Singlet-aided infinite resource reduction in the comparison of distant fields

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We present a task which can be faithfully solved with finite resources only when aided by particles prepared in a particular entangled state: the singlet state. The task consists of identifying the mutual parallelity or orthogonality of weak distant magnetic fields whose absolute directions are completely unknown.

It is well known that quantum mechanics helps to reduce the resources required to accomplish certain tasks [1–5]. Some problems can be solved with exponentially less resources when aided by quantum mechanics, as featured in Shor's factorization algorithm [2]. Other problems lead to a quadratic speed-up, such as Grover's search algorithm [3]. Moreover, the use of quantum entanglement can result in resource reduction in a variety of communication associated tasks. For example, in quantum dense coding [4], prior entanglement is used to increase the classical information capacity of a quantum bit by a factor of two. Sharing entanglement can also reduce the amount of communication needed to evaluate certain functions of distributed inputs [5]. In this paper, we present a task which illustrates the superiority of an entanglement-based strategy in very radical terms. We show that there is in fact an infinite gap in the resources required for accomplishing the task with or without the use of a certain entangled quantum state (a singlet). Without sharing a singlet state, the task requires infinitely many qubits for error free operation, while the use of shared singlets reduces the resource requirement to at most four qubits.

Consider the situation depicted in Figure 1., where two spatially separated and disconnected regions are occupied by distant partners Alice and Bob. A third person, Eve, who has access to both separated zones, may subject these two regions to two weak uniform magnetic fields of unit strength but otherwise random direction. However, she gives Alice and Bob an important promise: *the two fields are either parallel or orthogonal*. In other words, if she had chosen the direction $\hat{n}$ for the field applied on Alice's side, she chooses either $\hat{n}$ or any direction $\hat{n}_\perp$ orthogonal to $\hat{n}$ for the field applied on Bob's side. We also make the assumption that these fields are sufficiently weak so that they cannot be determined classically. The only way to determine the field direction is by means of detecting their action on quantum states. Alice and Bob are given the task of *faultlessly identifying (i.e. with unit probability of success), Eve's choice among the two alternative relative orientations of the fields*. Note that

this is strictly a "quantum task" in contrast to existing examples [1–5] in which quantum mechanics is used to reduce the resources required to accomplish a "classical task". Here, because of the weakness of the magnetic fields, there is no hope to accomplish the task classically. But, as we will show, even within the available quantum protocols, using entanglement leads to an infinite resource reduction.
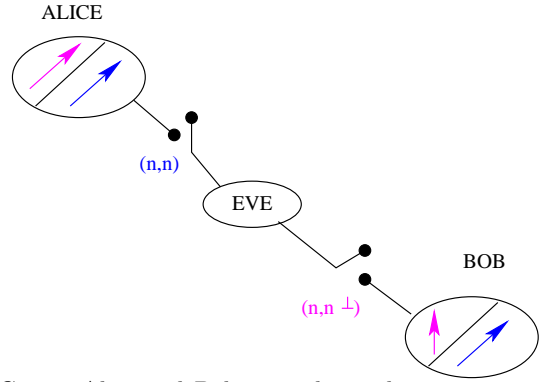

FIG. 1. Alice and Bob control two distant regions. Eve may subject these two regions to a weak magnetic field of unit strength but random direction. She gives Alice and Bob only the promise that the two fields are either parallel or orthogonal. Alice and Bob have the task of distinguishing the two cases locally.

We will first consider the case when Alice and Bob do not share any entanglement. Suppose Alice has a qubit A in an initial state $|\psi\rangle_A$ and Bob has a qubit B be in an initial state $|\psi\rangle_B$. Let $\vec{\sigma}^{(A)} = (\sigma_x^{(A)}, \sigma_y^{(A)}, \sigma_z^{(A)})$ and $\vec{\sigma}^{(B)} = (\sigma_x^{(B)}, \sigma_y^{(B)}, \sigma_z^{(B)})$, where $\sigma_i^{(A/B)}$ denotes the Pauli matrices of A/B. To distinguish between the two alternatives locally would thus require

$$\langle\psi|\vec{\sigma}^{(B)}.\hat{n} \otimes \vec{\sigma}^{(B)}.\hat{n}_\perp|\psi\rangle_B = 0. \qquad (1)$$

