \underset{z_{N-1}}{B} \right] \tag{38}$$

in $\mathcal{B}_{N-1}((z_N - z_{N-1})) \otimes M$. In the second step, we used the functoriality of coinvariants, Lemma 21.

It remains to show that this expression is equal to the expression on the right-hand side in the statement of Theorem 4. The latter is, first and foremost, a formal series in $(z_N - z_{N-1})^{\pm 1}$ whose coefficients belong to the space of coinvariants $F(\mathfrak{g}; \mathcal{B}_{N-1}; M_1, \ldots, M_{N-2}, \mathbb{C}) \cong_{\mathcal{B}_{N-1}} \mathcal{B}_{N-1} \otimes M$:

$$F(\mathfrak{g}; \mathcal{B}_{N-1}; M_1, \ldots, M_{N-2}, \mathbb{C}) \cong_{\mathcal{B}_{N-1}} \mathcal{B}_{N-1} \otimes M:$$

$$\left[m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \underset{z_{N-2}}{\otimes} Y(A; z_N - z_{N-1}) \underset{z_{N-1}}{B} \right]$$

$$= \sum_{k \in \mathbb{Z}} (z_N - z_{N-1})^{-k-1} \left[m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \underset{z_{N-2}}{\otimes} A_{(k)} \underset{z_{N-1}}{B} \right]$$

Smoothness of the module $\mathbb{V}$ ensures that for each fixed $A, B \in \mathbb{V}$ this is in fact a formal Laurent series, that is, $A_{(k)}B = 0$ for $k \gg 0$. Thus, it certainly belongs in $(\mathcal{B}_{N-1} \otimes M)((z_N - z_{N-1}))$. To show that it is equal to the expression in (38) we must show that for each $k \in \mathbb{Z}$, the coefficients of $(z_N - z_{N-1})^k$ agree. Consider any term (μ, ν) in the sum in (38). We have

$$\iota_{\substack{\text{base} \\ \text{change}}} \iota_{\text{far}} \overleftarrow{A}_\mu = \iota_{z_N \to z_{N-1}} \iota_{\text{far}} \overleftarrow{A}_\mu = \iota_{\text{far}} \iota_{z_N \to z_{N-1}} \overleftarrow{A}_\mu.$$

Since $(\iota_{\text{near}} \overleftarrow{A}_\nu)B \in (z_N - z_{N-1})^{-1} \mathcal{B}_{N-1}[(z_N - z_{N-1})^{-1}] \otimes \mathbb{V}$, only finitely many terms in the series $\iota_{z_N \to z_{N-1}} \overleftarrow{A}_\mu$ contribute to the overall coefficient of $(z_N - z_{N-1})^k$. The coefficients of these finitely many terms belong to $U(\mathfrak{g} \otimes \mathcal{B}_{N-1}[w, (w - z_{N-1})^{-1}]')$, and we can swap them over to the module at the marked point z_{N-1}, by definition of the space of coinvariants. After doing so we obtain

$$(-1)^n \sum_{m=0}^n (-1)^m \sum_{(\mu, \nu) \in \text{Unshf}^n_m} \left[m_1 \underset{z_1}{\otimes} \cdots \otimes m_{N-2} \underset{z_{N-2}}{\otimes} (\iota_{\text{near}} \iota_{z_N \to z_{N-1}} \overrightarrow{A}_\mu)(\iota_{\substack{\text{base} \\ \text{change}}} \iota_{\text{near}} \overleftarrow{A}_\nu) \underset{z_{N-1}}{B} \right]$$

where $\overrightarrow{A}_\mu := X^{\mu_1}_{-k_{\mu_1}} \ldots X^{\mu_m}_{-k_{\mu_m}}$. Here, we recognize the state-field map Y:

$$\sum_{m=0}^n (-1)^{n-m} \sum_{(\mu, \nu) \in \text{Unshf}^n_m} (\iota_{\text{near}} \iota_{z_N \to z_{N-1}} \overrightarrow{A}_\mu)(\iota_{\substack{\text{base} \\ \text{change}}} \iota_{\text{near}} \overleftarrow{A}_\nu)B = Y(A, z_N - z_{N-1})B.$$

(This expression can be checked by induction on n.)

