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$(\mathfrak{gl}_M, \mathfrak{gl}_N)$-Dualities in Gaudin Models with Irregular Singularities

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Abstract. We establish $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-dualities between quantum Gaudin models with irregular singularities. Specifically, for any $M, N \in \mathbb{Z}_{\geq 1}$ we consider two Gaudin models: the one associated with the Lie algebra $\mathfrak{gl}_M$ which has a double pole at infinity and N poles, counting multiplicities, in the complex plane, and the same model but with the roles of M and N interchanged. Both models can be realized in terms of Weyl algebras, i.e., free bosons; we establish that, in this realization, the algebras of integrals of motion of the two models coincide. At the classical level we establish two further generalizations of the duality. First, we show that there is also a duality for realizations in terms of free fermions. Second, in the bosonic realization we consider the classical cyclotomic Gaudin model associated with the Lie algebra $\mathfrak{gl}_M$ and its diagram automorphism, with a double pole at infinity and $2N$ poles, counting multiplicities, in the complex plane. We prove that it is dual to a non-cyclotomic Gaudin model associated with the Lie algebra $\mathfrak{sp}_{2N}$, with a double pole at infinity and M simple poles in the complex plane. In the special case $N = 1$ we recover the well-known self-duality in the Neumann model.

Key words: Gaudin models; dualities; irregular singularities

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1 Introduction

Fix a set of N distinct complex numbers $\{z_i\}_{i=1}^N \subset \mathbb{C}$, and an element $\lambda \in \mathfrak{gl}_M^*$. The quadratic Hamiltonians of the quantum Gaudin model [12, 13] associated to $\mathfrak{gl}_M$ are the following elements of $U(\mathfrak{gl}_M)^{\otimes N}$:

$$\mathcal{H}_i = \sum_{j \neq i} \sum_{a,b=1}^N \frac{\mathsf{E}_{ab}^{(i)} \mathsf{E}_{ba}^{(j)}}{z_i - z_j} + \sum_{a,b=1}^N \lambda(\mathsf{E}_{ab}) \mathsf{E}_{ba}^{(i)},$$

where $\{\mathsf{E}_{ab}\}_{a,b=1}^M$ denote the standard basis of $\mathfrak{gl}_M$ and $\mathsf{E}_{ab}^{(i)}$ means E_{ab} in the ith tensor factor. The $\mathcal{H}_i$ belong to a large commutative subalgebra $\mathcal{Z} \subset U(\mathfrak{gl}_M)^{\otimes N}$ called the Gaudin [11] or Bethe [19] subalgebra, for which an explicit set of generators is known [6, 19, 31].

If the element $\lambda \in \mathfrak{gl}_M^*$ is regular semisimple, i.e., if we can choose bases such that $\lambda(\mathsf{E}_{ab}) = \lambda_a \delta_{ab}$ for some distinct numbers $\{\lambda_a\}_{a=1}^M \subset \mathbb{C}$, then one can also consider the following elements of $U(\mathfrak{gl}_N)^{\otimes M}$:

$$\widetilde{\mathcal{H}}_a = \sum_{b \neq a} \sum_{i,j=1}^M \frac{\tilde{\mathsf{E}}_{ij}^{(a)} \tilde{\mathsf{E}}_{ji}^{(b)}}{\lambda_a - \lambda_b} + \sum_{i=1}^M z_i \tilde{\mathsf{E}}_{ii}^{(a)},$$

where $\{\tilde{\mathsf{E}}_{ij}\}_{i,j=1}^N$ denote the standard basis of $\mathfrak{gl}_N$. They belong to a large commutative subalgebra $\widetilde{\mathcal{Z}} \subset U(\mathfrak{gl}_N)^{\otimes M}$.

Let $\mathbb{C}^M$ denote the defining representation of $\mathfrak{gl}_M$. Then $\mathcal{Z}$ can be represented as a subalgebra of

$$\mathrm{End}\left(\left(\mathbb{C}^M\right)^{\otimes N}\right) \cong \mathrm{End}\left(\mathbb{C}^{NM}\right) \cong \mathrm{End}\left(\left(\mathbb{C}^N\right)^{\otimes M}\right).$$

So can $\widetilde{\mathcal{Z}}$. In fact their images in $\mathrm{End}\left(\mathbb{C}^{NM}\right)$ coincide. This is the $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality for quantum Gaudin models first observed between the quadratic Gaudin Hamiltonians and the dynamical Hamiltonians in [33], see also [32]. It was later proved in [21], see also [4]. (Under this realization the Hamiltonians $\widetilde{\mathcal{H}}_a \in \widetilde{\mathcal{Z}}$ of the dual model coincide with suitably defined *dynamical Hamiltonians* [10] of the original $\mathfrak{gl}_M$ Gaudin model. See [20, 21].) The classical counterpart of this duality goes back to the works of J. Harnad [1, 14].

In this paper we generalize this $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality in a number of ways, for both the quantum and classical Gaudin models. Let us describe first the main result. Two natural generalizations of the Gaudin model above are to

(a) models in which the quadratic Hamiltonians (and the Lax matrix, see below) have higher order singularities at the marked points $z_i \in \mathbb{C}$, $i = 1, \dots, N$. Such models are called Gaudin models *with irregular singularities.*[1]

(b) models in which $\lambda \in \mathfrak{gl}_M^*$ is not semisimple, i.e., has non-trivial Jordan blocks.[2]

We show that these two generalizations are natural $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duals to one another. Namely, we show that there is a correspondence among models generalized in both directions, (a) and (b), and that under this correspondence the sizes of the Jordan blocks get exchanged with the degrees of the irregular singularities at the marked points in the complex plane. See Theorem 4.8 below.

The heart of the proof is the observation that the generating functions for the generators of both algebras $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$ can be obtained by evaluating, in two different ways, the column-ordered determinant of a certain Manin matrix. (A similar trick was also used in [4, Proposition 8].) Given that observation, the duality between (a) and (b) above is essentially a consequence of the simple fact that the inverse of a Jordan block matrix

$$\begin{pmatrix} x & 0 & \dots & 0 \\ -1 & x & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -1 & x \end{pmatrix} \qquad \text{is of the form} \qquad \begin{pmatrix} x^{-1} & 0 & \dots & 0 \\ x^{-2} & x^{-1} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ x^{-k} & \dots & x^{-2} & x^{-1} \end{pmatrix};$$

here the higher-order poles in x will give rise to the irregular singularities of the dual Gaudin model.

Now let us give an overview of the results of the paper in more detail. Consider the direct sum of Lie algebras

$$\mathfrak{gl}_M^{(N)} := \bigoplus_{i=1}^N \mathfrak{gl}_M \oplus \mathfrak{gl}_M^{\mathrm{com}}, \tag{1.1}$$

[1]The reason for this terminology is that the spectrum of such models is described in terms of opers with irregular singularities; see [9] and also [35]. Strictly speaking, the term $\lambda(\mathsf{E}_{ab})\mathsf{E}_{ab}$ in $\mathcal{H}_i$ is already an irregular singularity of order 2 at ∞ in the same sense: namely, the opers describing the spectrum have a double pole at ∞. For that reason we refer to a Gaudin model with such terms in the Hamiltonians $\mathcal{H}_i$ as having a double pole at infinity.

[2]Let us note in passing that the case of λ semisimple but not regular is very rich; see for example [8, 23, 24].

where the Lie algebra $\mathfrak{gl}_M^{\rm com}$ in the last summand is isomorphic to $\mathfrak{gl}_M$ as a vector space but endowed with the trivial Lie bracket. Henceforth we denote the copy of E_{ab} in the $i^{\rm th}$ direct summand of $\mathfrak{gl}_M^{(N)}$ by $\mathsf{E}_{ab}^{(z_i)}$ and the copy in the last abelian summand $\mathfrak{gl}_M^{\rm com}$ by $\mathsf{E}_{ab}^{(\infty)}$. In terms of these data, the formal Lax matrix of the Gaudin model associated with $\mathfrak{gl}_M$, with a double pole at infinity and simple poles at each z_i, $i = 1, \dots, N$, is given by

$$\mathcal{L}(z)dz \coloneqq \sum_{a,b=1}^{M} E_{ba} \otimes \left(\mathsf{E}_{ab}^{(\infty)} + \sum_{i=1}^{N} \frac{\mathsf{E}_{ab}^{(z_i)}}{z - z_i} \right) dz. \tag{1.2}$$

Here $E_{ab} \coloneqq \rho(\mathsf{E}_{ab})$ where $\rho\colon \mathfrak{gl}_M \to \mathrm{Mat}_{M\times M}(\mathbb{C})$ is the defining representation.

Regarding $\mathcal{L}(z)$ as an $M \times M$ matrix with entries in the symmetric algebra $S\big(\mathfrak{gl}_M^{(N)}\big)$, the coefficients of its characteristic polynomial

$$\det\big(\lambda \mathbf{1}_{M\times M} - \mathcal{L}(z)\big)$$

span a large Poisson commutative subalgebra $\mathscr{Z}^{\rm cl}_{(z_i)}\big(\mathfrak{gl}_M^{(N)}\big)$ of $S\big(\mathfrak{gl}_M^{(N)}\big)$. Given a classical model described by a Poisson algebra $\mathcal{P}$ and Hamiltonian $H \in \mathcal{P}$, the latter becomes of particular interest if we have a homomorphism of Poisson algebras $\pi\colon S\big(\mathfrak{gl}_M^{(N)}\big) \to \mathcal{P}$ such that H lies in the image of $\mathscr{Z}^{\rm cl}_{(z_i)}\big(\mathfrak{gl}_M^{(N)}\big)$. Indeed, $\pi\big(\mathscr{Z}^{\rm cl}_{(z_i)}\big(\mathfrak{gl}_M^{(N)}\big)\big) \subset \mathcal{P}$ then consists of Poisson commuting integrals of motion of the model.

The Lax matrix (1.2) can also be used to describe quantum models by regarding it instead as an $M \times M$ matrix with entries in the universal enveloping algebra $U\big(\mathfrak{gl}_M^{(N)}\big)$. In this case, a large commutative subalgebra $\mathscr{Z}_{(z_i)}\big(\mathfrak{gl}_M^{(N)}\big) \subset U\big(\mathfrak{gl}_M^{(N)}\big)$, called the *Gaudin algebra*, is spanned by the coefficients in the partial fraction decomposition of the rational functions obtained as the coefficients of the differential operator

$$\mathrm{cdet}\big(\partial_z \mathbf{1}_{M\times M} - {}^t\mathcal{L}(z)\big),$$

where cdet is the column ordered determinant. Given a unital associative algebra $\mathcal{U}$ and a homomorphism $\hat{\pi}\colon U\big(\mathfrak{gl}_M^{(N)}\big) \to \mathcal{U}$, the image of $\mathscr{Z}_{(z_i)}\big(\mathfrak{gl}_M^{(N)}\big)$ provides a large commutative subalgebra of $\mathcal{U}$.

Let $\mathcal{U}$ be the Weyl algebra generated by the commuting variables x_i^a for $i = 1, \dots, N$ and $a = 1, \dots, M$ together with their partial derivatives $\partial_i^a \coloneqq \partial/\partial x_i^a$. We introduce another set $\{\lambda_a\}_{a=1}^M \subset \mathbb{C}$ of M distinct complex numbers. It is well known that

$$\hat{\pi}\big(\mathsf{E}_{ab}^{(\infty)}\big) = \lambda_a \delta_{ab}, \qquad \hat{\pi}\big(\mathsf{E}_{ab}^{(z_i)}\big) = x_i^a \partial_i^b \tag{1.3}$$

defines a homomorphism $\hat{\pi}\colon U\big(\mathfrak{gl}_M^{(N)}\big) \to \mathcal{U}$. Therefore, in particular, $\hat{\pi}\big(\mathscr{Z}_{(z_i)}\big(\mathfrak{gl}_M^{(N)}\big)\big)$ is a commutative subalgebra of $\mathcal{U}$. On the other hand, given the new set of complex numbers λ_a, $a = 1, \dots, M$, we may now equally consider the Gaudin model associated with $\mathfrak{gl}_N$, with a double pole at infinity and simple poles at each λ_a for $a = 1, \dots, M$. Its formal Lax matrix is defined as in (1.2), explicitly we let

$$\tilde{\mathcal{L}}(\lambda)d\lambda \coloneqq \sum_{i,j=1}^{N} \tilde{E}_{ji} \otimes \left(\tilde{\mathsf{E}}_{ij}^{(\infty)} + \sum_{a=1}^{M} \frac{\tilde{\mathsf{E}}_{ij}^{(\lambda_a)}}{\lambda - \lambda_a} \right) d\lambda.$$

We can define another homomorphism $\hat{\tilde{\pi}}\colon U\big(\mathfrak{gl}_N^{(M)}\big) \to \mathcal{U}$ as

$$\hat{\tilde{\pi}}\big(\tilde{\mathsf{E}}_{ij}^{(\infty)}\big) = z_i \delta_{ij}, \qquad \hat{\tilde{\pi}}\big(\tilde{\mathsf{E}}_{ij}^{(\lambda_a)}\big) = \partial_j^a x_i^a.$$

(Note here the order between ∂_j^a and x_i^a as compared, for instance, to [20, Section 5.1] where $\tilde{\mathsf{E}}_{ij}^{(\lambda_a)}$ is realised as $x_i^a \partial_j^a$.) The $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality between the above two Gaudin models associated with $\mathfrak{gl}_M$ and $\mathfrak{gl}_N$ can be formulated, in the present conventions, as the equality of differential polynomials

$$\hat{\pi}\left(\prod_{i=1}^{N}(z-z_i)\,\mathrm{cdet}\left(\partial_z \mathbf{1}_{M\times M} - {}^t\mathcal{L}(z)\right)\right) = \hat{\tilde{\pi}}\left(\prod_{a=1}^{M}(\partial_z-\lambda_a)\,\mathrm{cdet}\left(z\mathbf{1}_{N\times N} - \tilde{\mathcal{L}}(\partial_z)\right)\right),$$

whose coefficients are $\mathcal{U}$-valued polynomials in z. (See Section 4.2 for the precise definition of the expression appearing on the right hand side.) In the classical setting discussed above the same identity holds with ∂_z replaced everywhere by the spectral parameter λ, the Weyl algebra $\mathcal{U}$ is replaced by the Poisson algebra $\mathcal{P}$ defined as the polynomial algebra in the canonically conjugate variables (p_i^a, x_i^a) and column ordered determinants replaced by ordinary determinants.

