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A CLASSIFICATION OF DISJOINT UNIONS OF TWO OR THREE COPIES OF THE FREE MONOGENIC SEMIGROUP

N. ABU-GHAZALH, J. D. MITCHELL, Y. PÉRESSE, N. RUŠKUC

Abstract. We prove that, up to isomorphism and anti-isomorphism, there are only two semigroups which are the union of two copies of the free monogenic semigroup. Similarly, there are only nine semigroups which are the union of three copies of the free monogenic semigroup. We provide finite presentations for each of these semigroups.

1. Introduction and Preliminaries

There are several well-known examples of structural theorems for semigroups, which involve decomposing a semigroup into a disjoint union of subsemigroups. For example, up to isomorphism, the Rees Theorem states that every completely simple semigroup is a Rees matrix semigroup over a group G, and is thus a disjoint union of copies of G, see [6, Theorem 3.3.1]; every Clifford semigroup is a strong semilattice of groups and as such it is a disjoint union of its maximal subgroups, see [6, Theorem 4.2.1]; every commutative semigroup is a semilattice of archimedean semigroups, see [5, Theorem 2.2].

If S is a semigroup which can be decomposed into a disjoint union of subsemigroups, then it is natural to ask how the properties of the subsemigroups influence S. For example, if the subsemigroups are finitely generated, then so is S. There are several further examples in the literature where such questions are addressed: Araújo et al. [2] consider the finite presentability of semigroups which are the disjoint union of finitely presented subsemigroups; Golubov [3] showed that a semigroup which is the disjoint union of residually finite subsemigroups is residually finite; in [1] the authors proved that every semigroup which is a disjoint union of finitely many copies of $\mathbb{N}$ is finitely presented; further references are [4, 8].

In this paper we completely classify those semigroups which are the disjoint union of two or three copies of the free monogenic semigroup.

The main theorems of this paper are the following.

Theorem 1.1. *Let S be a semigroup. Then S is a disjoint union of two copies of the free monogenic semigroup if and only if S is isomorphic or anti-isomorphic to the semigroup defined by one of the following presentations:*

(i) $\langle\, a, b \mid ab = ba = a^k \,\rangle$ *for some* $k \geq 1$*;*
(ii) $\langle\, a, b \mid ab = a^2,\ ba = b^2 \,\rangle$.

Theorem 1.2. *Let S be a semigroup. Then S is a disjoint union of three copies of the free monogenic semigroup if and only if S is isomorphic or anti-isomorphic to the semigroup defined by one of the following presentations:*

(i) $\langle\, a, b, c \mid ab = a^i, ba = a^i, ac = a^j, ca = a^j, bc = a^k, cb = a^k \,\rangle$ *where* $i + j = k + 2$ *and* $i, j, k \in \mathbb{N}$*;*
(ii) $\langle\, a, b, c \mid ab = a^i, ba = a^i, ac = a^j, ca = a^j, bc = b^k, cb = b^k \,\rangle$ *where* $i + j + k - ik = 2$ *and* $i, j, k \in \mathbb{N}$*;*
(iii) $\langle\, a, b, c \mid ab = a^i, ba = a^i, ac = a^i, ca = a^i, bc = c^2, cb = b^2 \,\rangle$ *where* $i \in \mathbb{N}$*;*
(iv) $\langle\, a, b, c \mid ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = a^i \,\rangle$ *where* $i \in \mathbb{N}$*;*
(v) $\langle\, a, b, c \mid ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = c^i \,\rangle$ *where* $i \in \mathbb{N}$*;*
(vi) $\langle\, a, b, c \mid ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = b^2 \,\rangle$*;*
(vii) $\langle\, a, b, c \mid ab = b^2, ba = a^2, ac = c^2, ca = b^2, bc = c^2, cb = a^2 \,\rangle$*;*
(viii) $\langle\, a, b, c \mid ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = a^2 \,\rangle$*;*

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(ix) $\langle\ a,b,c \mid ab = b^2, ba = a^2, ac = b^i, ca = a^i, bc = a^i, cb = b^i\ \rangle$ *where* $i \in \mathbb{N}$.

We prove Theorems 1.1 and 1.2 in Sections 2 and 3 respectively.