7.2 Proof of Theorem 20

Now we return to the proof of Theorem 20. Where possible, we shall try to follow the proof above almost word-for-word.

We begin by establishing a raviolo analogue of the functoriality property of coinvariants, Lemma 21.

7.2.1 *Functoriality of raviolo coinvariants*

Once again, let us consider certain spaces of coinvariants with $N - 1$ movable marked points. We certainly have the dg $\mathcal{A}_{N-1}$-module

$$F(\mathfrak{g}; \mathcal{A}_{N-1}; M_1, \ldots, M_{N-1}).$$

But we may also choose to work over $\mathcal{A}_N$, or over $\mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}$. More precisely, we may consider the following cospans of commutative algebras

$$
\begin{array}{ccc}
\mathcal{A}_N\{w, (w - z_i)^{-1}\}'_{1 \leq i \leq N-1} & \longrightarrow & \mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}\{w, (w - z_i)^{-1}\}'_{1 \leq i \leq N-1} \\
\downarrow & & \downarrow \\
\bigoplus_{j=1}^{N-1} \mathcal{A}_N\{\{w - z_j\}\} & \longrightarrow & \bigoplus_{j=1}^{N-1} \mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}\{\{w - z_j\}\} \\
\uparrow & & \uparrow \\
\bigoplus_{j=1}^{N-1} \mathcal{A}_N[[w - z_j]] & \longrightarrow & \bigoplus_{j=1}^{N-1} \mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}[[w - z_j]]
\end{array}
\tag{39}
$$

—in dg $\mathcal{A}_N$-modules on the left and in dg $\mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}$-modules on the right. Here, the algebras appearing in the top line are defined in close analogy with the definition of $\mathcal{A}'_{N+1}$ in §5.1. Namely, we first let $\mathcal{A}_N\{w, (w - z_i)^{-1}\}_{1 \leq i \leq N-1}$ denote the commutative algebra in dg $\mathcal{A}_N$-modules given by the subalgebra of $\mathcal{A}_{N+1}$ consisting of forms that are regular in $w - z_N \equiv z_{N+1} - z_N$ everywhere:

$$\mathcal{A}_N\{w, (w - z_i)^{-1}\}_{1 \leq i \leq N-1}$$

$$:= \Big\{ \omega \in \mathcal{B}_N[w, (w - z_i)^{-1}]_{1 \leq i \leq N-1} \otimes \mathbb{C}[u_\sigma, du_\sigma]_{\sigma \in S_{N+1}} \Big/ \langle \sum_{\sigma \in S_{N+1}} u_\sigma - 1, \sum_{\sigma \in S_{N+1}} du_\sigma \rangle$$

$$: \text{for all distinct } i, j \in [1, N + 1] \text{ the pullback } \omega|_{\{u_\sigma = 0 \forall \sigma \in S_{N+1}^{ij}\}}$$

$$\text{is regular in } z_i - z_j \Big\}.$$

Similarly, we let $\mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}\{w, (w - z_i)^{-1}\}_{1 \leq i \leq N-1}$ denote the dg commutative algebra given by

$$\mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}\{w, (w - z_i)^{-1}\}_{1 \leq i \leq N-1}$$

$$:= \Big\{ \omega \in \mathcal{B}_{N-1}((z_N - z_{N-1}))[w, (w - z_i)^{-1}]_{1 \leq i \leq N-1} \otimes \mathbb{C}[v, dv]$$

$$\otimes \mathbb{C}[u_\sigma, du_\sigma]_{\sigma \in S_{[1,N-1] \cup \{N+1\}}} \Big/ \langle \sum_{\sigma \in S_{[1,N-1] \cup \{N+1\}}} u_\sigma - 1, \sum_{\sigma \in S_{[1,N-1] \cup \{N+1\}}} du_\sigma \rangle$$