We generalise this statement in a number of directions. Firstly, in both the classical and quantum cases, we consider Gaudin models with irregular singularities. Specifically, fix a positive integer $n \in \mathbb{Z}_{\geq 1}$ and let $\{\tau_i\}_{i=1}^n \subset \mathbb{Z}_{\geq 1}$ be such that $\sum\limits_{i=1}^{n}\tau_i = N$. We consider a $\mathfrak{gl}_M$-Gaudin model with a double pole at infinity and an irregular singularity of order τ_i at each z_i for $i = 1, \ldots, n$. The direct sum of Lie algebras (1.1) is replaced in this case by a direct sum of Takiff Lie algebras[3]

$$\mathfrak{gl}_M^{\mathcal{D}} \coloneqq \bigoplus_{i=1}^{n}\left(\mathfrak{gl}_M[\varepsilon]/\varepsilon^{\tau_i}\mathfrak{gl}_M[\varepsilon]\right) \oplus \mathfrak{gl}_M^{\mathrm{com}}, \tag{1.4}$$

where $\mathcal{D}$ is a divisor encoding the collection of points z_i for $i = 1, \ldots, n$ weighted by the integers τ_i for $i = 1, \ldots, n$. The formal Lax matrix $\mathcal{L}(z)$ of this Gaudin model is an $M \times M$ matrix with entries in the Lie algebra $\mathfrak{gl}_M^{\mathcal{D}}$, and the Gaudin algebra $\mathscr{Z}_{(z_i)}(\mathfrak{gl}_M^{\mathcal{D}})$ is spanned by the coefficients in the partial fraction decomposition of the rational functions obtained as the coefficients of the differential operator

$$\mathrm{cdet}\left(\partial_z \mathbf{1}_{M\times M} - {}^t\mathcal{L}(z)\right). \tag{1.5}$$

Let $\mathcal{U}$ be the same unital associative algebra as above. In order to define a suitable homomorphism $\hat{\pi}\colon U\left(\mathfrak{gl}_M^{\mathcal{D}}\right) \to \mathcal{U}$ we combine representations of the Takiff Lie algebras $\mathfrak{gl}_M[\varepsilon]/\varepsilon^{\tau_i}\mathfrak{gl}_M[\varepsilon] \to \mathcal{U}$ for each $i = 1, \ldots, n$, naturally generalising the representation $\mathfrak{gl}_M \to \mathcal{U}$, $\mathsf{E}_{ab} \mapsto x_i^a \partial_i^b$ in the above regular singularity case, together with a constant homomorphism $\mathfrak{gl}_M^{\mathrm{com}} \to \mathbb{C}1 \subset \mathcal{U}$. As before, the choice of the latter is what determines the position of the poles of the dual $\mathfrak{gl}_N$-Gaudin model. In fact, if instead of choosing a diagonal matrix as in (1.3) we let

$$\left(\hat{\pi}\left(\mathsf{E}_{ab}^{(\infty)}\right)\right)_{a,b=1}^{M} = \begin{pmatrix} \lambda_1 & & & & & & & & \\ 1 & \lambda_1 & & & & & 0 & & \\ & \ddots & \ddots & & & & & & \\ & & 1 & \lambda_1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_m & & & \\ & & & & & 1 & \lambda_m & & \\ & 0 & & & & & \ddots & \ddots & \\ & & & & & & & 1 & \lambda_m \end{pmatrix}$$

[3]These were introduced in the mathematics literature in [29] but have also been widely used in the mathematical physics literature though not by this name, see for instance [22].

be a direct sum of m Jordan blocks of size $\tilde{\tau}_a \in \mathbb{Z}_{\geq 1}$ with $\lambda_a \in \mathbb{C}$ along the diagonal for $a = 1, \dots, m$, such that $\sum\limits_{a=1}^{m} \tilde{\tau}_a = M$, then the dual Gaudin model associated with $\mathfrak{gl}_N$ will have a double pole at infinity and an irregular singularity at each λ_a of order $\tilde{\tau}_a$ for $a = 1, \dots, m$. Let $\tilde{\mathcal{D}}$ be the divisor corresponding to these data and $\mathfrak{gl}_N^{\tilde{\mathcal{D}}}$ the associated direct sum of Takiff algebras, cf. (1.4). After defining a corresponding homomorphism $\hat{\tilde{\pi}} \colon U\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big) \to \mathcal{U}$ for this Gaudin model, we prove a $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality similar to the one stated above for the regular singularity case, see Theorem 4.8. As before, a similar result also holds in the classical setting where π and $\tilde{\pi}$ in this case are homomorphisms from the symmetric algebras $S\big(\mathfrak{gl}_M^{\mathcal{D}}\big)$ and $S\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big)$, respectively, to the Poisson algebra $\mathcal{P}$, see Theorem 3.2.

In the classical setup of Section 3 we also consider fermionic generalisations of $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality. Specifically, for the Poisson algebra $\mathcal{P}$ we take instead the even part of the $\mathbb{Z}_2$-graded Poisson algebra generated by canonically conjugate Grassmann variable pairs (π_i^a, ψ_i^a). The corresponding homomorphisms of Poisson algebras $\pi_{\mathsf{f}} \colon S\big(\mathfrak{gl}_M^{\mathcal{D}}\big) \to \mathcal{P}$ and $\tilde{\pi}_{\mathsf{f}} \colon S\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big) \to \mathcal{P}$ are defined in Lemma 3.3. In this case we establish a different type of $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality between the same Gaudin models with irregular singularities and associated with $\mathfrak{gl}_M$ and $\mathfrak{gl}_N$ as above. Denoting by $\mathcal{L}(z)$ and $\tilde{\mathcal{L}}(\lambda)$ their respective Lax matrices, it takes the form

$$\pi_{\mathsf{f}}\big(\det\big(\lambda \mathbf{1}_{M\times M} - \mathcal{L}(z)\big)\big)\tilde{\pi}_{\mathsf{f}}\big(\det\big(z \mathbf{1}_{N\times N} - \tilde{\mathcal{L}}(\lambda)\big)\big) = \prod_{i=1}^{n} (z - z_i)^{\tau_i} \prod_{a=1}^{m} (\lambda - \lambda_a)^{\tilde{\tau}_a}.$$

See Theorem 3.4, the proof of which is completely analogous to that of Theorem 3.2 in the bosonic setting, using basic properties of the Berezinian of an $(M|N) \times (M|N)$ supermatrix. We leave the possible generalisation of such a fermionic $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality to the quantum setting for future work.

Finally, in Section 5 we consider extensions of these results to cyclotomic Gaudin models also in the classical setting. Specifically, we consider a $\mathbb{Z}_2$-cyclotomic $\mathfrak{gl}_M$-Gaudin model with a double pole at infinity as usual and with irregular singularities at the origin of order τ_0 and at points $z_i \in \mathbb{C}^\times$, with disjoint orbits under $z \mapsto -z$, of order τ_i for each $i = 1, \dots, n$. Let $N = \tau_0 + \sum\limits_{i=1}^{n} \tau_i$. Using the bosonic Poisson algebra $\mathcal{P}$ generated by canonically conjugate variables (p_i^a, x_i^a) we prove that this model is dual to a Gaudin model associated with the Lie algebra $\mathfrak{sp}_{2N}$, with a double pole at infinity and regular singularities at M points λ_a, $a = 1, \dots, M$, see Theorem 5.2. We show that the well know self-duality in the Neumann model is a particular example of the latter with $N = 1$. Generalisations of such $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-dualities involving cyclotomic Gaudin models to the quantum case are less obvious since it is known [34] that in this case the cyclotomic Gaudin algebra is not generated by a cdet-type formula as in (1.5), see Remark 5.3.

2 Gaudin models with irregular singularities

2.1 Lie algebras $\mathfrak{gl}_M^{\mathcal{D}}$ and $\mathfrak{gl}_N^{\tilde{\mathcal{D}}}$

Let $M, N \in \mathbb{Z}_{\geq 1}$. Denote by E_{ab} for $a, b = 1, \dots, M$ the standard basis of $\mathfrak{gl}_M$ and by $\tilde{\mathsf{E}}_{ij}$ for $i, j = 1, \dots, N$ the standard basis of $\mathfrak{gl}_N$.

Let $z_i \in \mathbb{C}$ for $i = 1, \dots, n$ and $\lambda_a \in \mathbb{C}$ for $a = 1, \dots, m$ be such that $z_i \neq z_j$ for $i \neq j$ and $\lambda_a \neq \lambda_b$ for $a \neq b$. Pick and fix integers $\tau_i \in \mathbb{Z}_{\geq 1}$ for each $i = 1, \dots, n$ and $\tilde{\tau}_a \in \mathbb{Z}_{\geq 1}$ for each $a = 1, \dots, m$. We call these the *Takiff degrees* at z_i and λ_a, respectively. Consider the effective divisors

$$\mathcal{D} = \sum_{i=1}^{n} \tau_i \cdot z_i + 2 \cdot \infty, \qquad \tilde{\mathcal{D}} = \sum_{a=1}^{m} \tilde{\tau}_a \cdot \lambda_a + 2 \cdot \infty.$$

(Recall that an *effective divisor* is a finite formal linear combination of points in some Riemann surface, here the Riemann sphere $\mathbb{C} \cup \{\infty\}$, with coefficients in $\mathbb{Z}_{\geq 0}$.)

We require that $\deg \mathcal{D} = N + 2$ and $\deg \tilde{\mathcal{D}} = M + 2$ or in other words,

$$\sum_{i=1}^{n} \tau_i = N \qquad \text{and} \qquad \sum_{a=1}^{m} \tilde{\tau}_a = M.$$

Note that if $\tau_i = 1 = \tilde{\tau}_a$ for all $i = 1, \dots, n$ and $a = 1, \dots, m$ then in fact we have $n = N$ and $m = M$. More generally, it will be convenient to break up the list of integers from 1 to N into n blocks of sizes τ_i, $i = 1, \dots, n$, and similarly for the list of integers from 1 to M. To that end, let us define

$$\nu_i := \sum_{j=1}^{i-1} \tau_j, \qquad \text{and} \qquad \tilde{\nu}_a := \sum_{b=1}^{a-1} \tilde{\tau}_b \tag{2.1}$$

for $i = 1, \dots, N$ and $a = 1, \dots, M$, so that

$$(1, \dots, N) = (1, \dots, \tau_1; \nu_2 + 1, \dots, \nu_2 + \tau_2; \dots; \nu_n + 1, \dots, \nu_n + \tau_n),$$
$$(1, \dots, M) = (1, \dots, \tilde{\tau}_1; \tilde{\nu}_2 + 1, \dots, \tilde{\nu}_2 + \tilde{\tau}_2; \dots; \tilde{\nu}_m + 1, \dots, \tilde{\nu}_m + \tilde{\tau}_m).$$

Note that $\nu_1 = \tilde{\nu}_1 = 0$.

Let $\mathfrak{gl}_M[\varepsilon] := \mathfrak{gl}_M \otimes \mathbb{C}[\varepsilon]$ denote the Lie algebra of polynomials in a formal variable ε with coefficients in $\mathfrak{gl}_M$. For any $k \in \mathbb{Z}_{\geq 1}$ we have the ideal $\varepsilon^k \mathfrak{gl}_M[\varepsilon] := \mathfrak{gl}_M \otimes \varepsilon^k \mathbb{C}[\varepsilon]$. The corresponding quotient $\mathfrak{gl}_M[\varepsilon]/\varepsilon^k := \mathfrak{gl}_M[\varepsilon]/\varepsilon^k \mathfrak{gl}_M[\varepsilon]$ is called a *Takiff* Lie algebra over $\mathfrak{gl}_M$. When $k \in \mathbb{Z}_{\geq 2}$, for every $n \in \mathbb{Z}_{\geq 1}$ with $n < k$ we have a non-trivial ideal in $\mathfrak{gl}_M[\varepsilon]/\varepsilon^k$ given by $\varepsilon^n \mathfrak{gl}_M[\varepsilon]/\varepsilon^k := \varepsilon^n \mathfrak{gl}_M[\varepsilon]/\varepsilon^k \mathfrak{gl}_M[\varepsilon]$, which by abuse of terminology we shall also refer to as a Takiff Lie algebra. We define direct sums of Takiff Lie algebras over $\mathfrak{gl}_M$ and $\mathfrak{gl}_N$, respectively, as

$$\mathfrak{gl}_M^{\mathcal{D}} := \varepsilon_\infty \mathfrak{gl}_M[\varepsilon_\infty]/\varepsilon_\infty^2 \oplus \bigoplus_{i=1}^{n} \mathfrak{gl}_M[\varepsilon_{z_i}]/\varepsilon_{z_i}^{\tau_i},$$
$$\mathfrak{gl}_N^{\tilde{\mathcal{D}}} := \tilde{\varepsilon}_\infty \mathfrak{gl}_N[\tilde{\varepsilon}_\infty]/\tilde{\varepsilon}_\infty^2 \oplus \bigoplus_{a=1}^{m} \mathfrak{gl}_N[\tilde{\varepsilon}_{\lambda_a}]/\tilde{\varepsilon}_{\lambda_a}^{\tilde{\tau}_a}.$$

Note that $\varepsilon_\infty \mathfrak{gl}_M[\varepsilon_\infty]/\varepsilon_\infty^2$ and $\tilde{\varepsilon}_\infty \mathfrak{gl}_N[\tilde{\varepsilon}_\infty]/\tilde{\varepsilon}_\infty^2$ are respectively isomorphic to the abelian Lie algebras $\mathfrak{gl}_M^{\mathrm{com}}$ and $\mathfrak{gl}_N^{\mathrm{com}}$ in the notation used in the introduction, see, e.g., (1.1).

We use the abbreviated notation $\mathsf{X}\varepsilon^k$ for an element $\mathsf{X} \otimes \varepsilon^k \in \mathfrak{gl}_M[\varepsilon]$ where $\mathsf{X} \in \mathfrak{gl}_M$ and $k \in \mathbb{Z}_{\geq 0}$, and likewise for elements of $\mathfrak{gl}_N[\varepsilon]$. Fix a basis of $\mathfrak{gl}_M^{\mathcal{D}}$ defined by

$$\mathsf{E}_{ab[r]}^{(z_i)} := \mathsf{E}_{ab} \varepsilon_{z_i}^r, \qquad \mathsf{E}_{ab[1]}^{(\infty)} := \mathsf{E}_{ab} \varepsilon_\infty$$

for $i = 1, \dots, N$, $a, b = 1, \dots, M$ and $r = 0, \dots, \tau_i - 1$. Let us note, in particular, that $\mathsf{E}_{ab[r]}^{(z_i)} = 0$ whenever $r \geq \tau_i$. Likewise, as a basis of $\mathfrak{gl}_N^{\tilde{\mathcal{D}}}$ we take

$$\tilde{\mathsf{E}}_{ij[s]}^{(\lambda_a)} := \tilde{\mathsf{E}}_{ij} \tilde{\varepsilon}_{\lambda_a}^s, \qquad \tilde{\mathsf{E}}_{ij[1]}^{(\infty)} := \tilde{\mathsf{E}}_{ij} \tilde{\varepsilon}_\infty$$

for $a = 1, \dots, M$, $i, j = 1, \dots, N$ and $s = 0, \dots, \tilde{\tau}_a - 1$. Here also $\tilde{\mathsf{E}}_{ij[s]}^{(\lambda_a)} = 0$ for $s \geq \tilde{\tau}_a$.

The set of non-trivial Lie brackets of these basis elements read

$$\big[\mathsf{E}^{(z_i)}_{ab[r]}, \mathsf{E}^{(z_j)}_{cd[s]}\big] = \delta_{ij}[\mathsf{E}_{ab}, \mathsf{E}_{cd}]^{(z_i)}_{[r+s]} = \delta_{ij}\delta_{bc}\mathsf{E}^{(z_i)}_{ad[r+s]} - \delta_{ij}\delta_{ad}\mathsf{E}^{(z_i)}_{cb[r+s]}, \tag{2.2}$$

for any $i, j = 1, \dots, n$ and $a, b, c, d = 1, \dots, M$, and

$$\big[\tilde{\mathsf{E}}^{(\lambda_a)}_{ij[r]}, \tilde{\mathsf{E}}^{(\lambda_b)}_{kl[s]}\big] = \delta_{ab}[\tilde{\mathsf{E}}_{ij}, \tilde{\mathsf{E}}_{kl}]^{(\lambda_a)}_{[r+s]} = \delta_{ab}\delta_{jk}\tilde{\mathsf{E}}^{(\lambda_a)}_{il[r+s]} - \delta_{ab}\delta_{il}\tilde{\mathsf{E}}^{(\lambda_a)}_{kj[r+s]},$$

for any $i, j, k, l = 1, \dots, N$ and $a, b = 1, \dots, m$. Note, in particular, that $\mathsf{E}^{(\infty)}_{ab[1]}$ and $\tilde{\mathsf{E}}^{(\infty)}_{ij[1]}$ are Casimirs of the Lie algebras $\mathfrak{gl}^{\mathcal{D}}_M$ and $\mathfrak{gl}^{\tilde{\mathcal{D}}}_N$, respectively.

2.2 Lax matrices

Let $\rho\colon \mathfrak{gl}_M \to \mathrm{Mat}_{M\times M}(\mathbb{C})$ and $\tilde{\rho}\colon \mathfrak{gl}_N \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ denote the defining representations of $\mathfrak{gl}_M$ and $\mathfrak{gl}_N$, respectively. We write $E_{ab} \coloneqq \rho(\mathsf{E}_{ab})$ and $\tilde{E}_{ij} \coloneqq \tilde{\rho}\big(\tilde{\mathsf{E}}_{ij}\big)$.

The sets $\{\mathsf{E}_{ab}\}^M_{a,b=1}$ and $\{\mathsf{E}_{ba}\}^M_{a,b=1}$ form dual bases of $\mathfrak{gl}_M$ with respect to the trace in the representation ρ since $\mathrm{tr}(E_{ab}E_{cd}) = \delta_{ad}\delta_{bc}$ for all $a, b, c, d = 1, \dots, M$. Likewise, dual bases of $\mathfrak{gl}_N$ with respect to the trace in the representation $\tilde{\rho}$ are given by $\big\{\tilde{\mathsf{E}}_{ij}\big\}^N_{i,j=1}$ and $\big\{\tilde{\mathsf{E}}_{ji}\big\}^N_{i,j=1}$.