Let A be a set, and let S be any semigroup. Then we denote by A^+ the *free semigroup* on A, which consists of the non-empty words over A. Any mapping $\psi : A \to S$ can be extended in a unique way to a homomorphism $\phi : A^+ \to S$, and A^+ is determined up to isomorphism by these properties. If A is a generating set for S, then the identity mapping on A induces an epimorphism $\pi : A^+ \to S$. The kernel $\ker(\pi)$ is a congruence on S; if $R \subseteq A^+ \times A^+$ generates this congruence we say that $\langle A \mid R\rangle$ is a presentation for S. We say that S *satisfies a relation* $(u,v) \in A^+ \times A^+$ if $\pi(u) = \pi(v)$; we write $u = v$ in this case. Suppose we are given a set $R \subseteq A^+ \times A^+$ and two words $u, v \in A^+$. We write $u \equiv v$ if u and v are equal as elements of A^+. We say that the relation $u = v$ is a *consequence* of R if there exist words $u \equiv w_1, w_2, \ldots, w_{k-1}, w_k \equiv v$ $(k \geq 1)$ such that for each $i = 1, \ldots, k-1$ we can write $w_i \equiv \alpha_i u_i \beta_i$ and $w_{i+1} \equiv \alpha_i v_i \beta_i$ where $(u_i, v_i) \in R$ or $(v_i, u_i) \in R$. We say that $\langle A \mid R\rangle$ is a presentation for S if and only if S satisfies all relations from R, and every relation that S satisfies is a consequence of R: see [7, Proposition 1.4.2]. If A and R are finite, then S is finitely presented.

Let ρ be a congruence on a semigroup S, and let $\phi : S \to T$ be a homomorphism such that $\rho \subseteq \ker\phi$. Then there is a unique homomorphism $\beta : S/\rho \to T$ defined by $s/\rho \mapsto \phi(s)$ and such that $\operatorname{im}\beta = \operatorname{im}\phi$; [6, Theorem 1.5.3]. Let S be the semigroup defined by the presentation $\langle A \mid R\rangle$. If T is any semigroup satisfying the relations R, then T is a homomorphic image of S.

Lemma 1.3. *Let A be a set, let ρ be a congruence on A^+, let $\varphi : A \to \mathbb{N}\cup\{0\}$ be any mapping, and let $\psi : A^+ \to \mathbb{N}\cup\{0\}$ be the unique homomorphism extending φ. If $\rho \subseteq \ker\psi$ and $a \in A$ such that $\varphi(a) \neq 0$, then $\langle a/\rho\rangle$ is an infinite subsemigroup of A^+/ρ.*

Proof. Since $\rho \subseteq \ker\psi$, it follows that $\overline{\psi} : A^+/\rho \to \mathbb{N}\cup\{0\}$ defined by $\overline{\psi}(w/\rho) = \psi(w)$ is a homomorphism. Homomorphisms map elements of finite order to elements of finite order, and since $\varphi(a) \neq 0$ does not have finite order, a/ρ must have infinite order in A^+/ρ. □

Let $\langle a\rangle$ be the free monogenic semigroup. Then any two non-empty subsemigroups S and T of $\langle a\rangle$ have non-empty intersection, since $a^i \in S$ and $a^j \in T$ implies $a^{ij} \in S \cap T$. Let S be a semigroup which is the disjoint union of $m \in \mathbb{N}$ copies of the free monogenic semigroup, and let $a_1, \ldots, a_m \in S$ be the generators of these copies. Suppose that S is also the disjoint union of $n \in \mathbb{N}$ copies of the free monogenic semigroup. Then there exist $b_1, \ldots, b_n \in S$ such that $\langle b_1\rangle, \ldots, \langle b_n\rangle$ are free, disjoint, and

$$S = \langle a_1\rangle \cup \cdots \cup \langle a_m\rangle = \langle b_1\rangle \cup \cdots \cup \langle b_n\rangle.$$

If $n > m$, say, then there exist i, j such that $b_i, b_j \in \langle a_k\rangle$ for some k. But then $\langle b_i\rangle \cap \langle b_j\rangle \neq \varnothing$, a contradiction. Hence a semigroup cannot be the disjoint union of m and n copies of the free monogenic semigroup when $n \neq m$.