$$: \text{for all distinct } i, j \in [1, N - 1] \cup \{N + 1\},$$

$$\text{the pullback } \omega|_{\{u_\sigma = 0 \forall \sigma \in S_{[1,N-1] \cup \{N+1\}}^{ij}\}} \text{ is regular in } z_i - z_j,$$

$$\text{and the pullbacks } \omega|_{v=0} \text{ and } \omega|_{v=1} \text{ are both regular in } z_N - z_{N-1} \Big\}.$$

It is a commutative algebra in dg $\mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}$-modules, by an argument similar to that in Lemma 8.

We again let prime $'$ denote the subalgebras of these algebras consisting of forms ω such that $\omega \to 0$ as $w \to \infty$.

We obtain corresponding spaces of coinvariants, which we denote respectively as

$$F(\mathfrak{g}; \mathcal{A}_N; M_1, \ldots, M_{N-1}) \quad \text{and} \quad F(\mathfrak{g}; \mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}; M_1, \ldots, M_{N-1}).$$

Moreover the algebra map

$$\mathcal{A}_N \to \mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}$$

from §5.1 induces the maps of commutative algebras in dg $\mathcal{A}_{N-1}$-modules shown as horizontal arrows in the diagram above. At the cost of overloading notation somewhat, let us continue to use $\iota_{\text{base change}}$ for that change-of-base map.

The diagram above is then a commuting diagram in the category of commutative algebras in dg $\mathcal{A}_{N-1}$-modules. In this way, one has the following functoriality property of coinvariants.

Lemma 22 (Base change commutes with taking coinvariants—raviolo case). The following diagram of dg $\mathcal{A}_{N-1}$-modules commutes:

$$
\begin{array}{ccc}
\bigotimes_{i=1}^{N-1} \mathscr{M}_i & \longrightarrow & \bigotimes_{i=1}^{N-1} \iota_{\text{base change}} \mathscr{M}_i \\
\downarrow & & \downarrow \\
\mathsf{F}(\mathfrak{g}; \mathcal{A}_N; M_1, \ldots, M_{N-1}) & \longrightarrow & \mathsf{F}(\mathfrak{g}; \mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}; M_1, \ldots, M_{N-1})
\end{array}
$$

We continue to reserve the notation $\iota_{z_N \to z_{N-1}}$ for the expansion map, in the sense of §5.1, which expands in small $z_N - z_{N-1}$ for fixed $z_1, \ldots, z_{N-1}$ *and* $w \equiv z_{N+1}$.

7.3 Propagation of vacua in the raviolo case

When $M_N = \mathbb{C}$ is the trivial module, there is an isomorphism, of dg $\mathcal{A}_N$-modules, between our initial space of coinvariants with N marked points and one with only $N-1$ marked points:

$$\mathsf{F}(\mathfrak{g}; \mathcal{A}_N; M_1, \ldots, M_{N-2}, \mathbb{C}, \mathbb{C}) \cong \mathsf{F}(\mathfrak{g}; \mathcal{A}_N; M_1, \ldots, M_{N-2}, \mathbb{C}).$$

One has the equality

$$\left[m_1 \otimes \cdots \otimes m_{N-2} \underset{z_{N-1}}{\otimes} B \underset{z_N}{\otimes} |0\rangle\right] = \left[m_1 \otimes \cdots \otimes m_{N-2} \underset{z_{N-1}}{\otimes} B\right], \tag{40}$$

which is the *propagation of vacua* property in the raviolo case.

7.4 Completion of the proof of Theorem 20

The equality in (40) establishes the statement of Theorem 20 in the special case that $A = |0\rangle$ is the vacuum state.