The Lax matrix of the Gaudin model associated with $\mathfrak{gl}^{\mathcal{D}}_M$ is given by

$$\mathcal{L}^{\mathcal{D}}(z)dz \coloneqq \sum_{a,b=1}^{M} E_{ba} \otimes \left(\mathsf{E}^{(\infty)}_{ab[1]} + \sum_{i=1}^{n}\sum_{r=0}^{\tau_i-1} \frac{\mathsf{E}^{(z_i)}_{ab[r]}}{(z-z_i)^{r+1}} \right) dz. \tag{2.3a}$$

It is an $M \times M$ matrix whose coefficients are rational functions of z valued in $\mathfrak{gl}^{\mathcal{D}}_M$. Likewise, the Lax matrix of the Gaudin model associated with $\mathfrak{gl}^{\tilde{\mathcal{D}}}_N$ reads

$$\mathcal{L}^{\tilde{\mathcal{D}}}(\lambda)d\lambda \coloneqq \sum_{i,j=1}^{N} \tilde{E}_{ji} \otimes \left(\tilde{\mathsf{E}}^{(\infty)}_{ij[1]} + \sum_{a=1}^{m}\sum_{s=0}^{\tilde{\tau}_a-1} \frac{\tilde{\mathsf{E}}^{(\lambda_a)}_{ij[s]}}{(\lambda-\lambda_a)^{s+1}} \right) d\lambda, \tag{2.3b}$$

and is an $N \times N$ matrix with entries rational functions of λ valued in $\mathfrak{gl}^{\tilde{\mathcal{D}}}_N$.

3 Classical $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality

3.1 Classical Gaudin model

The algebra of observables of the classical Gaudin model associated with $\mathfrak{gl}^{\mathcal{D}}_M$ is the symmetric tensor algebra $S\big(\mathfrak{gl}^{\mathcal{D}}_M\big)$. It is a Poisson algebra: the Poisson bracket is defined to be equal to the Lie bracket (2.2) on the subspace $\mathfrak{gl}^{\mathcal{D}}_M \hookrightarrow S\big(\mathfrak{gl}^{\mathcal{D}}_M\big)$ and then extended by the Leibniz rule to the whole of $S\big(\mathfrak{gl}^{\mathcal{D}}_M\big)$. Consider the quantity

$$\prod_{i=1}^{n}(z-z_i)^{\tau_i} \det\big(\lambda\mathbf{1}_{M\times M} - \mathcal{L}^{\mathcal{D}}(z)\big). \tag{3.1}$$

This is a polynomial of degree M in λ whose coefficients are rational functions in z with coefficients in $S\big(\mathfrak{gl}^{\mathcal{D}}_M\big)$. The *classical Gaudin algebra* $\mathscr{Z}^{\mathrm{cl}}\big(\mathfrak{gl}^{\mathcal{D}}_M\big)$ of the $\mathfrak{gl}^{\mathcal{D}}_M$-Gaudin model is by definition the linear subspace of $S\big(\mathfrak{gl}^{\mathcal{D}}_M\big)$ spanned by these coefficients. It is a Poisson-commutative subalgebra of $S\big(\mathfrak{gl}^{\mathcal{D}}_M\big)$.

The classical Gaudin algebra $\mathscr{Z}^{\mathrm{cl}}\big(\mathfrak{gl}^{\tilde{\mathcal{D}}}_N\big)$ of the $\mathfrak{gl}^{\tilde{\mathcal{D}}}_N$-Gaudin model is defined analogously in terms of the following polynomial of degree N in z with coefficients rational in λ,

$$\prod_{a=1}^{m}(\lambda-\lambda_a)^{\tilde{\tau}_a} \det\big(z\mathbf{1}_{N\times N} - \mathcal{L}^{\tilde{\mathcal{D}}}(\lambda)\big). \tag{3.2}$$

3.2 Bosonic realisation

Introduce the Poisson algebra $\mathcal{P}_{\mathsf{b}} := \mathbb{C}\big[x_i^a, p_j^b\big]_{i,j=1}^{N}{}_{a,b=1}^{M}$ with Poisson brackets

$$\big\{x_i^a, x_j^b\big\} = 0, \qquad \big\{p_i^a, x_j^b\big\} = \delta_{ij}\delta_{ab}, \qquad \big\{p_i^a, p_j^b\big\} = 0, \tag{3.3}$$

for $a, b = 1, \dots, M$ and $i, j = 1, \dots, N$. In the following we shall regard $\mathcal{P}_{\mathsf{b}}$ as a Lie algebra under the Poisson bracket.

For any $x \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 1}$ we denote by $J_k(x)$ the Jordan block of size $k \times k$ with x along the diagonal and -1's below the diagonal, namely

$$J_k(x) = \begin{pmatrix} x & 0 & \dots & 0 \\ -1 & x & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -1 & x \end{pmatrix}.$$

We note for later that if $x \neq 0$ then this is invertible and its inverse is given by

$$J_k(x)^{-1} = \begin{pmatrix} x^{-1} & 0 & \dots & 0 \\ x^{-2} & x^{-1} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ x^{-k} & \dots & x^{-2} & x^{-1} \end{pmatrix}. \tag{3.4}$$

Lemma 3.1. *The linear maps $\pi_{\mathsf{b}} \colon \mathfrak{gl}_M^{\mathcal{D}} \to \mathcal{P}_{\mathsf{b}}$ and $\tilde{\pi}_{\mathsf{b}} \colon \mathfrak{gl}_N^{\tilde{\mathcal{D}}} \to \mathcal{P}_{\mathsf{b}}$ defined by*

$$\pi_{\mathsf{b}}\big(\mathsf{E}^{(z_i)}_{ab[r]}\big) = \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r} x^a_{u+r} p^b_u, \qquad \pi_{\mathsf{b}}\big(\mathsf{E}^{(\infty)}_{ab[1]}\big) = -\left(\bigoplus_{c=1}^{m} J_{\tilde{\tau}_c}(-\lambda_c)\right)_{ba},$$

for every $r = 0, \dots, \tau_i - 1$, $i = 1, \dots, n$ and $a, b = 1, \dots, M$, and

$$\tilde{\pi}_{\mathsf{b}}\big(\tilde{\mathsf{E}}^{(\lambda_a)}_{ij[s]}\big) = \sum_{u=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a-s} p^u_j x^{u+s}_i, \qquad \tilde{\pi}_{\mathsf{b}}\big(\tilde{\mathsf{E}}^{(\infty)}_{ij[1]}\big) = -\left(\bigoplus_{k=1}^{n} J_{\tau_k}(-z_k)\right)_{ji},$$

for every $s = 0, \dots, \tilde{\tau}_a - 1$, $i, j = 1, \dots, N$ and $a = 1, \dots, m$, are homomorphisms of Lie algebras. They extend uniquely to homomorphisms of Poisson algebras $\pi_{\mathsf{b}} \colon S\big(\mathfrak{gl}_M^{\mathcal{D}}\big) \to \mathcal{P}_{\mathsf{b}}$ and $\tilde{\pi}_{\mathsf{b}} \colon S\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big) \to \mathcal{P}_{\mathsf{b}}$.

Proof. We will prove the corresponding result in the quantum case in detail below. See Lemma 4.7. That proof applies line-by-line here, with ∂ replaced by p. ∎

Let $\mathbb{C}(z)[\lambda]$ denote the algebra of polynomials in λ with coefficients rational in z. Given any Poisson algebra $\mathcal{P}$ we introduce the Poisson algebra $\mathcal{P}(z)[\lambda] := \mathcal{P} \otimes \mathbb{C}(z)[\lambda]$ with Poisson bracket defined using multiplication in the second tensor factor. Extend the homomorphisms π_{b} and $\tilde{\pi}_{\mathsf{b}}$ from Lemma 3.1 to homomorphisms of Poisson algebras

$$\pi_{\mathsf{b}} \colon\ S\big(\mathfrak{gl}_M^{\mathcal{D}}\big)(z)[\lambda] \longrightarrow \mathcal{P}_{\mathsf{b}}(z)[\lambda], \qquad \tilde{\pi}_{\mathsf{b}} \colon\ S\big(\mathfrak{gl}_M^{\tilde{\mathcal{D}}}\big)(\lambda)[z] \longrightarrow \mathcal{P}_{\mathsf{b}}(\lambda)[z],$$

by letting them act trivially on the tensor factors $\mathbb{C}(z)[\lambda]$ and $\mathbb{C}(\lambda)[z]$, respectively. In particular, we may apply these homomorphisms respectively to the expressions (3.1) and (3.2). It follows from Theorem 3.2 below that the resulting expressions in fact live in the common subalgebra $\mathcal{P}_{\mathsf{b}}[z, \lambda] := \mathcal{P}_{\mathsf{b}} \otimes \mathbb{C}[z, \lambda]$ of both $\mathcal{P}_{\mathsf{b}}(z)[\lambda]$ and $\mathcal{P}_{\mathsf{b}}(\lambda)[z]$, where $\mathbb{C}[z, \lambda]$ denotes the algebra of polynomials in the variables z and λ. The coefficients of these polynomials in $\mathcal{P}_{\mathsf{b}}[z,\lambda]$ span the images of the classical Gaudin algebras in $\mathcal{P}_{\mathsf{b}}$, namely

$$\pi_{\mathsf{b}}\big(\mathscr{Z}^{\mathrm{cl}}\big(\mathfrak{gl}_M^{\mathcal{D}}\big)\big) \subset \mathcal{P}_{\mathsf{b}} \qquad \text{and} \qquad \tilde{\pi}_{\mathsf{b}}\big(\mathscr{Z}^{\mathrm{cl}}\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big)\big) \subset \mathcal{P}_{\mathsf{b}},$$

respectively. The following theorem establishes that these Poisson-commutative subalgebras of $\mathcal{P}_{\mathsf{b}}$ coincide.

Theorem 3.2. *We have the following relation*

$$\pi_{\mathsf{b}}\left(\prod_{i=1}^{n}(z-z_i)^{\tau_i}\det\big(\lambda\mathbf{1}_{M\times M}-\mathcal{L}^{\mathcal{D}}(z)\big)\right) = \tilde{\pi}_{\mathsf{b}}\left(\prod_{a=1}^{m}(\lambda-\lambda_a)^{\tilde{\tau}_a}\det\big(z\mathbf{1}_{N\times N}-\mathcal{L}^{\tilde{\mathcal{D}}}(\lambda)\big)\right),$$

as an equality in $\mathcal{P}_{\mathsf{b}}[z,\lambda]$.

Proof. Introduce the $M\times M$ and $N\times N$ block diagonal matrices

$$\Lambda \coloneqq \bigoplus_{a=1}^{m} {}^tJ_{\tilde{\tau}_a}(\lambda-\lambda_a), \qquad Z \coloneqq \bigoplus_{i=1}^{n} J_{\tau_i}(z-z_i).$$

Also introduce the $M\times N$ matrices

$$P \coloneqq (p_i^a)_{a=1}^{M}{}_{i=1}^{N}, \qquad X \coloneqq (x_i^a)_{a=1}^{M}{}_{i=1}^{N}.$$

Consider the block matrix

$$M \coloneqq \begin{pmatrix} \Lambda & X \\ {}^tP & Z \end{pmatrix}, \tag{3.5}$$

with entries in the commutative algebra $\mathcal{P}_{\mathsf{b}}[\lambda,z]$. We may evaluate its determinant in two ways. On the one hand, we have

$$\begin{aligned} \det M &= \det\left(M\begin{pmatrix} 1 & -\Lambda^{-1}X \\ 0 & 1\end{pmatrix}\right) \\ &= \det\begin{pmatrix} \Lambda & 0 \\ {}^tP & Z-{}^tP\Lambda^{-1}X\end{pmatrix} = \det\Lambda\det\big(Z-{}^tP\Lambda^{-1}X\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} \det M &= \det\begin{pmatrix} Z & {}^tP \\ X & \Lambda\end{pmatrix} = \det\left(\begin{pmatrix} Z & {}^tP \\ X & \Lambda\end{pmatrix}\begin{pmatrix} 1 & -Z^{-1}\,{}^tP \\ 0 & 1\end{pmatrix}\right) \\ &= \det\begin{pmatrix} Z & 0 \\ X & \Lambda-XZ^{-1}\,{}^tP\end{pmatrix} = \det Z\det\big(\Lambda-XZ^{-1}\,{}^tP\big). \end{aligned}$$

Hence we obtain the relation

$$\det Z\det\big(\Lambda-XZ^{-1}\,{}^tP\big) = \det\Lambda\det\big(Z-{}^tP\Lambda^{-1}X\big). \tag{3.6}$$

It remains to note that the square matrices Z and Λ can be written as

$$Z = \sum_{i,j=1}^{N}\tilde{E}_{ij}\big(z\delta_{ij}-\pi_{\mathsf{b}}\big(\tilde{\mathsf{E}}_{ij[1]}^{(\infty)}\big)\big), \qquad \Lambda = \sum_{a,b=1}^{M}E_{ab}\big(\lambda\delta_{ab}-\pi_{\mathsf{b}}\big(\mathsf{E}_{ba[1]}^{(\infty)}\big)\big)$$

with π_{b} and $\tilde{\pi}_{\mathsf{b}}$ as defined in Lemma 3.1, and that their inverses are given by

$$Z^{-1} = \bigoplus_{i=1}^{n} J_{\tau_i}(z - z_i)^{-1}, \qquad \Lambda^{-1} = \bigoplus_{a=1}^{m} {}^t J_{\tilde{\tau}_a}(\lambda - \lambda_a)^{-1}.$$

Thus we have

$$\begin{aligned}\Lambda - XZ^{-1}\,{}^tP &= \sum_{a,b=1}^{M} E_{ab}\big(\Lambda - XZ^{-1}\,{}^tP\big)_{ab} \\ &= \lambda\mathbf{1} - \sum_{a,b=1}^{M} E_{ab}\left(\pi_{\mathsf{b}}\big(\mathsf{E}^{(\infty)}_{ab[1]}\big) + \sum_{i=1}^{n}\sum_{j,k=\nu_i+1}^{\nu_i+\tau_i} x^a_j\big(J_{\tau_i}(z-z_i)^{-1}\big)_{jk} p^b_k\right),\end{aligned}$$

which is nothing but $\lambda\mathbf{1} - \pi_{\mathsf{b}}\big({}^t\mathcal{L}^{\mathcal{D}}(z)\big)$ using Lemma 4.7, the expression (2.3a) for the Lax matrix $\mathcal{L}^{\mathcal{D}}(z)$ and (3.4) for the inverse of a Jordan block. Likewise

$$\begin{aligned}Z - {}^tP\Lambda^{-1}X &= \sum_{i,j=1}^{N} \tilde{E}_{ij}\big(Z - {}^tP\Lambda^{-1}X\big)_{ij} \\ &= z\mathbf{1} - \sum_{i,j=1}^{N} \tilde{E}_{ij}\left(\tilde{\pi}_{\mathsf{b}}\big(\tilde{\mathsf{E}}^{(\infty)}_{ji[1]}\big) + \sum_{a=1}^{m}\sum_{b,c=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a} p^b_i\big(J_{\tilde{\tau}_a}(\lambda-\lambda_a)^{-1}\big)_{cb} x^c_j\right),\end{aligned}$$

which coincides with $z\mathbf{1} - \tilde{\pi}_{\mathsf{b}}\big(\tilde{\mathcal{L}}(\lambda)\big)$, as required. Since $\det {}^tA = \det A$ for any square matrix A and noting that $\det Z = \prod_{i=1}^{n}(z-z_i)^{\tau_i}$ and $\det\Lambda = \prod_{a=1}^{m}(\lambda-\lambda_a)^{\tilde{\tau}_a}$, the result follows. ■

3.3 Fermionic realisation

Let $V := \mathrm{span}_{\mathbb{C}}\,\{\psi^a_i, \pi^b_j\}_{i,j=1}^{N}{}_{a,b=1}^{M}$ and define the exterior algebra $\mathcal{P}_{\mathsf{f}} := \bigwedge V = \bigoplus_{k=0}^{2MN}\bigwedge^k V$, whose skew-symmetric product we denote simply by juxtaposition. We refer to an element $u \in \bigwedge^k V$ as being homogeneous of degree k and write $|u| = k$. In particular, $|\psi^a_i| = |\pi^a_i| = 1$ for any $a = 1,\dots,M$ and $i = 1,\dots,N$. We endow $\mathcal{P}_{\mathsf{f}}$ with a $\mathbb{Z}_2$-graded Poisson structure defined by

$$\{\pi^a_i, \psi^b_j\}_+ = \{\psi^b_j, \pi^a_i\}_+ = \delta_{ij}\delta_{ab},$$

for any $a, b = 1,\dots,M$ and $i, j = 1,\dots,N$, and extended to the whole of $\mathcal{P}_{\mathsf{f}}$ by the $\mathbb{Z}_2$-graded skew-symmetry property and the $\mathbb{Z}_2$-graded Leibniz rule, i.e.,

$$\begin{aligned}&\{u, v\}_+ = -(-1)^{|u||v|}\{v, u\}_+, \\ &\{u, vw\}_+ = \{u, v\}_+ w + (-1)^{|u||v|} v\{u, w\}_+\end{aligned}$$

for any homogeneous elements $u, v, w \in \mathcal{P}_{\mathsf{f}}$.