Lemma 1.4. *Let S and T be semigroups which are the disjoint union of $m \in \mathbb{N}$ copies of the free monogenic semigroup, and let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be the generators of these copies in S and T, respectively. Then every homomorphism $\varphi : T \to S$ such that $\varphi(a_i) = b_i$ for all i is an isomorphism.*

Proof. Since φ is surjective, it follows that the function $f : \{1, \ldots, m\} \to \{1, \ldots, m\}$ defined by $\varphi(a_i) \in \langle b_{f(i)}\rangle$ is a bijection.

Suppose that there exist $x, y \in S$ such that $\varphi(x) = \varphi(y)$. Then there exist $a, b \in A$ such that $x = a^i$ and $y = b^j$ for some $i, j \in \mathbb{N}$. It follows that $\varphi(a)^i = \varphi(b)^j$, which implies that $\varphi(a), \varphi(b) \in \langle c\rangle$ for some $c \in B$. Hence $a = b$, since f is a bijection, and so $x = y$. □

Since the free monogenic semigroup is anti-isomorphic to itself, it follows that a semigroup S is the disjoint union of m copies of the free monogenic semigroups if and only if any semigroup anti-isomorphic to S has this property.

2. Two copies of the free monogenic semigroup

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. ($\Leftarrow$) To prove the converse implication, it suffices to show that the semigroups mentioned in Theorem 1.1 are disjoint unions of two copies of the free monogenic semigroup, since this is a property preserved by (anti-)isomorphisms.

Let $m \in \mathbb{N}$ be arbitrary and let S be the semigroup defined by the presentation $\langle a, b | ab = ba = a^m \rangle$. It is clear that every element of S is a power of a or b, and so $S = \langle a \rangle \cup \langle b \rangle$. Since there is no relation in the presentation that can be applied to a power of b, it follows that $\langle a \rangle \cap \langle b \rangle = \varnothing$ and $\langle b \rangle$ is infinite. We show that $\langle a \rangle$ is infinite using Lemma 1.3. Let ρ be the congruence on $\{a, b\}^+$ generated by the relations $ab = a^m$ and $ba = a^m$, let $\varphi : \{a, b\} \to \mathbb{N}$ be defined by $\varphi(a) = 1$, $\varphi(b) = m - 1$, and let $\psi : \{a, b\}^+ \to \mathbb{N}$ be the unique homomorphism extending φ. Then

$$\psi(ab) = \psi(a) + \psi(b) = 1 + m - 1 = m = \psi(a^m)$$

and, similarly, $\psi(ba) = \psi(a^m)$. Hence $\rho \subseteq \ker \psi$ and so $\langle a \rangle$ is infinite in S, by Lemma 1.3.

Let T be the semigroup defined by the presentation $\langle\, a, b \mid ab = a^2, ba = b^2 \,\rangle$. Then as above $T = \langle a \rangle \cup \langle b \rangle$. Any product of a and b equal to a power of a must start with a and any product equal to a power of b must start with b. Hence $\langle a \rangle \cap \langle b \rangle = \varnothing$. The proof that $\langle a \rangle$ and $\langle b \rangle$ are infinite follows using a similar argument as above but where $\varphi : \{a, b\} \to \mathbb{N}$ is defined by $\varphi(a) = 1 = \varphi(b)$.

($\Rightarrow$) Let S be a semigroup which is the disjoint union of the free semigroups $\langle a \rangle$ and $\langle b \rangle$. Clearly one of the following must hold:

(a) $ab, ba \in \langle a \rangle$,
(b) $ab, ba \in \langle b \rangle$,
(c) $ab \in \langle a \rangle$ and $ba \in \langle b \rangle$,
(d) $ab \in \langle b \rangle$ and $ba \in \langle a \rangle$.

In case (b), S is isomorphic to a semigroup satisfying (a) and in case (d), S is anti-isomorphic to a semigroup satisfying (c). Hence we may assume without loss of generality that (a) or (c) hold.

Case (a) There exist $m, n \in \mathbb{N}$ such that $ab = a^m$ and $ba = a^n$. Hence

$$a^{m+1} = a^m a = (ab)a = a(ba) = aa^n = a^{n+1}$$

and so $m = n$. So, in this case, S is a homomorphic image of the semigroup T defined by the presentation $\langle\, a, b \mid ab = ba = a^m \,\rangle$. It follows from Lemma 1.4 that S is isomorphic to T.