Next we shall show that for all states $A \in \mathcal{V}$ in the raviolo vacuum module $\mathcal{V}$, the class

$$\left[m_1 \otimes \cdots \otimes m_{N-2} \underset{z_{N-1}}{\otimes} B \underset{z_N}{\otimes} A\right] \in \mathcal{A}_N \otimes M$$

has a representative of the form

$$\sum_i \left[A_i^{(-)}.(m_1 \otimes \cdots \otimes m_{N-2}) \otimes A_i^{(+)}. B \underset{z_N}{\otimes} |0\rangle\right]$$

for some finite sum over i and for certain $A_i^{(-)}$ and $A_i^{(+)}$ belonging to $U(\mathfrak{g} \otimes \mathbb{C}\{\{w - z_N\}\}_-)$. Recall the algebra map

$$g_N : \mathcal{A}_N\{\{w - z_N\}\}_- \to \mathcal{A}'_{N+1}$$

from (26). When we write $A_i^{(-)}.(m_1 \otimes \ldots \otimes m_{N-2})$, the action is by definition via the map of dg Lie algebras

$$\iota_{\text{far}} : \mathfrak{g} \otimes \mathbb{C}\{\!\{w - z_N\}\!\}_- \hookrightarrow \mathfrak{g} \otimes \mathcal{A}_N\{\!\{w - z_N\}\!\}_-$$

$$\xrightarrow{g_N} \mathfrak{g} \otimes \mathcal{A}'_{N+1} \xrightarrow{(\iota_{w \to z_1}, \ldots, \iota_{w \to z_{N-2}})} \bigoplus_{i=1}^{N-2} \mathfrak{g} \otimes \mathcal{A}_N\{\!\{w - z_i\}\!\}_+,$$

which we continue to call ι_{far}. Similarly when we write $A_i^{(+)}.B$, the action is by definition via the map of dg Lie algebras

$$\iota_{\text{near}} : \mathfrak{g} \otimes \mathbb{C}\{\!\{w - z_N\}\!\}_- \hookrightarrow \mathfrak{g} \otimes \mathcal{A}_N\{\!\{w - z_N\}\!\}_-$$

$$\xrightarrow{g_N} \mathfrak{g} \otimes \mathcal{A}'_{N+1} \xrightarrow{\iota_{w \to z_{N-1}}} \mathfrak{g} \otimes \mathcal{A}_N\{\!\{w - z_{N-1}\}\!\}_+.$$

Indeed, we may suppose

$$A = X^1 \cdots X^n |0\rangle$$

for some number $n \geq 0$ of elements $X^i \in \mathfrak{g} \otimes \mathbb{C}\{\!\{w - z_N\}\!\}_-$, $1 \leq i \leq n$. By a straightforward induction on n, one checks that

$$\left[m_1 \otimes \cdots \otimes m_{N-2} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{A} \right]$$
$$= \sum_{m=0}^{n} \sum_{(\mu,\nu) \in \text{Unshf}_m^n} (-1)^{n+\chi} \left[(\iota_{\text{far}} \overleftarrow{A}_\mu)(m_1 \otimes \cdots \otimes m_{N-2}) \otimes (\iota_{\text{near}} \overleftarrow{A}_\nu) \underset{z_{N-1}}{B} \otimes \underset{z_N}{Y |0\rangle} \right] \quad (41)$$

where the inner sum is over unshuffles, as defined above after (37), and where we write

$$\overleftarrow{A}_\mu := X^{\mu_m} \cdots X^{\mu_1}, \qquad \overleftarrow{A}_\nu := X^{\nu_{n-m}} \cdots X^{\nu_1}.$$

In the expression above $(-1)^\chi$ denotes the appropriate Koszul sign coming from the braiding of the tensor product; it is implicitly a function of the grades of the factors X^i and of the states m_j (all of which without loss of generality we shall assume are homogeneous) and on the unshuffle (μ, ν). We don't need to work it out explicitly at this stage—many of the terms will cancel out in the next swapping step below; in particular the dependence on the $|m_j|$ will drop out.