Let $\mathcal{P}^{\bar{0}}_{\mathsf{f}} := \bigoplus_{k=0}^{MN}\bigwedge^{2k} V$ denote the even subspace of $\mathcal{P}_{\mathsf{f}}$. The restriction of the $\mathbb{Z}_2$-graded Poisson bracket $\{\cdot,\cdot\}_+$ to $\mathcal{P}^{\bar{0}}_{\mathsf{f}}$ defines a Lie algebra structure on $\mathcal{P}^{\bar{0}}_{\mathsf{f}}$.

Lemma 3.3. *The linear maps $\pi_{\mathsf{f}}\colon \mathfrak{gl}^{\mathcal{D}}_M \to \mathcal{P}^{\bar{0}}_{\mathsf{f}}$ and $\tilde{\pi}_{\mathsf{f}}\colon \mathfrak{gl}^{\tilde{\mathcal{D}}}_N \to \mathcal{P}^{\bar{0}}_{\mathsf{f}}$ defined by*

$$\pi_{\mathsf{f}}\big(\mathsf{E}^{(z_i)}_{ab[r]}\big) = \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r} \pi^a_{u+r}\psi^b_u, \qquad \pi_{\mathsf{f}}\big(\mathsf{E}^{(\infty)}_{ab[1]}\big) = -\left(\bigoplus_{c=1}^{m} J_{\tilde{\tau}_c}(-\lambda_c)\right)_{ab},$$

for every $i = 1, \dots, n$ and $a, b = 1, \dots, M$, and

$$\tilde{\pi}_{\mathsf{f}}\big(\tilde{\mathsf{E}}^{(\lambda_a)}_{ij[s]}\big) = \sum_{u=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a-s} \psi^u_i \pi^{u+s}_j, \qquad \tilde{\pi}_{\mathsf{f}}\big(\tilde{\mathsf{E}}^{(\infty)}_{ij[1]}\big) = -\left(\bigoplus_{k=1}^{n} J_{\tau_k}(-z_k)\right)_{ij},$$

for every $i, j = 1, \dots, N$ and $a = 1, \dots, m$, are homomorphisms of Lie algebras.

Proof. For each $i, j = 1, \dots, n$ and $a, b = 1, \dots, M$ we have

$$\begin{aligned}
\big\{\pi_{\mathsf{f}}\big(\mathsf{E}^{(z_i)}_{ab[r]}\big), \pi_{\mathsf{f}}\big(\mathsf{E}^{(z_j)}_{cd[s]}\big)\big\}_+ &= \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r} \sum_{v=\nu_j+1}^{\nu_j+\tau_j-s} \{\pi^a_{u+r}\psi^b_u, \pi^c_{v+s}\psi^d_v\}_+ \\
&= \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r} \sum_{v=\nu_i+1}^{\nu_i+\tau_i-s} \big(\pi^a_{u+r}\{\psi^b_u, \pi^c_{v+s}\}_+\psi^d_v - \pi^c_{v+s}\{\pi^a_{u+r}, \psi^d_v\}_+\psi^b_u\big)\delta_{ij} \\
&= \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r-s} \big(\delta_{cb}\pi^a_{u+r+s}\psi^d_u - \delta_{ad}\pi^c_{u+r+s}\psi^b_u\big)\delta_{ij} \\
&= \big(\delta_{bc}\pi_{\mathsf{f}}\big(\mathsf{E}^{(z_i)}_{ad[r+s]}\big) - \delta_{ad}\pi_{\mathsf{f}}\big(\mathsf{E}^{(z_i)}_{cb[r+s]}\big)\big)\delta_{ij} = \pi_{\mathsf{f}}\big(\big\{\mathsf{E}^{(z_i)}_{ab[r]}, \mathsf{E}^{(z_j)}_{cd[s]}\big\}\big).
\end{aligned}$$

Likewise, for each $i, j = 1, \dots, N$ and $a, b = 1, \dots, m$ one shows that

$$\big\{\tilde{\pi}_{\mathsf{f}}\big(\tilde{\mathsf{E}}^{(\lambda_a)}_{ij[r]}\big), \tilde{\pi}_{\mathsf{f}}\big(\tilde{\mathsf{E}}^{(\lambda_b)}_{kl[s]}\big)\big\}_+ = \tilde{\pi}_{\mathsf{f}}\big(\big\{\tilde{\mathsf{E}}^{(\lambda_a)}_{ij[r]}, \tilde{\mathsf{E}}^{(\lambda_b)}_{kl[s]}\big\}\big),$$

and all Poisson brackets involving the generators at infinity are also easily seen to be preserved by the linear maps π_{f} and $\tilde{\pi}_{\mathsf{f}}$ since $z_i \in \mathbb{C}$ and $\lambda_a \in \mathbb{C}$ are central in $\mathcal{P}^{\bar{0}}_{\mathsf{f}}$. ■

Theorem 3.4. *We have the following relation*

$$\pi_{\mathsf{f}}\big(\det\big(\lambda \mathbf{1}_{M\times M} - \mathcal{L}^{\mathcal{D}}(z)\big)\big)\tilde{\pi}_{\mathsf{f}}\big(\det\big(z\mathbf{1}_{N\times N} - \mathcal{L}^{\tilde{\mathcal{D}}}(\lambda)\big)\big) = \prod_{i=1}^{n}(z - z_i)^{\tau_i} \prod_{a=1}^{m}(\lambda - \lambda_a)^{\tilde{\tau}_a}.$$

Proof. Consider the same $M \times M$ and $N \times N$ block diagonal matrices Z and Λ as in the proof of Theorem 3.2. Introduce the $M \times N$ and $N \times M$ matrices

$$\Pi := (\pi^a_i)^{M}_{a=1}{}^{N}_{i=1}, \qquad \Psi := (\psi^a_i)^{N}_{i=1}{}^{M}_{a=1},$$

and consider the following even supermatrix

$$M := \left(\begin{array}{c|c} \Lambda & \Pi \\ \hline \Psi & Z \end{array}\right).$$

Since Λ and Z are both invertible, we can define the Berezinian, or superdeterminant, of M which is given by $\operatorname{Ber} M = \det \Lambda\big(\det\big(Z - \Psi\Lambda^{-1}\Pi\big)\big)^{-1}$. Alternatively, the Berezinian of M can equally be expressed as $\operatorname{Ber} M = \det\big(\Lambda - \Pi Z^{-1}\Psi\big)(\det Z)^{-1}$, see for instance [2]. Equating these two expressions of $\operatorname{Ber} M$ we obtain the relation

$$\det\big(\Lambda - \Pi Z^{-1}\Psi\big)\det\big(Z - \Psi\Lambda^{-1}\Pi\big) = \det Z \det \Lambda.$$

Recalling the expressions for the square matrices Z and Λ and their inverses given in the proof of Theorem 3.2, we can write

$$\Lambda - \Pi Z^{-1}\Psi = \sum_{a,b=1}^{M} E_{ab}\big(\Lambda - \Pi Z^{-1}\Psi\big)_{ab}$$

$$= \lambda \mathbf{1} - \sum_{a,b=1}^{M} E_{ab} \left(\pi_{\mathsf{f}}\big(\mathsf{E}^{(\infty)}_{ba[1]}\big) + \sum_{i=1}^{n} \sum_{j,k=\nu_i+1}^{\nu_i+\tau_i} \pi^a_j \big(J_{\tau_i}(z-z_i)^{-1}\big)_{jk} \psi^b_k \right),$$

which is nothing but $\lambda \mathbf{1} - \pi_{\mathsf{f}}\big(\mathcal{L}^{\mathfrak{D}}(z)\big)$. Likewise

$$Z - \Psi\Lambda^{-1}\Pi = \sum_{i,j=1}^{N} E_{ij} \big(Z - \Psi\Lambda^{-1}\Pi\big)_{ij}$$

$$= z\mathbf{1} - \sum_{i,j=1}^{N} E_{ij} \left(\tilde{\pi}_{\mathsf{f}}\big(\mathsf{E}^{(\infty)}_{ij[1]}\big) + \sum_{a=1}^{m} \sum_{b,c=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a} \psi^b_i \big(J_{\tilde{\tau}_a}(\lambda-\lambda_a)^{-1}\big)_{cb} \pi^c_j \right),$$

which is $z\mathbf{1} - \tilde{\pi}_{\mathsf{f}}\big({}^t\mathcal{L}^{\tilde{\mathfrak{D}}}(\lambda)\big)$. The result now follows as in the proof of Theorem 3.2. ∎

4 Quantum $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality

There is a natural quantum version of Theorem 3.2. In order to state it, we first need a short digression on Manin matrices. In this section we do not consider the fermionic counterpart of Theorem 3.2, namely Theorem 3.4, but leave this for future work.

4.1 Manin matrices

Let $\mathcal{A}$ be an associative (but possibly noncommutative) algebra over $\mathbb{C}$. Suppose $M = (M_{ij})$ is a matrix with entries in $\mathcal{A}$.

Definition 4.1. The matrix M is a *Manin matrix* if

(i) $[M_{ij}, M_{kj}] = 0$ for all i, j, k, and

(ii) $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$ for all i, j, k, l.

That is, elements of the same column must commute amongst themselves, and commutators of cross terms of 2×2 submatrices must be equal (for example $[M_{11}, M_{22}] = [M_{21}, M_{12}]$). Actually the second of these conditions implies the first (set $j = l$) but it is convenient to think of them separately.

In the literature Manin matrices have been also called *right quantum* matrices [15, 16, 17, 18] or *row-pseudo-commutative* matrices [3]. For a review of their properties, and further references, see [5].

Definition 4.2. The *column(-ordered) determinant* of an $N \times N$ matrix M is

$$\operatorname{cdet} M := \sum_{\sigma\in S_N} (-1)^{|\sigma|} M_{\sigma(1)1} M_{\sigma(2)2} \cdots M_{\sigma(N)N}.$$

Lemma 4.3. *The column determinant* $\operatorname{cdet} M$ *changes only by a sign under the exchange of any two rows of* M*. If* M *is Manin, then* $\operatorname{cdet} M$ *also changes only by a sign under the exchange of any two columns of* M*.*

Proof. The first part is manifest. See [5, Section 3.4] for the second. ∎

Proposition 4.4. *Let* M *be an* $N\times N$ *Manin matrix with coefficients in* $\mathcal{A}$*. Let* X *be a* $k\times(N-k)$ *matrix with coefficients in* $\mathcal{A}$*, for some* $0 \le k \le N$*. Then*

$$\operatorname{cdet} M = \operatorname{cdet}\left(M \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \right).$$

Proof. See [5, Section 5.1]. ■

This has the following corollary which will be important for us.

Proposition 4.5. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be the block form of an* $N \times N$ *Manin matrix with coefficients in* $\mathcal{A}$.

(*i*) *Suppose* $\mathcal{A}$ *is a subalgebra of a (possibly larger) algebra* $\mathcal{A}'$ *over which* A *has a right inverse, i.e.,* $AA^{-1} = 1$ *for some matrix* A^{-1} *with coefficients in* $\mathcal{A}'$. *Then*

$$\operatorname{cdet} M = \operatorname{cdet} A \operatorname{cdet}\left(D - CA^{-1}B\right)$$

as an equality in $\mathcal{A}$.

(*ii*) *Suppose* $\mathcal{A}$ *is a subalgebra of a (possibly larger) algebra* $\mathcal{A}''$ *over which* D *has a right inverse, i.e.,* $DD^{-1} = 1$ *for some matrix* D^{-1} *with coefficients in* $\mathcal{A}''$. *Then*

$$\operatorname{cdet} M = \operatorname{cdet} D \operatorname{cdet}\left(A - BD^{-1}C\right)$$

as an equality in $\mathcal{A}$.

Proof. We work initially over $\mathcal{A}'$. Suppose A has a right inverse. By Proposition 4.4 we have

$$\begin{aligned}\operatorname{cdet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \operatorname{cdet}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & -A^{-1}B \\ 0 & 1 \end{pmatrix}\right) \\ &= \operatorname{cdet}\begin{pmatrix} A & 0 \\ C & D - CA^{-1}B \end{pmatrix} = \operatorname{cdet} A \operatorname{cdet}\left(D - CA^{-1}B\right)\end{aligned}$$

as an equality in $\mathcal{A}'$. But $\operatorname{cdet} M$ belongs to $\mathcal{A}$, so in fact this is an equality in $\mathcal{A}$. This establishes part (i).

For part (ii) note that, by Lemma 4.3, $\operatorname{cdet} M$ is invariant under the exchange of any pair of rows followed by the exchange of the corresponding pair of columns. So we can rearrange the blocks to find

$$\operatorname{cdet} M = \operatorname{cdet}\begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

and then argue as for part (i). ■

Remark 4.6. The proposition above is the first half of [5, Proposition 10], specifically lines (5.17) and (5.18). The subsequent lines (5.19) and (5.20) appear to contain misprints. For example, if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 Manin matrix with d invertible then $\operatorname{cdet} M = ad - cb = \left(a - cbd^{-1}\right)d = \left(a - cd^{-1}b\right)d$ whereas [5, line (5.20)] gives $\operatorname{cdet} M = \left(a - bd^{-1}c\right)d$, which is not in general the same.

4.2 Quantum Gaudin model

The algebra of observables of the quantum Gaudin model associated with $\mathfrak{gl}_M^{\mathcal{D}}$ is the enveloping algebra $U\left(\mathfrak{gl}_M^{\mathcal{D}}\right)$, equipped with its usual associative product. Let $\partial_z := \frac{\partial}{\partial z}$ and consider the same Lax matrix given by (2.3a), as in the classical model we considered above but now regarded as taking values in $\mathfrak{gl}_M^{\mathcal{D}} \hookrightarrow U\left(\mathfrak{gl}_M^{\mathcal{D}}\right)$. Its transpose is

$${}^t\mathcal{L}^{\mathcal{D}}(z)dz = \sum_{a,b=1}^{M} E_{ab} \otimes \left(\mathsf{E}_{ab[1]}^{(\infty)} + \sum_{i=1}^{n}\sum_{r=0}^{\tau_i-1} \frac{\mathsf{E}_{ab[r]}^{(z_i)}}{(z-z_i)^{r+1}}\right) dz.$$

Recall the definition of the column-ordered determinant, Definition 4.2, and consider the quantity

$$\prod_{i=1}^{n}(z-z_i)^{\tau_i}\,\mathrm{cdet}\big(\partial_z\mathbf{1}_{M\times M}-{}^t\mathcal{L}^{\mathfrak{D}}(z)\big)=:\sum_{k=0}^{M}S_k(z)\partial_z^k. \tag{4.1}$$

This is a differential operator in z of order M. For each $0\le k\le M$, the coefficient $S_k(z)$ of ∂_z^k is a rational function in z valued in $U\big(\mathfrak{gl}_M^{\mathfrak{D}}\big)$.