Case (c) There exist $m, n \in \mathbb{N}$ such that $ab = a^m$ and $ba = b^n$. So, in this case,

$$a^{m+1} = a^m a = (ab)a = a(ba) = ab^n = a^m b^{n-1} = \cdots = a^{n(m-1)+1}$$

and so $m = n(m-1)$, which implies that $m = n = 2$. In this case, it follows that S is a homomorphic image of the semigroup defined by the presentation $\langle\, a, b \mid ab = a^2, ba = b^2 \,\rangle$, and so by Lemma 1.4, S is isomorphic to this semigroup. □

3. Disjoint unions of three copies of the free monogenic semigroup

Let S be a semigroup which is the disjoint union of the free monogenic semigroups $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$. We will show that S is determined, in some sense, by the values of the products ab, ba, ac, ca, bc, cb. To this end, define the *type* of S to be (A, B, C, D, E, F) where $A, B, C, D, E, F \in \{a, b, c\}$ if $ab \in \langle A \rangle$, $ba \in \langle B \rangle$, $ac \in \langle C \rangle$, $ca \in \langle D \rangle$, $bc \in \langle E \rangle$, $cb \in \langle F \rangle$. There are $3^6 = 729$ different types and so, potentially, 729 different cases to consider in the proof of Theorem 1.2. In order to bring this number down to a more manageable 9 cases, we require the following observations and lemma.

If S has type (A, B, C, D, E, F), then reversing the order of multiplication in S defines a semigroup anti-isomorphic to S of type (B, A, D, C, F, E). We will say that the types (A, B, C, D, E, F)

and (B, A, D, C, F, E) are anti-isomorphic. Similarly, by renaming the generators of S according to some permutation σ of the set $\{a, b, c\}$, we obtain a semigroup isomorphic to S of type (U, V, W, X, Y, Z). We will say that (A, B, C, D, E, F) is isomorphic to (U, V, W, X, Y, Z) via σ. For example, (b, a, b, a, c, b) is isomorphic to (b, a, c, a, c, a) via the permutation (cba). One step in the proof of Theorem 1.2 is to show that every S is isomorphic or anti-isomorphic to a semigroup of one of only 9 types.

Lemma 3.1. *Let S be a semigroup which is the disjoint union of the free monogenic semigroups $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$. Then one of $\langle a\rangle \cup \langle b\rangle$, $\langle a\rangle \cup \langle c\rangle$, or $\langle b\rangle \cup \langle c\rangle$ is a subsemigroup of S.*

Proof. Seeking a contradiction suppose that none of $\langle a\rangle\cup\langle b\rangle$, $\langle a\rangle\cup\langle c\rangle$, or $\langle b\rangle\cup\langle c\rangle$ is a subsemigroup of S. Then ab or $ba \in \langle c\rangle$, and ac or $ca \in \langle b\rangle$, and bc or $cb \in \langle a\rangle$. In each of these cases we will show that some power of a, say, equals a power of b or c, which will yield the required contradiction.

If $ab = c^i$ and $bc = a^j$, then $a^{j+1} = abc = c^{i+1}$. If $ab = c^i$ and $ca = b^j$, then $b^{j+1} = cab = c^{i+1}$. If $ac = b^i$ and $cb = a^j$, then $b^{i+1} = acb = a^{j+1}$. The remaining cases follow by symmetry. □

Proof of Theorem 1.2. (⇐) We will show that the semigroup defined by any of the presentations in Theorem 1.2, and therefore any semigroup (anti-)isomorphic to it, is the disjoint union of three copies of the free monogenic semigroup. It is straightforward to verify that every element of a semigroup defined by any of the presentations is a power of a, b or c. It therefore suffices to show that $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$ are pairwise disjoint and infinite.

As in the proof of Theorem 1.1, we show that $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$ are infinite by applying Lemma 1.3 to the respective congruences ρ on $\{a, b, c\}^+$ generated by the relations in the presentation of the relevant case, and the functions $\varphi : \{a, b, c\} \to \mathbb{N}$ defined by $\varphi(a) = 1$ and

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)
$\varphi(b)$	$i-1$	$i-1$	$i-1$	$i-1$	$i-1$	1	1	1	1
$\varphi(c)$	$j-1$	$j-1$	$i-1$	1	1	1	1	1	$i-1$

To conclude this part of the proof we must show that $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$ are pairwise disjoint. There are several cases to consider. In each of these cases, we let S denote the semigroup defined by the presentation in that case.