By propagation of vacua as in (40), we may regard the right-hand side in (41) as an element of the space of coinvariants $F(\mathfrak{g}; \mathcal{A}_N; M_1, \ldots, M_{N-2}, \mathbb{C}, \mathbb{C})$. We may apply the change-of-base map $\iota_{\substack{\text{base} \\ \text{change}}}$ to both sides to obtain the equality

$$\iota_{\substack{\text{base} \\ \text{change}}} \left[m_1 \otimes \cdots \otimes m_{N-2} \otimes \underset{z_{N-1}}{B} \otimes \underset{z_N}{A} \right]$$

$$= \sum_{m=0}^{n} \sum_{(\mu,\nu) \in \text{Unshf}_m^n} (-1)^{n+\chi} \iota_{\substack{\text{base} \\ \text{change}}} \left[(\iota_{\text{far}} \overleftarrow{A}_\mu)(m_1 \otimes \cdots \otimes m_{N-2}) \otimes (\iota_{\text{near}} \overleftarrow{A}_\nu) \underset{z_{N-1}}{B} \right]$$

$$= \sum_{m=0}^{n} \sum_{(\mu,\nu) \in \text{Unshf}_m^n} (-1)^{n+\chi} \left[\iota_{\substack{\text{base} \\ \text{change}}} (\iota_{\text{far}} \overleftarrow{A}_\mu)(m_1 \otimes \cdots \otimes m_{N-2}) \otimes \iota_{\substack{\text{base} \\ \text{change}}} (\iota_{\text{near}} \overleftarrow{A}_\nu) \underset{z_{N-1}}{B} \right]$$

$$= \sum_{m=0}^{n} \sum_{(\mu,\nu) \in \text{Unshf}_m^n} (-1)^{n+\chi} \left[(\iota_{\substack{\text{base} \\ \text{change}}} \iota_{\text{far}} \overleftarrow{A}_\mu) \iota_{\substack{\text{base} \\ \text{change}}} (m_1 \otimes \cdots \otimes m_{N-2}) \otimes \iota_{\substack{\text{base} \\ \text{change}}} (\iota_{\text{near}} \overleftarrow{A}_\nu) \underset{z_{N-1}}{B} \right]. \quad (42)$$

In the second step, we used Lemma 22.

The space of coinvariants $F(\mathfrak{g}, \mathcal{A}_{N-1}\{\!\{z_N - z_{N-1}, M_1, \ldots, M_{N-2}, \mathbb{C}, \mathbb{C})$ is a quotient of the free module $\mathcal{A}_{N-1}\{\!\{z_N - z_{N-1}\}\!\} \otimes M$, cf. (29). Consider a representative in that free module. An element of $\mathcal{A}_{N-1}\{\!\{z_N - $

z_{N-1}}} $\otimes M$ is by definition an element of

$$\mathcal{A}_{N-1}((z_N - z_{N-1})) \otimes \mathbb{C}[v, dv] \otimes M$$

whose pullbacks to $v = 0$ and $v = 1$ are regular in $z_N - z_{N-1}$. So it is a Laurent series in $z_N - z_{N-1}$ (albeit one obeying certain extra conditions), and thus to specify it it is enough to give the coefficient of $(z_N - z_{N-1})^k$ for every $k \in \mathbb{Z}$. So, let $k \in \mathbb{Z}$ and consider the coefficient of $(z_N - z_{N-1})^k$ in the expression in (42). Consider any term (μ, v) in the sum. We still have (In more detail: it is still the case that $\iota_{w \to z_s}$ and $\iota_{z_N \to z_{N-1}}$ commute whenever $s \leq N - 2$, where these are now the expansion maps defined as in Definition-Proposition10. Indeed, one checks that the maps $p^*_{N+1 \to s}$ and $p^*_{N \to N-1}$ (the latter defined by obvious analogy with the former) commute for all $s \leq N - 2$. It is interesting to note also that they do *not* commute for $s = N - 1$.)