The *quantum Gaudin algebra* $\mathscr{Z}\big(\mathfrak{gl}_M^{\mathfrak{D}}\big)$ of the $\mathfrak{gl}_M^{\mathfrak{D}}$-Gaudin model is by definition the unital subalgebra of $U\big(\mathfrak{gl}_M^{\mathfrak{D}}\big)$ generated by the coefficients in the partial fraction decomposition of these rational functions $S_k(z)$. It is a commutative subalgebra of $U\big(\mathfrak{gl}_M^{\mathfrak{D}}\big)$, [19, 30].[4]

The quantum Gaudin algebra $\mathscr{Z}\big(\mathfrak{gl}_N^{\tilde{\mathfrak{D}}}\big)$ of the $\mathfrak{gl}_N^{\tilde{\mathfrak{D}}}$-Gaudin model is defined in exactly the same way in terms of the N^{th} order differential operator in λ,

$$\prod_{a=1}^{m}(\lambda-\lambda_a)^{\tilde{\tau}_a}\,\mathrm{cdet}\big(\partial_\lambda\mathbf{1}_{N\times N}-{}^t\mathcal{L}^{\tilde{\mathfrak{D}}}(\lambda)\big),$$

where, cf. (2.3b),

$${}^t\mathcal{L}^{\tilde{\mathfrak{D}}}(\lambda)d\lambda=\sum_{i,j=1}^{N}\tilde{E}_{ij}\otimes\left(\tilde{\mathsf{E}}_{ij[1]}^{(\infty)}+\sum_{a=1}^{m}\sum_{s=0}^{\tilde{\tau}_a-1}\frac{\tilde{\mathsf{E}}_{ij[s]}^{(\lambda_a)}}{(\lambda-\lambda_a)^{s+1}}\right)d\lambda.$$

There is an automorphism of $\mathfrak{gl}_N^{\mathfrak{D}}$ defined by $\mathcal{L}^{\tilde{\mathfrak{D}}}(\lambda)\mapsto-{}^t\mathcal{L}^{\tilde{\mathfrak{D}}}(\lambda)$. The Gaudin algebra is stabilized by this automorphism. (This statement follows from applying a tensor product of evaluation homomorphisms of Takiff algebras to the statement of [19, Proposition 8.4]). Therefore we may equivalently consider the N^{th} order differential operator

$$\prod_{a=1}^{m}(\lambda-\lambda_a)^{\tilde{\tau}_a}\,\mathrm{cdet}\big(\partial_\lambda\mathbf{1}_{N\times N}+\mathcal{L}^{\tilde{\mathfrak{D}}}(\lambda)\big)=:\sum_{k=0}^{N}\tilde{S}_k(\lambda)\partial_\lambda^k \tag{4.2}$$

and define the quantum Gaudin algebra $\mathscr{Z}\big(\mathfrak{gl}_N^{\tilde{\mathfrak{D}}}\big)$ to be the unital subalgebra of $U\big(\mathfrak{gl}_N^{\tilde{\mathfrak{D}}}\big)$ generated by the coefficients in the partial fraction decomposition of the rational functions $\tilde{S}_k(\lambda)$ in λ. It is a commutative subalgebra of $U\big(\mathfrak{gl}_N^{\tilde{\mathfrak{D}}}\big)$.

To state our result on quantum $(\mathfrak{gl}_M,\mathfrak{gl}_N)$-duality, it will be convenient to write (4.2) in the equivalent form

$$\prod_{a=1}^{m}(\partial_z-\lambda_a)^{\tilde{\tau}_a}\,\mathrm{cdet}\big(-z\mathbf{1}_{N\times N}+\mathcal{L}^{\tilde{\mathfrak{D}}}(\partial_z)\big)=\sum_{k=0}^{N}\tilde{S}_k(\partial_z)(-z)^k.$$

Let us explain the meaning of the expression

$$\mathrm{cdet}\big(-z\mathbf{1}_{N\times N}+\mathcal{L}^{\tilde{\mathfrak{D}}}(\partial_z)\big).$$

The quantity

$$\mathrm{cdet}\big(\partial_\lambda\mathbf{1}_{N\times N}+\mathcal{L}^{\tilde{\mathfrak{D}}}(\lambda)\big),$$

[4]It is shown in [19] that $\mathrm{cdet}\big(\partial_z\mathbf{1}_{M\times M}-E_{ab}\otimes\sum_{n=0}^{\infty}(\mathsf{E}_{ab}\otimes t^n)z^{-n-1}\big)$ generates a commutative subalgebra of $U(\mathfrak{gl}_M[t])$. The algebra $\mathscr{Z}\big(\mathfrak{gl}_M^{\mathfrak{D}}\big)$ is a homomorphic image of this algebra in $U\big(\mathfrak{gl}_M^{\mathfrak{D}}\big)$.

which appears in (4.2), belongs to the algebra $U\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big)(\lambda)[\partial_\lambda]$ of differential operators in λ whose coefficients are rational functions of λ with coefficients in $U\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big)$. Here λ and ∂_λ can be regarded as formal generators obeying the commutation relation $[\partial_\lambda, \lambda] = 1$. We can relabel these generators as we wish, provided we preserve this relation. In particular, we may send $(\partial_\lambda, \lambda) \mapsto (-z, \partial_z)$, since $[-z, \partial_z] = 1$. Thus $\operatorname{cdet}\big(-z\mathbf{1}_{N\times N} + \mathcal{L}^{\tilde{\mathcal{D}}}(\partial_z)\big)$ is an element of the algebra $U\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big)(\partial_z)[z]$.

More precisely, we shall be concerned in what follows with the quantity

$$\prod_{a=1}^{m} (\partial_z - \lambda_a)^{\tilde{\tau}_a} \operatorname{cdet}\big(z\mathbf{1}_{N\times N} - \mathcal{L}^{\tilde{\mathcal{D}}}(\partial_z)\big) = \sum_{k=0}^{N} (-1)^{N-k} \tilde{S}_k(\partial_z) z^k. \tag{4.3}$$

4.3 Bosonic realisation

We consider realisations of $U\big(\mathfrak{gl}_M^{\mathcal{D}}\big)$ and $U\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big)$ acting by differential operators on the polynomial algebra $\mathbb{C}[x_i^a]_{i=1}^{N}{}_{a=1}^{M}$. Namely, let $\partial_i^a := \frac{\partial}{\partial x_i^a}$ and let us denote by $\mathcal{U}_\mathsf{b}$ the unital associative algebra generated by $\{x_i^a\}_{i=1}^{N}{}_{a=1}^{M}$ and $\{\partial_i^a\}_{i=1}^{N}{}_{a=1}^{M}$ subject to the commutation relations

$$[x_i^a, x_j^b] = 0, \quad [\partial_i^a, x_j^b] = \delta_{ij}\delta_{ab}, \quad [\partial_i^a, \partial_j^b] = 0,$$

for $a, b = 1, \dots, M$ and $i, j = 1, \dots, N$.

$\mathcal{U}_\mathsf{b}$ is in particular a Lie algebra, with the Lie bracket given by the commutator.

Lemma 4.7. *The linear maps $\hat{\pi}_\mathsf{b} \colon \mathfrak{gl}_M^{\mathcal{D}} \to \mathcal{U}_\mathsf{b}$ and $\hat{\tilde{\pi}}_\mathsf{b} \colon \mathfrak{gl}_N^{\tilde{\mathcal{D}}} \to \mathcal{U}_\mathsf{b}$ defined by*

$$\hat{\pi}_\mathsf{b}\big(\mathsf{E}_{ab[r]}^{(z_i)}\big) = \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r} x_{u+r}^a \partial_u^b, \qquad \hat{\pi}_\mathsf{b}\big(\mathsf{E}_{ab[1]}^{(\infty)}\big) = -\left(\bigoplus_{c=1}^{m} J_{\tilde{\tau}_c}(-\lambda_c)\right)_{ba},$$

for every $r = 0, \dots, \tau_i - 1$, $i = 1, \dots, n$ and $a, b = 1, \dots, M$, and

$$\hat{\tilde{\pi}}_\mathsf{b}\big(\tilde{\mathsf{E}}_{ij[s]}^{(\lambda_a)}\big) = \sum_{u=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a-s} \partial_j^u x_i^{u+s}, \qquad \hat{\tilde{\pi}}_\mathsf{b}\big(\tilde{\mathsf{E}}_{ij[1]}^{(\infty)}\big) = -\left(\bigoplus_{k=1}^{n} J_{\tau_k}(-z_k)\right)_{ji},$$

for every $s = 0, \dots, \tilde{\tau}_a - 1$, $i, j = 1, \dots, N$ and $a = 1, \dots, m$, are homomorphisms of Lie algebras. They extend uniquely to homomorphisms of associative algebras $\hat{\pi}_\mathsf{b} \colon U\big(\mathfrak{gl}_M^{\mathcal{D}}\big) \to \mathcal{U}_\mathsf{b}$ and $\hat{\tilde{\pi}}_\mathsf{b} \colon U\big(\mathfrak{gl}_N^{\tilde{\mathcal{D}}}\big) \to \mathcal{U}_\mathsf{b}$.

Proof. For each $i, j = 1, \dots, n$ and $a, b = 1, \dots, M$ we have

$$\begin{aligned}
\big[\hat{\pi}_\mathsf{b}\big(\mathsf{E}_{ab[r]}^{(z_i)}\big), \hat{\pi}_\mathsf{b}\big(\mathsf{E}_{cd[s]}^{(z_j)}\big)\big] &= \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r} \sum_{v=\nu_j+1}^{\nu_j+\tau_j-s} \big[x_{u+r}^a \partial_u^b, x_{v+s}^c \partial_v^d\big] \\
&= \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r} \sum_{v=\nu_i+1}^{\nu_i+\tau_i-s} \big(x_{u+r}^a \big[\partial_u^b, x_{v+s}^c\big] \partial_v^d + x_{v+s}^c \big[x_{u+r}^a, \partial_v^d\big] \partial_u^b\big) \delta_{ij} \\
&= \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r-s} \big(\delta_{bc} x_{u+r+s}^a \partial_u^d - \delta_{ad} x_{u+r+s}^c \partial_u^b\big) \delta_{ij} \\
&= \big(\delta_{bc} \hat{\pi}_\mathsf{b}\big(\mathsf{E}_{ad[r+s]}^{(z_i)}\big) - \delta_{ad} \hat{\pi}_\mathsf{b}\big(\mathsf{E}_{cb[r+s]}^{(z_i)}\big)\big) \delta_{ij} = \hat{\pi}_\mathsf{b}\big(\big[\mathsf{E}_{ab[r]}^{(z_i)}, \mathsf{E}_{cd[s]}^{(z_j)}\big]\big).
\end{aligned}$$

In the second equality we have used the fact that if $i \neq j$ then all commutators vanish due to the restriction in the range of values in the sums over u and v.

Likewise, for all $i,j=1,\dots,N$ and $a,b=1,\dots,m$ we find

$$\begin{aligned}
\big[\hat{\tilde{\pi}}_{\mathsf{b}}\big(\tilde{\mathsf{E}}^{(\lambda_a)}_{ij[r]}\big),\hat{\tilde{\pi}}_{\mathsf{b}}\big(\tilde{\mathsf{E}}^{(\lambda_b)}_{kl[s]}\big)\big] &= \sum_{u=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a-r}\sum_{v=\tilde{\nu}_b+1}^{\tilde{\nu}_b+\tilde{\tau}_b-s}\big[\partial^u_j x^{u+r}_i,\partial^v_l x^{v+s}_k\big]\\
&= \sum_{u=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a-r}\sum_{v=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a-s}\big(\partial^u_j\big[x^{u+r}_i,\partial^v_l\big]x^{v+s}_k+\partial^v_l\big[\partial^u_j,x^{v+s}_k\big]x^{u+r}_i\big)\delta_{ab}\\
&= \sum_{u=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a-r-s}\big(-\delta_{il}\partial^u_j x^{u+r+s}_k+\delta_{jk}\partial^u_l x^{u+r+s}_i\big)\delta_{ab}\\
&= \big(\delta_{jk}\hat{\tilde{\pi}}_{\mathsf{b}}\big(\tilde{\mathsf{E}}^{(\lambda_a)}_{il[r+s]}\big)-\delta_{il}\hat{\tilde{\pi}}_{\mathsf{b}}\big(\tilde{\mathsf{E}}^{(\lambda_a)}_{kj[r+s]}\big)\big)\delta_{ab}=\hat{\tilde{\pi}}_{\mathsf{b}}\big(\big[\tilde{\mathsf{E}}^{(\lambda_a)}_{ij[r]},\tilde{\mathsf{E}}^{(\lambda_b)}_{kl[s]}\big]\big),
\end{aligned}$$

as required. Moreover, all the commutators involving the generators at infinity are also easily seen to be preserved by the linear maps $\hat{\pi}_{\mathsf{b}}$ and $\hat{\tilde{\pi}}_{\mathsf{b}}$ since $z_i\in\mathbb{C}$ and $\lambda_a\in\mathbb{C}$ are central in $\mathcal{U}_{\mathsf{b}}$. ■

Given any unital associative algebra $\mathcal{U}$ we denote by $\mathcal{U}[z,\partial_z]$ the tensor product of unital associative algebras $\mathcal{U}\otimes\mathbb{C}[z,\partial_z]$. As in the classical setting of Section 3.2, consider also the unital associative algebras $\mathcal{U}(z)[\partial_z]:=\mathcal{U}\otimes\mathbb{C}(z)[\partial_z]$ and $\mathcal{U}(\partial_z)[z]:=\mathcal{U}\otimes\mathbb{C}(\partial_z)[z]$, both containing $\mathcal{U}[z,\partial_z]$ as a subalgebra. We extend the homomorphisms $\hat{\pi}_{\mathsf{b}}$ and $\hat{\tilde{\pi}}_{\mathsf{b}}$ from Lemma 4.7 to homomorphisms of tensor product algebras,

$$\hat{\pi}_{\mathsf{b}}\colon\ U\big(\mathfrak{gl}^{\mathcal{D}}_M\big)(z)[\partial_z]\to\mathcal{U}_{\mathsf{b}}(z)[\partial_z],\qquad \hat{\tilde{\pi}}_{\mathsf{b}}\colon\ U\big(\mathfrak{gl}^{\tilde{\mathcal{D}}}_M\big)(\partial_z)[z]\to\mathcal{U}_{\mathsf{b}}(\partial_z)[z],$$

respectively. Applying these homomorphisms respectively to the expressions given by (4.1) and (4.3), Theorem 4.8 below shows that the resulting expressions in fact live in the common subalgebra $\mathcal{U}_{\mathsf{b}}[z,\partial_z]$. The coefficients of the resulting differential operators in z span the respective images of the quantum Gaudin algebras in $\mathcal{U}_{\mathsf{b}}$, namely

$$\hat{\pi}_{\mathsf{b}}\big(\mathscr{Z}\big(\mathfrak{gl}^{\mathcal{D}}_M\big)\big)\subset\mathcal{U}_{\mathsf{b}}\qquad\text{and}\qquad\hat{\tilde{\pi}}_{\mathsf{b}}\big(\mathscr{Z}\big(\mathfrak{gl}^{\tilde{\mathcal{D}}}_N\big)\big)\subset\mathcal{U}_{\mathsf{b}}.$$

The following theorem establishes that these commutative subalgebras of $\mathcal{U}_{\mathsf{b}}$ coincide.

Theorem 4.8. *We have*

$$\begin{aligned}
&\hat{\pi}_{\mathsf{b}}\left(\prod_{i=1}^{n}(z-z_i)^{\tau_i}\operatorname{cdet}\big(\partial_z\mathbf{1}_{M\times M}-{}^t\mathcal{L}^{\mathcal{D}}(z)\big)\right)\\
&\qquad=\hat{\tilde{\pi}}_{\mathsf{b}}\left(\prod_{a=1}^{m}(\partial_z-\lambda_a)^{\tilde{\tau}_a}\operatorname{cdet}\big(z\mathbf{1}_{N\times N}-\mathcal{L}^{\tilde{\mathcal{D}}}(\partial_z)\big)\right),
\end{aligned}$$

as an equality of polynomial differential operators in z.

Proof. Introduce the $M\times M$ and $N\times N$ block diagonal matrices

$$\Lambda:=\bigoplus_{a=1}^{m}{}^tJ_{\tilde{\tau}_a}(\partial_z-\lambda_a),\qquad Z:=\bigoplus_{i=1}^{n}J_{\tau_i}(z-z_i).$$

Also introduce the $M\times N$ matrices

$$D:=(\partial^a_i)^{M}_{a=1}{}^{N}_{i=1},\qquad X:=(x^a_i)^{M}_{a=1}{}^{N}_{i=1}.$$

Consider the block matrix

$$M:=\begin{pmatrix}\Lambda & X\\ {}^tD & Z\end{pmatrix},$$

with entries in the noncommutative algebra $\mathcal{A} \coloneqq \mathcal{U}_{\mathsf{b}}[z, \partial_z]$. The key observation is that this is a Manin matrix. Indeed, the only non-trivial check is for the 2×2 submatrices of the form

$$\begin{pmatrix} \partial_z - \lambda_a & x_i^a \\ \partial_i^a & z - z_i \end{pmatrix}$$

and for these we have $[\partial_z - \lambda_a, z - z_i] = 1 = [\partial_i^a, x_i^a]$ as required. This fact means that we can follow the proof of Theorem 3.2, with suitable modifications, as follows.