Case (i). In the semigroup defined by the presentation in case (i), no relation can be applied to a power of b or c, and so $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$ are disjoint (and the latter two are infinite).

Cases (ii) to (vi) and (ix). The proofs that $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$ are pairwise disjoint in the semigroups defined by the presentations in cases (ii) to (vi) and (ix) are similar, and so we present the proofs simultaneously. In each of these cases, let T be the semigroup defined by the respective multiplication table:

(ii)	a'	b'	c'
a'	a'	a'	a'
b'	a'	b'	b'
c'	a'	b'	c'

(iii)	a'	b'	c'
a'	a'	a'	a'
b'	a'	b'	c'
c'	a'	b'	c'

(iv)	a'	b'	c'
a'	a'	a'	c'
b'	a'	b'	c'
c'	a'	a'	c'

(v)	a'	b'	c'
a'	a'	a'	c'
b'	a'	b'	c'
c'	a'	c'	c'

(vi)	a'	b'	c'
a'	a'	b'	c'
b'	a'	b'	c'
c'	a'	b'	c'

,

(xi)	1	a'	b'	c'
1	1	a'	b'	c'
a'	a'	a'	b'	b'
b'	b'	a'	b'	a'
c'	c'	a'	b'	1

and let $\sigma : \{a, b, c\} \to T$ be defined by $\sigma(a) = a'$, $\sigma(b) = b'$, and $\sigma(c) = c'$. If $\tau : \{a, b, c\}^+ \to T$ is the unique homomorphism extending σ, then it is routine to verify that $\ker(\tau)$ contains the congruence ρ generated by the relations in the presentation for the corresponding case. Thus, in each case, the function $w/\rho \mapsto \tau(w)$ is a homomorphism from $S = \{a, b, c\}^+/\rho$ onto T (by [6, Theorem 1.5.3]). Therefore $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$ are pairwise disjoint.

Case (vii). It is routine to verify that by applying any relation in presentation (vii) to $w \in \{a,b,c\}^+$ ending in a^2, cb, or ba we obtain another word ending a^2, cb, or ba. Hence $\langle a\rangle$ is disjoint from $\langle b\rangle \cup \langle c\rangle$. Similarly, by considering words ending ab, ca, or b^2, it can be shown that $\langle b\rangle$ is disjoint from $\langle a\rangle \cup \langle c\rangle$. Hence $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$ are pairwise disjoint.

Case (viii). As in the previous case, it can be shown that applying any relation from the presentation in (viii) to $w \in \{a,b,c\}^+$ ending c we obtain another such word. Hence $\langle c\rangle$ is disjoint from $\langle a\rangle \cup \langle b\rangle$. Similarly, by considering words ending ab or b^2, it can be shown that $\langle b\rangle$ is disjoint from $\langle a\rangle \cup \langle c\rangle$, as required.

($\Rightarrow$) By Lemma 3.1, we may assume without loss of generality that $\langle a\rangle \cup \langle b\rangle$ is a subsemigroup of S. Furthermore, we may assume that S has type $(a,a,*,*,*,*)$ or $(b,a,*,*,*,*)$, since type $(b,b,*,*,*,*)$ is isomorphic to $(a,a,*,*,*,*)$ via (ab) and $(a,b,*,*,*,*)$ is anti-isomorphic to $(b,a,*,*,*,*)$.

We will show that, up to isomorphism and anti-isomorphism, the only possible types for S are: (a,a,a,a,a,a), (a,a,a,a,b,b), (a,a,a,a,c,b), (a,a,c,a,c,a), (a,a,c,a,c,c), (b,a,c,a,c,b), (b,a,c,b,c,a), (b,a,c,a,c,a), (b,a,b,a,a,b).