$$\iota_{\substack{\text{base} \\ \text{change}}} \iota_{\text{far}} \overleftarrow{A}_\mu = \iota_{z_N \to z_{N-1}} \iota_{\text{far}} \overleftarrow{A}_\mu = \iota_{\text{far}} \iota_{z_N \to z_{N-1}} \overleftarrow{A}_\mu.$$

By smoothness of the vacuum module $\mathcal{V}$, we have that

$$\iota_{\text{near}} \overleftarrow{A}_v B \in \mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}_- \otimes \mathcal{V}.$$

(To stress the point: on grading grounds, at most finitely many terms in the series $\iota_{\text{near}} X \in \mathfrak{g} \otimes \mathcal{A}_N[[w - z_{N-1}]]$ are nonzero when acting on B, for any given state $B \in \mathcal{V}$ and $X \in \mathfrak{g} \otimes \mathbb{C}\{\{w - z_N\}\}$. It follows that there is a lower bound on the powers of $z_N - z_{N-1}$ that appear in $\iota_{\text{near}} \overleftarrow{A}_v B$. This logic is exactly as in the usual case discussed in §7.1.)

Therefore only finitely many terms in the series $\iota_{z_N \to z_{N-1}} \overleftarrow{A}_\mu$ contribute to the overall coefficient of $(z_N - z_{N-1})^k$. The coefficients of these finitely many terms belong to $U(\mathfrak{g} \otimes \mathcal{A}_{N-1}\{\{z_N - z_{N-1}\}\}_-)$, and we can swap them over to the module at the marked point z_{N-1}, by definition of the space of coinvariants. After doing so we obtain

$$(-1)^{|A||B|} \big[m_1 \otimes \cdots \otimes m_{N-2} \otimes Y_{\text{Rav}}(A, z_N - z_{N-1}) B \big]$$
$$\underset{z_1}{\uparrow} \qquad\qquad \underset{z_{N-2}}{\uparrow} \qquad\qquad\qquad \underset{z_{N-1}}{\uparrow}$$

where we recognize the expression for the raviolo state-field map Y_{Rav} from Lemma 1. This completes the proof of Theorem 20.

Acknowledgments

Appendix A. Semisimplicial Objects and the Thom–Sullivan Functor

A.1 Semisimplicial objects

Let Δ denote the category whose objects are the finite nonempty totally-ordered sets

$$[n] := \{0 < 1 < \cdots < n\}, \qquad n \in \mathbb{Z}_{\geq 0},$$

and whose morphisms are the strictly order-preserving maps $\theta : [n] \to [N]$. Such maps are generated by *coface maps*,

$$d_j : [n] \to [n+1]; \quad i \mapsto \begin{cases} i & i < j \\ i+1 & i \geq j \end{cases} \quad \text{for} \quad j = 0, 1, \ldots, n+1.$$

A *semicosimplicial object* A in a category $\mathcal{C}$ is a functor $A : \Delta \to \mathcal{C}$. Similarly, a *semisimplicial object* Z in a category $\mathcal{C}$ is a functor $Z : \Delta^{\mathrm{op}} \to \mathcal{C}$. The maps $\partial_i^n := Z(d_i^n) : Z([n+1]) \to Z([n])$ are the *face maps* of Z. One thinks of the category Δ as follows:

$$\cdots [2] \mathrel{\substack{\textstyle\longleftarrow \\ \textstyle\longleftarrow \\ \textstyle\longleftarrow}} [1] \mathrel{\substack{\textstyle\longleftarrow \\ \textstyle\longleftarrow}} [0]$$

and so a semisimplicial object Z in $\mathcal{C}$ defines a diagram in $\mathcal{C}$ of the form

$$\cdots Z([2]) \mathrel{\substack{\textstyle\longrightarrow \\ \textstyle\longrightarrow \\ \textstyle\longrightarrow}} Z([1]) \mathrel{\substack{\textstyle\longrightarrow \\ \textstyle\longrightarrow}} Z([0]).$$