The square matrices Z and Λ with entries in $\mathbb{C}[z, \partial_z] \subset \mathcal{U}_{\mathsf{b}}[z, \partial_z]$ have (two-sided) inverses in the enlarged algebras $\mathcal{A}'' \coloneqq \mathcal{U}_{\mathsf{b}}(z)[\partial_z]$ and $\mathcal{A}' \coloneqq \mathcal{U}_{\mathsf{b}}(\partial_z)[z]$, respectively, both of which contain $\mathcal{A}$ as a subalgebra. These inverses are given explicitly by

$$Z^{-1} = \bigoplus_{i=1}^{n} J_{\tau_i}(z - z_i)^{-1}, \qquad \Lambda^{-1} = \bigoplus_{a=1}^{m} {}^t J_{\tilde{\tau}_a}(\partial_z - \lambda_a)^{-1}.$$

We are therefore in the setup of Proposition 4.5. We may apply it to evaluate $\operatorname{cdet} M$ in two different ways. We obtain

$$\operatorname{cdet} \Lambda \operatorname{cdet} \big(Z - {}^tD\Lambda^{-1}X\big) = \operatorname{cdet} Z \operatorname{cdet} \big(\Lambda - XZ^{-1}\, {}^tD\big), \tag{4.4}$$

as an equality in $\mathcal{A} = \mathcal{U}_{\mathsf{b}}[z, \partial_z]$, namely this is an equality of polynomial differential operators in z with coefficients in $\mathcal{U}_{\mathsf{b}}$.

It remains to evaluate both sides of (4.4) more explicitly. We have

$$\operatorname{cdet} Z = \prod_{i=1}^{n} (z - z_i)^{\tau_i}, \qquad \operatorname{cdet} \Lambda = \prod_{a=1}^{m} (\partial_z - \lambda_a)^{\tilde{\tau}_a},$$

where the order of the products on the right of these equalities does not matter. Now Z and Λ can be written explicitly as follows

$$Z = \sum_{i,j=1}^{N} \tilde{E}_{ij}\big(z\delta_{ij} - \hat{\tilde{\pi}}_{\mathsf{b}}\big(\tilde{\mathsf{E}}^{(\infty)}_{ji[1]}\big)\big), \qquad \Lambda = \sum_{a,b=1}^{M} E_{ab}\big(\partial_z\delta_{ab} - \hat{\pi}_{\mathsf{b}}\big(\mathsf{E}^{(\infty)}_{ab[1]}\big)\big)$$

with $\hat{\pi}_{\mathsf{b}}$ and $\hat{\tilde{\pi}}_{\mathsf{b}}$ as defined in Lemma 4.7. In terms of these expressions we can write

$$\begin{aligned} \Lambda - XZ^{-1}\, {}^tD &= \sum_{a,b=1}^{M} E_{ab}\big(\Lambda - XZ^{-1}\, {}^tD\big)_{ab} \\ &= \partial_z \mathbf{1} - \sum_{a,b=1}^{M} E_{ab}\left(\hat{\pi}_{\mathsf{b}}\big(\mathsf{E}^{(\infty)}_{ab[1]}\big) + \sum_{i=1}^{n} \sum_{j,k=\nu_i+1}^{\nu_i+\tau_i} x_j^a \big(J_{\tau_i}(z - z_i)^{-1}\big)_{jk} \partial_k^b\right). \end{aligned}$$

The latter expression is exactly $\partial_z \mathbf{1} - \hat{\pi}_{\mathsf{b}}\big({}^t\mathcal{L}^{\mathcal{D}}(z)\big)$ by virtue of Lemma 4.7, the expression (2.3a) for the Lax matrix $\mathcal{L}^{\mathcal{D}}(z)$ and the expression (3.4) for the inverse of a Jordan block. Likewise

$$\begin{aligned} Z - {}^tD\Lambda^{-1}X &= \sum_{i,j=1}^{N} \tilde{E}_{ij}\big(Z - {}^tD\Lambda^{-1}X\big)_{ij} \\ &= z\mathbf{1} - \sum_{i,j=1}^{N} \tilde{E}_{ij}\left(\hat{\tilde{\pi}}_{\mathsf{b}}\big(\tilde{\mathsf{E}}^{(\infty)}_{ji[1]}\big) + \sum_{a=1}^{m} \sum_{b,c=\tilde{\nu}_a+1}^{\tilde{\nu}_a+\tilde{\tau}_a} \partial_i^b \big(J_{\tilde{\tau}_a}(\partial_z - \lambda_a)^{-1}\big)_{cb} x_j^c\right), \end{aligned}$$

which coincides with $z\mathbf{1} - \hat{\tilde{\pi}}_{\mathsf{b}}\big(\mathcal{L}^{\tilde{\mathcal{D}}}(\partial_z)\big)$. The result now follows. ■

In the special case of no Jordan blocks and no non-trivial Takiff algebras, Theorem 4.8 can be found in [21]. See also [4, Proposition 8], where it is noted that the relation $\operatorname{cdet} M = \det Z \operatorname{cdet} \big(\Lambda - XZ^{-1}\, {}^tD\big)$ leads to a relation between the classical spectral curve and the "quantum spectral curve".

5 $\mathbb{Z}_2$-cyclotomic Gaudin models with irregular singularities

Another possible class of generalisations of Gaudin models are those whose Lax matrix is equivariant under an action of the cyclic group, determined by a choice of automorphism of the Lie algebra (here $\mathfrak{gl}_M$). Such models were considered in [25, 26, 27] and in [7] for automorphisms of order 2, and for automorphisms of arbitrary finite order in [34, 35].

It is natural to ask whether $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-dualities also exist, in the sense of Section 3, between cyclotomic Gaudin models. Theorem 5.2, which can be deduced from the results of [1], establishes a duality between a cyclotomic $\mathfrak{gl}_M$-Gaudin model associated with the diagram automorphism of $\mathfrak{gl}_M$ and a non-cyclotomic $\mathfrak{sp}_N$-Gaudin model.

5.1 $\mathbb{Z}_2$-cyclotomic Lax matrix for the diagram automorphism

Let $z_i \in \mathbb{C}$ for $i = 1, \dots, n$ be such that $0 \neq z_i \neq \pm z_j$ for $i \neq j$. Pick and fix integers $\tau_i \in \mathbb{Z}_{\geq 1}$ for $i = 0$ and for each $i = 1, \dots, n$. Consider the effective divisor

$$\mathcal{C} = 2\tau_0 \cdot 0 + \sum_{i=1}^{n} \tau_i \cdot z_i + \sum_{i=1}^{n} \tau_i \cdot (-z_i) + 2 \cdot \infty.$$

Note, in particular, that the Takiff degree at the origin is always even. Let $N \in \mathbb{Z}_{\geq 1}$. We require that $\deg \mathcal{C} = 2N + 2$ or in other words,

$$\tau_0 + \sum_{i=1}^{n} \tau_i = N.$$

Let $M \in \mathbb{Z}_{\geq 1}$. As before, cf. Section 2.1, denote by E_{ab} for $a, b = 1, \dots, M$ the standard basis of $\mathfrak{gl}_M$. There is an automorphism σ of $\mathfrak{gl}_M$ defined by

$$\sigma(\mathsf{E}_{ab}) := -\mathsf{E}_{ba}.$$

We call this the *diagram automorphism* of $\mathfrak{gl}_M$. The Lie algebra $\mathfrak{gl}_M$ decomposes into the direct sum of the ± 1 eigenspaces of σ,

$$\mathfrak{gl}_M = \mathfrak{so}_M \oplus \mathfrak{p}_M.$$

Here the subalgebra of invariants, i.e., the $(+1)$-eigenspace, is a copy of the Lie algebra $\mathfrak{so}_M$. The (-1)-eigenspace $\mathfrak{p}_M$ is a copy of the symmetric second rank tensor representation of $\mathfrak{so}_M$. We shall write

$$\mathsf{E}^{\pm}_{ab} := \mathsf{E}_{ab} \pm \mathsf{E}_{ba},$$

so that $\mathsf{E}^{+}_{ab} \in \mathfrak{so}_M$ and $\mathsf{E}^{-}_{ab} \in \mathfrak{p}_M$, for all $a, b = 1, \dots, M$. We introduce the pair of maps $\Pi_{(0)} \colon \mathfrak{gl}_M \to \mathfrak{so}_M$, $\mathsf{E}_{ab} \mapsto \mathsf{E}^{-}_{ab}$ and $\Pi_{(1)} \colon \mathfrak{gl}_M \to \mathfrak{p}_M$, $\mathsf{E}_{ab} \mapsto \mathsf{E}^{+}_{ab}$. More generally, for $r \in \mathbb{Z}_{\geq 0}$ we define $\Pi_{(r)} := \Pi_{(r \bmod 2)} \colon \mathfrak{gl}_M \to \mathfrak{gl}_M$, so that $\Pi_{(r)}\mathsf{E}_{ab} = \mathsf{E}_{ab} - (-1)^r \mathsf{E}_{ba}$.

There is an extension of the automorphism σ to an automorphism of the polynomial algebra $\mathfrak{gl}_M[\varepsilon]$ defined by

$$\mathsf{X}\varepsilon^k \mapsto \sigma(\mathsf{X})(-\varepsilon)^k.$$

Let $\mathfrak{gl}_M[\varepsilon]^\sigma$ denote the subalgebra of invariants. As vector spaces, we have

$$\mathfrak{gl}_M[\varepsilon]^\sigma \cong \mathfrak{so}_M\left[\varepsilon^2\right] \oplus \varepsilon\mathfrak{p}_M\left[\varepsilon^2\right].$$

Define $\mathfrak{gl}_M^{\mathcal{C}}$ to be the direct sum of Takiff Lie algebras

$$\mathfrak{gl}_M^{\mathcal{C}} \coloneqq (\varepsilon_\infty \mathfrak{gl}_M[\varepsilon_\infty])^\sigma/\varepsilon_\infty^2 \oplus \bigoplus_{i=1}^n \mathfrak{gl}_M[\varepsilon_{z_i}]/\varepsilon_{z_i}^{\tau_i} \oplus \mathfrak{gl}_M[\varepsilon_0]^\sigma/\varepsilon_0^{2\tau_0}.$$

Note that as a vector space the Takiff algebra attached to the point at infinity is simply $(\varepsilon_\infty \mathfrak{gl}_M[\varepsilon_\infty])^\sigma/\varepsilon_\infty^2 \cong \mathfrak{p}_M \varepsilon_\infty$.

As before we let $\rho\colon \mathfrak{gl}_M \to \mathrm{Mat}_{M\times M}(\mathbb{C})$ denote the defining representation of $\mathfrak{gl}_M$ and write $E_{ab} \coloneqq \rho(\mathsf{E}_{ab})$. The formal Lax matrix of the $\mathbb{Z}_2$-cyclotomic Gaudin model associated with $\mathfrak{gl}_M^{\mathcal{C}}$ is the $M \times M$ matrix with entries consisting of $\mathfrak{gl}_M^{\mathcal{C}}$-valued rational functions of z, given by

$$\begin{aligned}\tilde{\mathcal{L}}^{\mathcal{C}}(z)dz \coloneqq \sum_{a,b=1}^M E_{ba} \otimes \Bigg(& \mathsf{E}_{ab[1]}^{+(\infty)} + \sum_{r=0}^{2\tau_0-1} \frac{(\Pi_{(r)}\mathsf{E}_{ab})_{[r]}^{(0)}}{z^{r+1}} \\ & + \sum_{i=1}^n \sum_{r=0}^{\tau_i-1} \frac{\mathsf{E}_{ab[r]}^{(z_i)}}{(z-z_i)^{r+1}} + \sum_{i=1}^n \sum_{r=0}^{\tau_i-1} \frac{(-1)^{r+1}\mathsf{E}_{ba[r]}^{(z_i)}}{(z+z_i)^{r+1}} \Bigg) dz. \end{aligned} \tag{5.1}$$

It obeys the following Lax algebra

$$\big[\tilde{\mathcal{L}}_{\mathtt{1}}^{\mathcal{C}}(z), \tilde{\mathcal{L}}_{\mathtt{2}}^{\mathcal{C}}(w)\big] = \big[r_{\mathtt{12}}(z,w), \tilde{\mathcal{L}}_{\mathtt{1}}^{\mathcal{C}}(z)\big] - \big[r_{\mathtt{21}}(w,z), \tilde{\mathcal{L}}_{\mathtt{2}}^{\mathcal{C}}(w)\big] \tag{5.2}$$

where $r_{\mathtt{12}}(z,w)$ denotes the (non-skew-symmetric) classical r-matrix

$$r_{\mathtt{12}}(z,w) \coloneqq \sum_{a,b=1}^M \left(\frac{E_{ba}\otimes E_{ab}}{w-z} - \frac{E_{ba}\otimes E_{ba}}{w+z} \right).$$

Consider the quantity

$$\left(z^{2\tau_0} \prod_{i=1}^n (z-z_i)^{\tau_i}(z+z_i)^{\tau_i} \right) \det\left(\lambda \mathbf{1}_{M\times M} - \tilde{\mathcal{L}}^{\mathcal{C}}(z)\right)$$

This is a polyomial in λ of order M. For each $0 \le k \le M$, the coefficient of λ^k is a rational function in z valued in $S\big(\mathfrak{gl}_M^{\mathcal{C}}\big)$. The *classical cyclotomic Gaudin algebra* $\mathscr{Z}\big(\mathfrak{gl}_M^{\mathcal{C}}\big)$ associated with the divisor $\mathcal{C}$ and the diagram automorphism σ is by definition the Poisson subalgebra of $S\big(\mathfrak{gl}_M^{\mathcal{C}}\big)$ generated by the coefficients of these rational functions. It follows from (5.2) that $\mathscr{Z}\big(\mathfrak{gl}_M^{\mathcal{C}}\big)$ is a Poisson-commutative subalgebra of $S\big(\mathfrak{gl}_M^{\mathcal{C}}\big)$.

5.2 Lax matrix of $\mathfrak{sp}_{2N}$-Gaudin model with regular singularities

Denote by $\tilde{\mathsf{E}}_{IJ}$ the standard basis of $\mathfrak{gl}_{2N}$, where, for convenience, we shall let I, J run over the index set $\mathcal{I} \coloneqq \{-N, \dots, -1, 1, \dots, N\}$. There is a subalgebra of $\mathfrak{gl}_{2N}$, isomorphic to the Lie algebra $\mathfrak{sp}_{2N}$, spanned by

$$\bar{\mathsf{E}}_{IJ} \coloneqq \tilde{\mathsf{E}}_{IJ} - \sigma_I \sigma_J \tilde{\mathsf{E}}_{-J,-I},$$

for all $I, J \in \mathcal{I}$. Here we denote by σ_I the sign of I, equal to 1 if $I > 0$ and to -1 if $I < 0$. We have the relation $\bar{\mathsf{E}}_{-J,-I} = -\sigma_I\sigma_J \bar{\mathsf{E}}_{IJ}$ for every $I, J \in \mathcal{I}$. Let

$$\mathcal{I}_2 \coloneqq \big\{(I,J) \in \mathcal{I}\times\mathcal{I} \,\big|\, I, J > 0 \text{ or } \sigma_I\sigma_J = -1 \text{ with } |I| \le |J|\big\}.$$

Then $\{\bar{\mathsf{E}}_{IJ}\}_{(I,J)\in\mathcal{I}_2}$ is a basis of the subalgebra $\mathfrak{sp}_{2N}$. A dual basis with respect to half the trace in the fundamental representation is given by $\{\bar{\mathsf{E}}^{IJ}\}_{(I,J)\in\mathcal{I}_2}$ where

$$\bar{\mathsf{E}}^{IJ} := \tilde{\mathsf{E}}_{JI} - \sigma_I\sigma_J\tilde{\mathsf{E}}_{-I,-J}, \qquad \bar{\mathsf{E}}^{I,-I} := \tilde{\mathsf{E}}_{-I,I},$$

for any $I,J\in\mathcal{I}$ with $J\neq -I$. Indeed, if we let $\bar{E}_{IJ} := \rho(\bar{\mathsf{E}}_{IJ})$ and $\bar{E}^{IJ} := \rho(\bar{\mathsf{E}}^{IJ})$ for all $I,J\in\mathcal{I}$ then we have $\frac{1}{2}\operatorname{tr}\big(\bar{E}_{IJ}\bar{E}^{KL}\big) = \delta_{IL}\delta_{JK}$ for all $(I,J),(K,L)\in\mathcal{I}_2$.