Suppose S has type $(a,a,*,*,*,*)$. By Theorem 1.1, it follows that $ab = a^i = ba$ for some $i \in \mathbb{N}$. If $ac = b^j$ for some $j \in \mathbb{N}$, then $b(ac) = b^{j+1}$ and $(ba)c = a^i c$. If $i = 1$, then $(ba)c = b^j$ and so $j = j+1$, a contradiction. If $i > 1$ them $(ba)c = a^i c = a^{i-1}b^j \in \langle a\rangle$, which is also a contradiction. By symmetry, we obtain a contradiction under the assumption that $ca \in \langle b\rangle$. It follows that $ac, ca \in \langle a\rangle \cup \langle c\rangle$.

In the case that $ac, ca \in \langle a\rangle$, we will show that $bc, cb \in \langle a\rangle$ or $bc, cb \in \langle b\rangle \cup \langle c\rangle$. Suppose not. Then, say, $bc \in \langle a\rangle$ and $cb \in \langle b\rangle \cup \langle c\rangle$. It follows that $(cb)(cb) \in \langle b\rangle \cup \langle c\rangle$ but $(c(bc))b \in \langle a\rangle$. The case that $cb \in \langle a\rangle$ and $bc \in \langle b\rangle \cup \langle c\rangle$ follows by a similar argument. We have shown that the only S of type $(a,a,a,a,*,*)$ are (a,a,a,a,a,a), (a,a,a,a,b,b), (a,a,a,a,b,c), (a,a,a,a,c,b), and (a,a,a,a,c,c).

If $ac \in \langle c\rangle$ and $bc \in \langle a\rangle \cup \langle b\rangle$, then $a(bc) \in \langle a\rangle$ and $(ab)c \in \langle c\rangle$, a contradiction. Similarly, if $ca \in \langle c\rangle$ and $cb \in \langle a\rangle \cup \langle b\rangle$, then $(cb)a \in \langle a\rangle$ but $c(ba) \in \langle c\rangle$. It follows that if S is of type $(a,a,c,*,*,*)$, then S is of type $(a,a,c,*,c,*)$ and if S is of type $(a,a,*,c,*,*)$, then S is of type $(a,a,*,c,*,c)$. It follows that the only S of type $(a,a,c,c,*,*)$ are of type (a,a,c,c,c,c).

If $ca \in \langle c\rangle$ and $bc \in \langle b\rangle$, then $b(ca) \in \langle b\rangle$ and $(bc)a \in \langle a\rangle$, a contradiction. Hence the only S of type $(a,a,a,c,*,c)$ are of type (a,a,a,c,a,c) or (a,a,a,c,c,c). It follows by symmetry that the only S of type $(a,a,c,a,c,*)$ are of type (a,a,c,a,c,a) or (a,a,c,a,c,c).

Therefore if S is a semigroup of type $(a,a,*,*,*,*)$, then S has one of the following types: (a,a,a,a,a,a), (a,a,a,a,b,b), (a,a,a,a,c,c), (a,a,a,a,b,c), (a,a,a,a,c,b), (a,a,c,c,c,c), (a,a,c,a,c,a), (a,a,c,a,c,c), (a,a,a,c,a,c), (a,a,a,c,c,c).

However, (a,a,a,a,b,c), (a,a,a,c,a,c) and (a,a,a,c,c,c) are anti-isomorphic to (a,a,a,a,c,b), (a,a,c,a,c,a) and (a,a,c,a,c,c), respectively. Moreover, (a,a,a,a,c,c) is isomorphic to (a,a,a,a,b,b) via (bc) and (a,a,c,c,c,c) is isomorphic to (a,a,a,a,b,b) via (cab). This leaves the tuples given at the start of this part of the proof.

Suppose that S has a type $(b,a,*,*,*,*)$ which is not isomorphic to $(a,a,*,*,*,*)$. Then S does not have type $(*,*,a,a,*,*)$, $(*,*,c,c,*,*)$, $(*,*,*,*,b,b)$ or $(*,*,*,*,c,c)$. We prove that the only possible S of type $(b,a,*,*,*,*)$ are:

$$(b,a,b,a,a,b), (b,a,b,a,c,b), (b,a,c,b,c,a), (b,a,c,b,c,b), (b,a,c,a,c,a), (b,a,c,a,a,b), (b,a,c,a,c,b).$$

Since (b,a,b,a,c,b), (b,a,c,b,c,b) and (b,a,c,a,a,b) are isomorphic to (b,a,c,a,c,a) via (cba), (ab) and (bc), respectively, this will leave the tuples given at the start of this part of the proof. It suffices to prove the following:

(a) if S has type $(b,a,a,*,*,*)$, then it has already been considered;
(b) if S has type $(b,a,b,*,*,*)$, then it has type $(b,a,b,a,*,b)$;
(c) if S has type $(b,a,c,b,*,*)$, then it has type $(b,a,c,b,c,*)$;
(d) if S has type $(b,a,c,a,*,a)$, then it has type (b,a,c,a,c,a);
(e) S cannot have type $(b,a,c,a,*,c)$.