A.2 Polynomial differential forms on the standard algebro-geometric simplex

There is a simplicial dg commutative algebra

$$\Omega : \Delta^{\mathrm{op}} \to \mathbf{Alg}^{Com}(\mathbf{dgVect}_{\mathbb{C}})$$

defined as follows. For each $n \geq 0$, $\Omega([n])$ is the dg commutative algebra

$$\Omega([n]) := \mathbb{C}[t_0, \dots, t_n; dt_0, \dots dt_n] / \langle \sum_{i=0}^{n} t_i - 1, \sum_{i=0}^{n} dt_i \rangle$$

with t_i in degree 0 and dt_i in degree 1, for each i, and equipped with the usual de Rham differential. For any map $\phi : [n] \to [N]$ of Δ,

$$\Omega(\phi) : \Omega([N]) \to \Omega([n])$$

is the map of dg commutative algebras defined by $t_i \mapsto \sum_{j \in \phi^{-1}(i)} t_j$. One should think of $\Omega([n])$ as the complex of polynomial differential forms on the *standard algebro-geometric n-simplex*,

$$\Delta_{\mathbb{C}}^n := \operatorname{Spec} \mathbb{C}[t_0, \dots, t_n] / \langle \sum_{i=0}^{n} t_i - 1 \rangle \hookrightarrow \mathbb{A}_{\mathbb{C}}^{n+1}.$$

A.3 The functor Th

Suppose we are given a functor $A : \Delta \to \mathbf{Alg}^{Com}(\mathbf{Vect}_{\mathbb{C}})$; that is, suppose we are given a semicosimplicial object in commutative algebras in vector spaces. One can construct a commutative algebra in dg vector spaces, given by the graded vector space

$$\mathrm{Th}^{\bullet}(A) := \Bigg\{ \mathbf{a} = (a_n)_{n \geq 0} \in \prod_{n \geq 0} A([n]) \otimes \Omega^{\bullet}([n]) :$$

$$\big(A(\phi) \otimes \mathrm{id} \otimes\big) a_n = \big(\mathrm{id} \otimes \Omega(\phi)\big) a_m \quad \text{in} \quad \Omega([n]) \otimes A([m])$$

$$\text{for all maps } \phi : [n] \to [m] \text{ of } \Delta \Bigg\}, \tag{A.1}$$

equipped with the de Rham differential $\mathbf{d} = \mathbf{d}_{\text{de Rham}} \otimes \mathrm{id}$ and the graded commutative product given by $(\omega \otimes a)(\tau \otimes b) := \omega \wedge \tau \otimes ab$. This defines the action on objects of a functor, called the *Thom–Sullivan* [30] or *Thom–Whitney* [22] functor, from semicosimplicial commutative algebras to dg commutative algebras,

$$\mathrm{Th} : [\Delta, \mathbf{Alg}^{Com}(\mathbf{Vect}_{\mathbb{C}})] \to \mathbf{Alg}^{Com}(\mathbf{dgVect}_{\mathbb{C}}).$$

There is a quasi-isomorphism of dg vector spaces $\int : \mathrm{Th}^{\bullet}(A) \xrightarrow{\sim} \mathcal{C}^{\bullet}(A)$ to the unnormalized cochain complex $\mathcal{C}^{\bullet}(A)$ associated A, namely the complex with $C^n(A) := A([n])$ for $n \geq 0$, $C^n(A) = 0$ for $n < 0$, and

differential $\mathsf{d}_{\mathcal{C}}^n := \sum_{j=0}^{n+1}(-1)^j d_j$. (This quasi-isomorphism is defined by integrating over the simplices; see [30,§5.2.6].)

(By suitably totalizing, the definition $\mathsf{Th}^\bullet$ extends to a functor from semicosimplicial dg commutative algebras to dg commutative algebras, which is how $\mathsf{Th}^\bullet$ is more commonly presented; but the semicosimplicial commutative algebras we encounter in the present paper are all concentrated in degree 0, so (A.1) suffices for our purposes.)

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