Let $\bar{\mathcal{D}}$ denote the special case of the effective divisor $\tilde{\mathcal{D}}$ of Section 2.1 obtained by setting $\tilde{\tau}_a = 1$ for each $a=1,\dots,m$, and hence $m=M$. That is,

$$\bar{\mathcal{D}} = \sum_{a=1}^{M}\lambda_a + 2\cdot\infty. \tag{5.3}$$

Introduce the direct sum of Lie algebras

$$\mathfrak{sp}_{2N}^{\bar{\mathcal{D}}} := \tilde{\varepsilon}_\infty\mathfrak{sp}_{2N}[\tilde{\varepsilon}_\infty]/\tilde{\varepsilon}_\infty^2 \oplus \bigoplus_{a=1}^{M}\mathfrak{sp}_{2N}.$$

The Lax matrix of the classical Gaudin model associated with the divisor $\bar{\mathcal{D}}$ is the $2N\times 2N$ matrix of $\mathfrak{sp}_{2N}^{\bar{\mathcal{D}}}$-valued rational functions of λ given by

$$\mathcal{L}^{\bar{\mathcal{D}}}(\lambda)d\lambda := \sum_{(I,J)\in\mathcal{I}_2}\bar{E}^{IJ}\otimes\left(\bar{\mathsf{E}}_{IJ}^{(\infty)} + \sum_{a=1}^{M}\frac{\bar{\mathsf{E}}_{IJ}^{(\lambda_a)}}{\lambda-\lambda_a}\right)d\lambda, \tag{5.4}$$

where by abuse of notation we drop the subscript on the Takiff generators, namely we define $\bar{\mathsf{E}}_{IJ}^{(\lambda_a)} := \bar{\mathsf{E}}_{IJ[0]}^{(\lambda_a)}$ for all $a=1,\dots,M$ and $\bar{\mathsf{E}}_{IJ}^{(\infty)} := \bar{\mathsf{E}}_{IJ[1]}^{(\infty)}$. It obeys the Lax algebra

$$\big[\mathcal{L}_{\mathtt{1}}^{\bar{\mathcal{D}}}(\lambda), \mathcal{L}_{\mathtt{2}}^{\bar{\mathcal{D}}}(\mu)\big] = \big[\bar{r}_{\mathtt{12}}(\lambda,\mu), \mathcal{L}_{\mathtt{1}}^{\bar{\mathcal{D}}}(\lambda) + \mathcal{L}_{\mathtt{2}}^{\bar{\mathcal{D}}}(\mu)\big] \tag{5.5}$$

where $\bar{r}_{\mathtt{12}}(\lambda,\mu)$ is the standard skew-symmetric classical r-matrix with spectral parameter for the Lie algebra $\mathfrak{sp}_{2N}$, namely

$$\bar{r}_{\mathtt{12}}(\lambda,\mu) := \sum_{(I,J)\in\mathcal{I}_2}\frac{\bar{E}^{IJ}\otimes\bar{E}_{IJ}}{\mu-\lambda}.$$

Just as in Section 3.1 we may consider the subalgebra $\mathscr{Z}\big(\mathfrak{sp}_{2N}^{\bar{\mathcal{D}}}\big)$ of the Poisson algebra $S\big(\mathfrak{sp}_{2N}^{\bar{\mathcal{D}}}\big)$ generated by the coefficients rational functions in λ obtained as the coefficients of the polynomial in z defined by

$$\prod_{a=1}^{M}(\lambda-\lambda_a)\det\big(z\mathbf{1}_{N\times N} - \mathcal{L}^{\bar{\mathcal{D}}}(\lambda)\big),$$

which is Poisson-commutative by virtue of the relation (5.5).

5.3 Bosonic realisation

Consider the Poisson algebra $\mathcal{P}_{\mathsf{b}} := \mathbb{C}[x_i^a, p_j^b]_{i,j=1}^{N}{}_{a,b=1}^{M}$, as in Section 3.2, with Poisson brackets given by (3.3).

We now want to break up the list of integers from 1 to N into $n+1$ blocks of size τ_i for each $i = 0, 1, \dots, n$. Define the integers ν_i by – in contrast to (2.1) –

$$\nu_i \coloneqq \sum_{j=0}^{i-1} \tau_j,$$

for each $i = 0, \dots, N$ (note in particular that now $\nu_0 = 0$), so that

$$(1, \dots, N) = (1, \dots, \tau_0; \nu_1 + 1, \dots, \nu_1 + \tau_1; \dots; \nu_n + 1, \dots, \nu_n + \tau_n).$$

Lemma 5.1. *Let $\mu \in \mathbb{C}$ be arbitrary and define a pair of linear maps $\pi_{\mathsf{b}} \colon \mathfrak{gl}_M^{\mathcal{C}} \to \mathcal{P}_{\mathsf{b}}$ and $\bar{\pi}_{\mathsf{b}} \colon \mathfrak{sp}_{2N}^{\bar{\mathcal{D}}} \to \mathcal{P}_{\mathsf{b}}$ by*

$$\pi_{\mathsf{b}}\big(\mathsf{E}^{(z_i)}_{ab[r]}\big) = \sum_{u=\nu_i+1}^{\nu_i+\tau_i-r} x^a_{u+r} p^b_u, \qquad \pi_{\mathsf{b}}\big(\mathsf{E}^{+(\infty)}_{ab[1]}\big) = \lambda_a \delta_{ab},$$
$$\pi_{\mathsf{b}}\big((\Pi_{(s)}\mathsf{E}_{ab})^{(0)}_{[s]}\big) = \sum_{u=1}^{\tau_0-s} \big(x^a_{u+s} p^b_u - (-1)^s x^b_{u+s} p^a_u\big) - \mu \sum_{\substack{u,v=1\\u+v=s+1}}^{\tau_0} (-1)^v x^a_u x^b_v$$

for every $r = 0, \dots, \tau_i - 1$, $s = 0, \dots, 2\tau_0 - 1$, $i = 1, \dots, n$ and $a, b = 1, \dots, M$, and

$$\bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{ij}\big) = p^a_j x^a_i \qquad \bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{i,-j}\big) = -x^a_j x^a_i, \qquad \bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{-i,j}\big) = p^a_j p^a_i,$$
$$\bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\infty)}_{IJ}\big) = -\left(\bigoplus_{i=n}^{1} \big(-J_{\tau_i}(-z_i)\big) \oplus \big(-J_{\tau_0}(0)\big) \oplus J_{\tau_0}(0) \oplus \bigoplus_{i=1}^{n} J_{\tau_i}(-z_i) + \mu \tilde{E}_{1,-1}\right)_{JI},$$

for every $i, j = 1, \dots, N$, $I, J \in \mathcal{I}$ and $a = 1, \dots, m$. These maps are homomorphisms of Lie algebras. They extend uniquely to homomorphisms of Poisson algebras $\pi_{\mathsf{b}} \colon S\big(\mathfrak{gl}_M^{\mathcal{C}}\big) \to \mathcal{P}_{\mathsf{b}}$ and $\bar{\pi}_{\mathsf{b}} \colon S\big(\mathfrak{sp}_{2N}^{\bar{\mathcal{D}}}\big) \to \mathcal{P}_{\mathsf{b}}$.

Proof. We first show that π_{b} is a homomorphism. It follows, exactly as in the proof of Lemma 3.1 (see Lemma 4.7) that

$$\big\{\pi_{\mathsf{b}}\big(\mathsf{E}^{(z_i)}_{ab[r]}\big), \pi_{\mathsf{b}}\big(\mathsf{E}^{(z_j)}_{cd[s]}\big)\big\} = \pi_{\mathsf{b}}\big(\big[\mathsf{E}^{(z_i)}_{ab[r]}, \mathsf{E}^{(z_j)}_{cd[s]}\big]\big), \tag{5.6}$$

for any $r, s = 0, \dots, \tau_i - 1$, $i, j = 1, \dots, n$ and $a, b, c, d = 1, \dots, M$. We also clearly have

$$\big\{\pi_{\mathsf{b}}\big((\Pi_{(s)}\mathsf{E}_{ab})^{(0)}_{[s]}\big), \pi_{\mathsf{b}}\big(\mathsf{E}^{(z_i)}_{cd[r]}\big)\big\} = 0$$

for any $r = 0, \dots, \tau_i - 1$ $i = 1, \dots, n$ and $a, b, c, d = 1, \dots, M$ since the canonical variables entering each argument of the Poisson brackets mutually commute.

To simplify the notation, introduce $y^{ab}_r \coloneqq \sum\limits_{u=1}^{\tau_0-r} \big(x^a_{u+r} p^b_u - (-1)^r x^b_{u+r} p^a_u\big)$. We can then write

$$\pi_{\mathsf{b}}\big((\Pi_{(s)}\mathsf{E}_{ab})^{(0)}_{[s]}\big) = y^{ab}_s - \mu \sum_{\substack{u,v=1\\u+v=s+1}}^{\tau_0} (-1)^v x^a_u x^b_v.$$

By a similar computation to the one leading to (5.6), we find that

$$\big\{y^{ab}_r, y^{cd}_s\big\} = \delta_{bc} y^{ad}_{r+s} + (-1)^s \delta_{ac} y^{db}_{r+s} + (-1)^r \delta_{bd} y^{ca}_{r+s} + (-1)^{r+s} \delta_{ad} y^{bc}_{r+s}.$$

Likewise, we have

$$\begin{aligned}
&- \sum_{\substack{v,w=1\\ v+w=s+1}}^{\tau_0} (-1)^w \big\{ y^{ab}_r, x^c_v x^d_w \big\} \\
&\qquad = - \sum_{\substack{u=r+1\\ u+w=r+s+1}}^{\tau_0} \sum_{w=1}^{s} (-1)^w \big(\delta_{bc} x^a_u x^d_w + (-1)^r \delta_{bd} x^c_u x^a_w + (-1)^s \delta_{ac} x^d_u x^b_w + (-1)^{r+s} \delta_{ad} x^b_u x^c_w \big).
\end{aligned}$$

and also by symmetry we obtain

$$\begin{aligned}
&- \sum_{\substack{v,w=1\\ v+w=r+1}}^{\tau_0} (-1)^w \big\{ x^a_v x^b_w, y^{cd}_s \big\} = \sum_{\substack{v,w=1\\ v+w=r+1}}^{\tau_0} (-1)^w \big\{ y^{cd}_s, x^a_v x^b_w \big\} \\
&\qquad = - \sum_{\substack{u=1\\ u+w=r+s+1}}^{r} \sum_{w=s+1}^{\tau_0} (-1)^w \big(\delta_{bc} x^a_u x^d_w + (-1)^r \delta_{bd} x^c_u x^a_w + (-1)^s \delta_{ac} x^d_u x^b_w + (-1)^{r+s} \delta_{ad} x^b_u x^c_w \big).
\end{aligned}$$

It now follows by combining all the above that

$$\begin{aligned}
&\big\{ \pi_{\mathsf{b}}\big((\Pi_{(r)} \mathsf{E}_{ab})^{(0)}_{[r]} \big), \pi_{\mathsf{b}}\big((\Pi_{(s)} \mathsf{E}_{cd})^{(0)}_{[s]} \big) \big\} \\
&\quad = \delta_{bc} \pi_{\mathsf{b}}\big((\Pi_{(r)} \mathsf{E}_{ad})^{(0)}_{[r+s]} \big) + (-1)^s \delta_{ac} \pi_{\mathsf{b}}\big((\Pi_{(r)} \mathsf{E}_{db})^{(0)}_{[r+s]} \big) \\
&\qquad + (-1)^r \delta_{bd} \pi_{\mathsf{b}}\big((\Pi_{(r)} \mathsf{E}_{ca})^{(0)}_{[r+s]} \big) + (-1)^{r+s} \delta_{ad} \pi_{\mathsf{b}}\big((\Pi_{(r)} \mathsf{E}_{bc})^{(0)}_{[r+s]} \big) \\
&\quad = \pi_{\mathsf{b}}\big(\big[(\Pi_{(r)} \mathsf{E}_{ab})^{(0)}_{[r]}, (\Pi_{(s)} \mathsf{E}_{cd})^{(0)}_{[s]} \big] \big),
\end{aligned}$$

as required. And finally, since $\mathsf{E}^{+(\infty)}_{ab[1]}$ is a Casimir and is sent to a constant under π_{b}, all Poisson brackets involving it are preserved by π_{b}.

We now turn to showing that $\bar{\pi}_{\mathsf{b}}$ is also a homomorphism. Define q^a_I for each $I \in \mathcal{I}$ and $a = 1, \dots, M$ by letting $q^a_i := x^a_i$ and $q^a_{-i} := p^a_i$ for every $i = 1, \dots, N$. In this notation the Poisson brackets (3.3) can be rewritten more uniformly as

$$\big\{ q^a_I, q^b_J \big\} = \sigma_J \delta_{I,-J} \delta_{ab},$$

for all $I, J \in \mathcal{I}$ and $a, b = 1, \dots, M$. Moreover, we also have $\bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{IJ} \big) = \sigma_J q^a_I q^a_{-J}$ for all $I, J \in \mathcal{I}$ and $a = 1, \dots, M$. We then have

$$\begin{aligned}
&\big\{ \bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{IJ} \big), \bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_b)}_{KL} \big) \big\} \\
&\quad = \sigma_J \sigma_L \big(\sigma_K \delta_{I,-K} q^a_{-J} q^a_{-L} + \sigma_{-L} \delta_{I,L} q^a_K q^a_{-J} + \sigma_K \delta_{J,K} q^a_I q^a_{-L} + \sigma_{-L} \delta_{J,-L} q^a_I q^a_K \big) \delta_{ab} \\
&\quad = \sigma_J \sigma_K \big(\bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{IL} \big) \delta_{J,K} + \bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{-J,-K} \big) \delta_{I,L} + \bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{I,-K} \big) \delta_{-J,L} + \bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{-J,L} \big) \delta_{K,-I} \big) \delta_{ab} \\
&\quad = \bar{\pi}_{\mathsf{b}}\big(\big[\bar{\mathsf{E}}^{(\lambda_a)}_{IJ}, \bar{\mathsf{E}}^{(\lambda_b)}_{KL} \big] \big),
\end{aligned}$$

where in the second equality we have made use of the fact that $\sigma_I \sigma_{-I} = -1$ for any $I \in \mathcal{I}$. Finally, the Poisson brackets involving the generators $\bar{\mathsf{E}}^{(\infty)}_{IJ}$ attached to infinity are all trivially preserved by $\bar{\pi}_{\mathsf{b}}$. ■

We are now in a position to prove the analogue of Theorem 3.2 in the present context.

Theorem 5.2. *For any $\mu \in \mathbb{C}$ as in Lemma 5.1, we have the relation*

$$\pi_{\mathsf{b}}\left(z^{2\tau_0}\prod_{i=1}^{n}(z-z_i)^{\tau_i}(z+z_i)^{\tau_i}\det\left(\lambda\mathbf{1}_{M\times M}-\tilde{\mathcal{L}}^{\mathcal{C}}(z)\right)\right)$$
$$=\bar{\pi}_{\mathsf{b}}\left(\prod_{a=1}^{M}(\lambda-\lambda_a)\det\left(z\mathbf{1}_{N\times N}-\mathcal{L}^{\tilde{\mathcal{D}}}(\lambda)\right)\right).$$

Proof. We follow the argument given in the proof of Theorem 3.2 very closely. Consider the $M\times M$ and $2N\times 2N$ block matrices

$$\Lambda := \big((\lambda-\lambda_a)\delta_{ab}\big)_{a,b=1}^{M},$$
$$Z := \bigoplus_{i=n}^{1}\big(-J_{\tau_i}(-z-z_i)\big)\oplus\big(-J_{\tau_0}(-z)\big)\oplus J_{\tau_0}(z)\oplus\bigoplus_{i=1}^{n}J_{\tau_i}(z-z_i)+\mu\tilde{E}_{1,-1}.$$

We use here the convention, cf. Section 5.2, that indices on components of the $2N\times 2N$ matrix Z run through the index set $\mathcal{I}=\{-N,\dots,-1,1,\dots,N\}$. As an example of the form of the matrix Z, if $n=2$, $\tau_0=2$, $\tau_1=1$ and $\tau_2=2$ then we have

$$Z=\begin{pmatrix}
z+z_2 & 0 & & & & & & & & & \\
1 & z+z_2 & & & & & & & & 0 & \\
 & & z+z_1 & & & & & & & & \\
 & & & z & 0 & 0 & 0 & & & & \\
 & & & 1 & z & 0 & 0 & & & & \\
 & & & 0 & \mu & z & 0 & & & & \\
 & & & 0 & 0 & -1 & z & & & & \\
 & & & & & & & z-z_1 & & & \\
 & 0 & & & & & & & z-z_2 & 0 \\
 & & & & & & & & -1 & z-z_2
\end{pmatrix}.$$

We define a pair of $M\times 2N$ matrices P and X, whose columns are also indexed by the set $\mathcal{I}$, as

$$^{t}P := \begin{pmatrix}
-x_N^1 & \dots & -x_N^M\\
\vdots & \ddots & \vdots\\
-x_1^1 & \dots & -x_1^M\\
p_1^1 & \dots & p_1^M\\
\vdots & \ddots & \vdots\\
p_N^1 & \dots & p_N^M
\end{pmatrix},\qquad
X := \begin{pmatrix}
p_N^1 & \dots & p_1^1 & x_1^1 & \dots & x_N^1\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
p_N^M & \dots & p_1^M & x_1^M & \dots & x_N^M
\end{pmatrix}.$$

Consider now the block $(M+2N)\times(M+2N)$ square matrix (3.5) with Λ, Z, X and P defined as above. Now the derivation leading to the equation (3.6) from the proof of Theorem 3.2 still holds and so it just remains to compute the determinants appearing on both sides of this identity.