Case (a). If $cb \in \langle a\rangle \cup \langle c\rangle$, then $a(cb) \in \langle a\rangle$ but $(ac)b \in \langle b\rangle$, a contradiction. Hence $cb \in \langle b\rangle$. If $ca \in \langle c\rangle$, then $(cb)a \in \langle a\rangle$ but $c(ba) \in \langle c\rangle$. If $ca \in \langle b\rangle$, then $a(ca) \in \langle b\rangle$ but $(ac)a \in \langle a\rangle$. Hence $ca \in \langle a\rangle$ and so S has type $(b, a, a, a, *, *)$. It follows that S is isomorphic or anti-isomorphic to a semigroup of type $(a, a, *, *, *, *)$.

Case (b). If $ca \in \langle b\rangle \cup \langle c\rangle$, then $a(ca) \in \langle b\rangle$ but $(ac)a \in \langle a\rangle$. Hence $ca \in \langle a\rangle$ and S has type $(b, a, b, a, *, *)$. If $cb \in \langle c\rangle$, then $c(ab) \in \langle c\rangle$ but $(ca)b \in \langle b\rangle$. If $cb \in \langle a\rangle$, then $a(cb) \in \langle a\rangle$ and $(ac)b \in \langle b\rangle$. Thus S has type $(b, a, b, a, *, b)$, as required.

Case (c). If $bc \in \langle a\rangle$, then $b(ca) \in \langle b\rangle$ but $(bc)a \in \langle a\rangle$. If $bc \in \langle b\rangle$, then $b(ac) \in \langle b\rangle$ but $(ba)c \in \langle c\rangle$. Hence S has type $(b, a, c, b, c, *)$, as required.

Case (d). If $bc \in \langle a\rangle \cup \langle b\rangle$, then $b(cb) \in \langle a\rangle$ but $(bc)b \in \langle b\rangle$. Therefore S has type (b, a, c, a, c, a), as required.

Case (e). If S has type $(b, a, c, a, *, c)$, then $c(ab) \in \langle c\rangle$ but $(ca)b \in \langle b\rangle$, a contradiction.

It remains to show that if S has one of the types given at the start of the proof, then S is isomorphic to a semigroup defined by one of the presentations in the theorem. By Lemma 1.4, it suffices to show that the generators a, b, and c of S satisfy the relations in one of the presentations.

Suppose that S has type (a, a, a, a, a, a). Then $\langle a\rangle \cup \langle b\rangle$ and $\langle a\rangle \cup \langle c\rangle$ are subsemigroups of S and hence by Theorem 1.1 $ab = a^i = ba$ and $ac = a^j = ca$ for some $i, j \in \mathbb{N}$. If $bc = a^k$ and $cb = a^l$ in S, then $a^{k-1+i} = a^k b = bcb = ba^l = a^{l-1+i}$ and so $k = l$. Also $a^{i+j} = a(bc)a = a^{k+2}$ and so $i + j = k + 2$. Thus the presentation in (i) defines a semigroup isomorphic to S.

If S has type (a, a, a, a, b, b), then $\langle a\rangle \cup \langle b\rangle$, $\langle a\rangle \cup \langle c\rangle$, and $\langle b\rangle \cup \langle c\rangle$ are subsemigroups of S, and so $ab = ba = a^i$, $ac = ca = a^j$, and $bc = cb = b^k$ for some $i, j, k \in \mathbb{N}$ (by Theorem 1.1). Also $a^{i+j} = abca = ab^k a = a^{ik-k+2}$, and so $i + j = ik - k + 2$. So, the presentation in (ii) defines S.