On the one hand, we have

$$\Lambda - XZ^{-1}\,{}^{t}P=\sum_{a,b=1}^{M}E_{ab}\big(\Lambda-XZ^{-1}\,{}^{t}P\big)_{ab}$$
$$=\lambda\mathbf{1}-\sum_{a,b=1}^{M}E_{ab}\left(\pi_{\mathsf{b}}\big(\mathsf{E}_{ab[1]}^{+(\infty)}\big)+\sum_{i=1}^{n}\sum_{j,k=\nu_i+1}^{\nu_i+\tau_i}x_j^a\big(Z^{-1}\big)_{jk}p_k^b+\sum_{j,k=1}^{\tau_0}x_j^a\big(Z^{-1}\big)_{jk}p_k^b\right.$$

$$-\sum_{j,k=1}^{\tau_0} p^a_j\big(Z^{-1}\big)_{-j,-k} x^b_k - \sum_{j,k=1}^{\tau_0} x^a_j\big(Z^{-1}\big)_{j,-k} x^b_k - \sum_{i=1}^{n}\sum_{j,k=\nu_i+1}^{\nu_i+\tau_i} p^a_j\big(Z^{-1}\big)_{-j,-k} x^b_k\Bigg).$$

For each $i=1,\dots,n$ we note using the expression (3.4) for the inverse of a Jordan block together with Lemma 5.1 that

$$\sum_{j,k=\nu_i+1}^{\nu_i+\tau_i} x^a_j\big(Z^{-1}\big)_{jk} p^b_k = \sum_{r=0}^{\tau_i-1} \frac{\pi_{\mathsf{b}}\big(\mathsf{E}^{(z_i)}_{ab[r]}\big)}{(z-z_i)^{r+1}},$$

$$-\sum_{j,k=\nu_i+1}^{\nu_i+\tau_i} p^a_j\big(Z^{-1}\big)_{-j,-k} x^b_k = \sum_{r=0}^{\tau_i-1} \frac{(-1)^{r+1}\pi_{\mathsf{b}}\big(\mathsf{E}^{(z_i)}_{ba[r]}\big)}{(z+z_i)^{r+1}}.$$

Next, for the two terms in the middle line above, corresponding to the origin, we find

$$\sum_{j,k=1}^{\tau_0} \big(x^a_j\big(Z^{-1}\big)_{jk} p^b_k - p^a_j\big(Z^{-1}\big)_{-j,-k} x^b_k\big) = \sum_{s=0}^{\tau_0-1} \frac{1}{z^{s+1}} \sum_{u=1}^{\tau_0-s} \big(x^a_{u+s}p^b_u - (-1)^s x^b_{u+s} p^a_u\big).$$

Finally, for the remaining term we have

$$-\sum_{j,k=1}^{\tau_0} x^a_j\big(Z^{-1}\big)_{j,-k} x^b_k = -\sum_{s=1}^{2\tau_0-1} \frac{\mu}{z^{s+1}} \sum_{\substack{u,v=1\\ u+v=s+1}}^{\tau_0} (-1)^v x^a_u x^b_v.$$

Putting all the above together we deduce that $\Lambda - XZ^{-1}\,{}^tP = \lambda\mathbf{1} - \pi_{\mathsf{b}}\big({}^t\tilde{\mathcal{L}}^{\mathcal{C}}(z)\big)$.

On the other hand, we have

$$\begin{aligned} Z - {}^tP\Lambda^{-1}X &= \sum_{I,J\in\mathcal{I}} \tilde{E}_{IJ}\big(Z - {}^tP\Lambda^{-1}X\big)_{IJ} \\ &= z\mathbf{1} - \sum_{(I,J)\in\mathcal{I}_2} \bar{E}^{IJ}\left(\bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\infty)}_{IJ}\big) - \sum_{a=1}^{M} \frac{\bar{\pi}_{\mathsf{b}}\big(\bar{\mathsf{E}}^{(\lambda_a)}_{IJ}\big)}{\lambda-\lambda_a}\right) = z\mathbf{1} - \bar{\pi}_{\mathsf{b}}\big(\mathcal{L}^{\bar{\mathcal{D}}}(\lambda)\big). \end{aligned}$$

To see the second equality we note that setting $z=0$ in $Z - {}^tP\Lambda^{-1}X$ yields a $2N\times 2N$ symplectic matrix, i.e., of the block form

$$M = \begin{pmatrix} A & B \\ C & -\widetilde{A} \end{pmatrix}$$

with $\widetilde{B} = B$ and $\widetilde{C} = C$, where for an $N\times N$ matrix A we denote by $\widetilde{A}$ the transpose of A along the minor diagonal. And for any such matrix M we have

$$\begin{aligned} M &= \sum_{I,J\in\mathcal{I}} \tilde{E}_{IJ}M_{IJ} = \sum_{i,j=1}^{N} \big(\big(\tilde{E}_{ij} - \tilde{E}_{-j,-i}\big)A_{ij} + \tilde{E}_{i,-j}C_{ij} - \tilde{E}_{-i,j}B_{ij}\big) \\ &= \sum_{i,j=1}^{N} \bar{E}^{ij}A_{ji} + \sum_{\substack{i,j=1\\ i\le j}}^{N} \bar{E}^{-i,j}C_{ji} - \sum_{\substack{i,j=1\\ i\le j}}^{N} \bar{E}^{i,-j}B_{ji} = \sum_{(I,J)\in\mathcal{I}_2} \bar{E}^{IJ}M_{IJ}. \end{aligned}$$

Lastly, we clearly have $\det\Lambda = \prod_{a=1}^{M}(\lambda-\lambda_a)$ and $\det Z = z^{2\tau_0}\prod_{i=1}^{n}(z-z_i)^{\tau_i}(z+z_i)^{\tau_i}$ from which the result now follows, using again the fact that $\det{}^tA = \det A$ for any square matrix A, as in the proof of Theorem 3.2. ∎

Remark 5.3. Consider replacing p by ∂ in the $(M+2N)\times(M+2N)$ square matrix

$$\begin{pmatrix} \lambda-\lambda_1 & & 0 & p_N^1 & \dots & p_1^1 & x_1^1 & \dots & x_N^1 \\ & \ddots & & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & & \lambda-\lambda_M & p_N^M & \dots & p_1^M & x_1^M & \dots & x_N^M \\ -x_N^1 & \dots & -x_N^M & & & & & & \\ \vdots & \ddots & \vdots & & & & & & \\ -x_1^1 & \dots & -x_1^M & & & & & & \\ p_1^1 & \dots & p_1^M & & & Z & & & \\ \vdots & \ddots & \vdots & & & & & & \\ p_N^1 & \dots & p_N^M & & & & & & \end{pmatrix}$$

used in the proof of Theorem 5.2. The resulting square matrix with non-commutative entries is not Manin since, for example, the entries of the first column are not mutually commuting. Consequently, we do not immediately obtain a quantum analogue of the classical relation in Theorem 5.2.

A related remark is that in the quantum case, higher Gaudin Hamiltonians for cyclotomic Gaudin models do exist but they are not in general given by a simple cdet-type formula. See [34, 36] (and especially Remark 2.5 in [34]).

Remark 5.4. Note that we did not allow irregular singularities on the $\mathfrak{sp}_{2N}$ side (appart from the double pole at infinity).

From the point of view of $(\mathfrak{gl}_M, \mathfrak{gl}_N)$-duality, the absence of irregular singularities in the $\mathfrak{sp}_{2N}$-Gaudin model is controlled by the fact that the matrix

$$\big(\pi_{\mathfrak{b}}\big(\mathsf{E}^{+(\infty)}_{ab[1]}\big)\big)_{a,b=1}^{M}, \tag{5.7}$$

representing the Casimir generators attached to infinity in the cyclotomic $\mathfrak{gl}_M$-Gaudin model, is purely diagonal and in particular has no Jordan blocks, as in Lemma 5.1. Yet this is forced on us since the matrix (5.7) is symmetric.

Alternatively, note that if one naively attempts to run the arguments above for the divisor $\tilde{\mathcal{D}}$ in place of $\bar{\mathcal{D}}$, one does not obtain a homomorphism $\mathfrak{sp}_{2N}^{\tilde{\mathcal{D}}} \to \mathcal{P}_{\mathfrak{b}}$. For example, Poisson brackets of the form $\big\{-\sum_u x_i^u x_j^{u+1}, \sum_v p_k^v p_l^{v+1}\big\}$ produce two sorts of terms: "good" terms like $\sum_u x_i^u p_l^{u+2}\delta_{jk}$, which respect the gradation of the Takiff algebra, but also "bad" terms like $\sum_u x_j^{u+1} p_l^{u+1}\delta_{ik}$, which do not.

5.4 Example: Neumann model

We end this section by considering the special case of Theorem 5.2 when $N=1$ and $\mu=-1$.

Specifically, for the $\mathbb{Z}_2$-cyclotomic Gaudin model of Section 5.1 we take $n=0$ and $\tau_0=1$. The formal Lax matrix (5.1) of the corresponding cyclotomic $\mathfrak{gl}_M$-Gaudin model with effective divisor $\mathcal{C} = 2\cdot 0 + 2\cdot\infty$ then reduces to

$$\tilde{\mathcal{L}}^{\mathcal{C}}(z)dz = \sum_{a,b=1}^{M} E_{ba}\otimes\left(\mathsf{E}^{+(\infty)}_{ab[1]} + \frac{\mathsf{E}^{-(0)}_{ab[0]}}{z} + \frac{\mathsf{E}^{+(0)}_{ab[1]}}{z^2}\right)dz. \tag{5.8}$$

When $N=1$ in Section 5.2 we have the canonical isomorphism $\mathfrak{sp}_2 \simeq \mathfrak{sl}_2$ given by $\bar{\mathsf{E}}_{11}\mapsto -\mathsf{H}$, $\bar{\mathsf{E}}_{1,-1}\mapsto 2\mathsf{F}$ and $\bar{\mathsf{E}}_{-1,1}\mapsto 2\mathsf{E}$. The dual basis elements are sent under this isomorphism to

$\bar{\mathsf{E}}^{11} = \bar{\mathsf{E}}_{11} \mapsto -\mathsf{H}$, $\bar{\mathsf{E}}^{1,-1} = \frac{1}{2}\bar{\mathsf{E}}_{-1,1} \mapsto \mathsf{E}$ and $\bar{\mathsf{E}}^{-1,1} = \frac{1}{2}\bar{\mathsf{E}}_{1,-1} \mapsto \mathsf{F}$. The formal Lax matrix (5.4) of the $\mathfrak{sl}_2$-Gaudin model with effective divisor (5.3) then becomes,

$$\mathcal{L}^{\bar{\mathcal{D}}}(\lambda)d\lambda = \Bigg(H \otimes \mathsf{H}^{(\infty)} + 2E \otimes \mathsf{F}^{(\infty)} + 2F \otimes \mathsf{E}^{(\infty)} \\ + \sum_{a=1}^{M} \frac{H \otimes \mathsf{H}^{(\lambda_a)} + 2E \otimes \mathsf{F}^{(\lambda_a)} + 2F \otimes \mathsf{E}^{(\lambda_a)}}{\lambda - \lambda_a} \Bigg) d\lambda, \tag{5.9}$$

where we have used the notation

$$E \coloneqq \rho(\mathsf{E}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F \coloneqq \rho(\mathsf{F}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H \coloneqq \rho(\mathsf{H}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Poisson algebra $\mathcal{P}_{\mathsf{b}}$ in the present context is simply $\mathbb{C}[x_a, p_a]_{a,b=1}^M$ where we have dropped the subscript 1 from the canonical variables by defining $x_a \coloneqq x_1^a$ and $p_a \coloneqq p_1^a$. In terms of this notation, the representation $\pi_{\mathsf{b}} \colon \mathfrak{gl}_M^{\mathcal{C}} \to \mathcal{P}_{\mathsf{b}}$ from Theorem 5.2 reads

$$\pi_{\mathsf{b}}\big(\mathsf{E}^{+(\infty)}_{ab[1]}\big) = \lambda_a \delta_{ab}, \qquad \pi_{\mathsf{b}}\big(\mathsf{E}^{-(0)}_{ab[0]}\big) = x_a p_b - x_b p_a, \qquad \pi_{\mathsf{b}}\big(\mathsf{E}^{+(0)}_{ab[1]}\big) = -x_a x_b,$$

recalling that $\mu = -1$. Correspondingly, the map $\bar{\pi}_{\mathsf{b}} \colon \mathfrak{sp}_2^{\mathcal{C}} \to \mathcal{P}_{\mathsf{b}}$ takes the form

$$\bar{\pi}_{\mathsf{b}}\big(\mathsf{E}^{(\infty)}\big) = \tfrac{1}{2}, \qquad \bar{\pi}_{\mathsf{b}}\big(\mathsf{F}^{(\infty)}\big) = 0, \qquad \bar{\pi}_{\mathsf{b}}\big(\mathsf{H}^{(\infty)}\big) = 0,$$
$$\bar{\pi}_{\mathsf{b}}\big(\mathsf{E}^{(\lambda_a)}\big) = \tfrac{1}{2}p_a^2, \qquad \bar{\pi}_{\mathsf{b}}\big(\mathsf{F}^{(\lambda_a)}\big) = -\tfrac{1}{2}x_a^2, \qquad \bar{\pi}_{\mathsf{b}}\big(\mathsf{H}^{(\lambda_a)}\big) = x_a p_a.$$

Applying the first representation π_{b} to the formal Lax matrix (5.8) we find

$$\tilde{L}(z)dz \coloneqq \pi_{\mathsf{b}}\big(\tilde{\mathcal{L}}^{\mathcal{C}}(z)\big)dz \\ = \left(\sum_{a=1}^{M} \lambda_a E_{aa} - z^{-1} \sum_{a,b=1}^{M} (x_a p_b - x_b p_a) E_{ab} - z^{-2} \sum_{a,b=1}^{M} x_a x_b E_{ab} \right) dz.$$

If we introduce variables ω_a, $a = 1, \dots, M$ such that $\omega_a^2 = \lambda_a$ then the above coincides with the $M \times M$ Lax matrix of the Neumann model, with Hamiltonian

$$H = \frac{1}{4} \sum_{\substack{a,b=1 \\ a \neq b}}^{M} (x_a p_b - x_b p_a)^2 + \frac{1}{2} \sum_{a=1}^{M} \omega_a^2 x_a^2,$$

describing the motion of a particle constrained to the sphere $\sum_{a=1}^{M} x_a^2 = 1$ in $\mathbb{R}^M$ and subject to harmonic forces with frequency ω_a along the a^{th} axis. On the other hand, applying $\bar{\pi}_{\mathsf{b}}$ to the formal Lax matrix (5.9) yields

$$L(\lambda)d\lambda \coloneqq \bar{\pi}_{\mathsf{b}}\big(\mathcal{L}^{\bar{\mathcal{D}}}(\lambda)\big)d\lambda = 2 \begin{pmatrix} \sum\limits_{a=1}^{M} \frac{x_a p_a}{\lambda - \lambda_a} & \sum\limits_{a=1}^{M} \frac{-x_a^2}{\lambda - \lambda_a} \\ 1 + \sum\limits_{a=1}^{M} \frac{p_a^2}{\lambda - \lambda_a} & -\sum\limits_{a=1}^{M} \frac{x_a p_a}{\lambda - \lambda_a} \end{pmatrix} d\lambda,$$

which coincides with the expression for the 2×2 Lax matrix of the same model. The statement of Theorem 5.2 corresponds to the well known relation between the above two Lax formulations of the Neumann model (see, e.g., [28, Section 12])

$$z^2 \det\big(\lambda \mathbf{1}_{M\times M} - \tilde{L}(z)\big) = \prod_{a=1}^{M} (\lambda - \lambda_a) \det\big(z \mathbf{1}_{2\times 2} - L(\lambda)\big).$$

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