If S has type (a, a, a, a, c, b), then again $\langle a\rangle \cup \langle b\rangle$, $\langle a\rangle \cup \langle c\rangle$, and $\langle b\rangle \cup \langle c\rangle$ are subsemigroups of S. Hence, by Theorem 1.1, $ab = ba = a^i$, $ac = ca = a^j$, $bc = c^2$, and $cb = b^2$. Also $a^{2i-1} = a^i b = ab^2 = acb = a^j b = a^{j-1+i}$ and so $i = j$, and S is defined by the presentation in (iii).

If S has type (a, a, c, a, c, a), then $\langle a\rangle \cup \langle b\rangle$ and $\langle a\rangle \cup \langle c\rangle$ are subsemigroups of S. Hence $ab = ba = a^i$ for some $i \in \mathbb{N}$, $ac = c^2$, and $ca = a^2$. If $bc = c^j$ and $cb = a^k$ for some $j, k \in \mathbb{N}$, then $c^{j+2} = acbc = a^{k+1}c = c^{k+2}$ and so $j = k$. Furthermore, $a^{i+k} = abcb = ac^j b = c^{j+1}b = c^j a^k = a^{j+k}$ and so $i = j$, and S is defined by the presentation in (iv).

If S has type (a, a, c, a, c, c), then $\langle a\rangle \cup \langle b\rangle$, $\langle a\rangle \cup \langle c\rangle$, and $\langle b\rangle \cup \langle c\rangle$ are subsemigroups of S. Hence $ab = ba = a^i$, $ac = c^2$, $ca = a^2$, and $bc = cb = c^j$ for some $i, j \in \mathbb{N}$. It follows that $a^{i+2} = abca = ac^j a = a^{j+2}$ and so $i = j$. This implies that S is defined by the presentation in (v).

If S has type (b, a, c, a, c, b), then $\langle a\rangle \cup \langle b\rangle$, $\langle a\rangle \cup \langle c\rangle$, and $\langle b\rangle \cup \langle c\rangle$ are subsemigroups of S and so, by Theorem 1.1, S is defined by the presentation in (vi).

If S has type (b, a, c, b, c, a), then $\langle a\rangle \cup \langle b\rangle$ is a subsemigroup of S and so $ab = b^2$ and $ba = a^2$ by Theorem 1.1. Suppose that $ac = c^i$, $ca = b^j$, $bc = c^k$, and $cb = a^l$. Then $c^{2k-1} = bc^k = b^2 c = (ab)c = a(bc) = ac^k = c^{i+k-1}$ which implies that $2k = i + k$ and so $i = k$. Also $c^{il-l+1} = a^{l-1}c^i = a^l c = (cb)c = c(bc) = cc^k = c^{k+1}$ and so $k = il - l$. Thus, since $i = k$, it follows that $k = (k-1)l$ and so $i = k = l = 2$. Finally, $a^{l+1} = (cb)a = c(ba) = ca^2 = b^j a = a^{j+1}$ and so $j = l = 2$. We have shown that S is defined by the presentation in (vii).

If S has type (b, a, c, a, c, a), then $\langle a\rangle \cup \langle b\rangle$ and $\langle a\rangle \cup \langle c\rangle$ are subsemigroups of S and so $ab = b^2$, $ba = a^2$, $ac = c^2$, and $ca = a^2$ by Theorem 1.1. If $bc = c^k$ and $cb = a^l$, then $a^{l+1} = ba^l = b(cb) = (bc)b = c^k b = c^{k-1}a^l = a^{k+l-1}$ and so $k = 2$. Also $c^3 = c(bc) = (cb)c = a^l c = c^{l+1}$ which implies that $l = 2$. It follows that S is defined by the presentation in (viii).

If S has type (b, a, b, a, a, b), then $\langle a\rangle \cup \langle b\rangle$ is a subsemigroup of S and so $ab = b^2$ and $ba = a^2$ by Theorem 1.1. Suppose $ac = b^i$, $ca = a^j$, $bc = a^k$, and $cb = b^l$. Then $a^{j+2} = baca = b^{i+1}a = a^{i+2}$ and so $i = j$. Also $b^{l+2} = abcb = a^{k+1}b = b^{k+2}$ and so $k = l$. Finally, $b^{k+2} = a^k b^2 = a^k ab = b(ca)b = ba^j b =$

$a^{j+1}b = b^{j+2}$ and so $j = k$. It follows that S is defined by the presentation in (ix), and the proof is complete. □

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