The only solution for this is for $|\psi\rangle_B$ to be an eigenstate of the operator $\vec{\sigma}^{(B)}.\hat{n}$, where the direction $\hat{n}$ is completely unknown. Therefore, in order to account for error free detection, Bob will need to have a number of qubits, each in an eigenstate of $\vec{\sigma}^{(B)}.\hat{n}$ corresponding to a different $\hat{n}$. As

there are an infinite number of choices of $\hat{n}$, an error free detection scheme requires Bob to hold an infinite number of qubits. It is important to note that having any or all of Alice's or Bob's qubits in classically correlated mixed states of the type $\sum_i p_i |\psi^i\rangle_A \langle\psi^i|_A \otimes |\psi^i\rangle_B \langle\psi^i|_B$ will also not help in perfect discrimination of $\vec{\sigma}^{(A)}.\hat{n} \otimes \vec{\sigma}^{(B)}.\hat{n}$ and $\vec{\sigma}^{(A)}.\hat{n} \otimes \vec{\sigma}^{(B)}.\hat{n}_\perp$. If that were the case, then one would have been able to choose three mutually perpendicular directions and perform quantum dense coding of capacity $\log_2 3$ bits per qubit. But this is not possible with a disentangled state, as shown in Ref. [6].

Let us now describe a strategy where Alice and Bob initially share entanglement. Imagine that the qubits A and B possessed by Alice and Bob are prepared in a singlet state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{2}$$

Now Eve subjects Alice's and Bob's qubits to her chosen unitary transformations. Suppose she chose the pair $\{\vec{\sigma}^{(A)}.\hat{n}, \vec{\sigma}^{(B)}.\hat{n}\}$, (i.e. parallel fields). Then the state shared by Alice and Bob evolves to

$$\begin{aligned} |\psi\rangle &= \vec{\sigma}^{(A)}.\hat{n} \otimes \vec{\sigma}^{(B)}.\hat{n}|\psi^-\rangle \\ &= |\psi^-\rangle, \end{aligned} \tag{3}$$

where we have used the fact that a singlet state is invariant under operations $U^{(A)} \otimes U^{(B)}$ (i.e. when the same unitary operation U is applied to both qubits). On the other hand, if Eve decided to apply the pair $\{\vec{\sigma}^{(A)}.\hat{n}, \vec{\sigma}^{(B)}.\hat{n}_\perp\}$ (i.e. perpendicular fields) the singlet will evolve to a coherent superposition of the three triplet states $|\psi^+\rangle, |\Phi^+\rangle$, and $|\Phi^-\rangle$. This can easily be seen from the fact there is always a unitary transformation $U(\hat{n})$ such that

$$\vec{\sigma}^{(A)}.\hat{n} \otimes \vec{\sigma}^{(B)}.\hat{n}_\perp = U(\hat{n})\sigma_x^{(A)}U(\hat{n})^\dagger \otimes U(\hat{n})\sigma_{y/z}^{(B)}U(\hat{n})^\dagger \tag{4}$$

and therefore

$$\langle\psi^-|\vec{\sigma}^{(A)}.\hat{n} \otimes \vec{\sigma}^{(B)}.\hat{n}_\perp|\psi^-\rangle = \langle\psi^-|\sigma_x^{(A)} \otimes \sigma_{y/z}^{(B)}|\psi^-\rangle = 0. \tag{5}$$

As a result, Alice and Bob can now easily check which of the two possible relative orientations Eve has chosen. If the parallel configuration $\{\vec{\sigma}^{(A)}.\hat{n}, \vec{\sigma}^{(B)}.\hat{n}\}$ was applied, Alice and Bob still share a singlet state. On the contrary, if they were subject to the orthogonal configuration $\{\vec{\sigma}^{(A)}.\hat{n}, \vec{\sigma}^{(B)}.\hat{n}_\perp\}$, Alice and Bob now hold a state that is orthogonal to the singlet state.
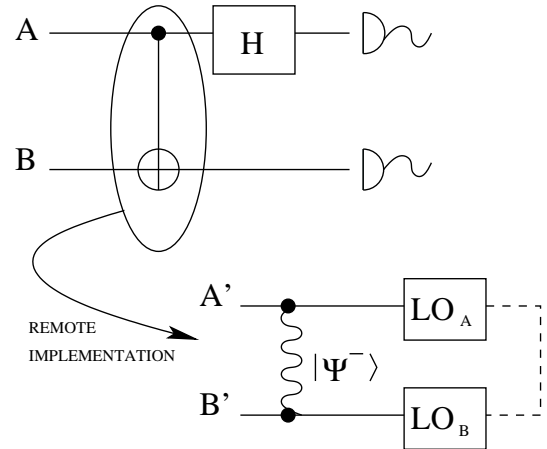

FIG. 2. A controlled-NOT and an inverse Hadamard suffice to determine whether one has a singlet state or a triplet state. A controlled-NOT in turn can be implemented consuming one extra singlet [7] which has been shielded from the action of the magnetic field and using only local operations (LO) supplemented by classical communication, as illustrated by the dashed line in the diagram.

For the determination which state Alice and Bob are holding, two scenarios are possible. In one, we may assume that Alice and Bob are allowed to send each other quantum particles. In this case Alice simply sends Bob her particle, and Bob then measures the projection operator onto the singlet space. If this projection is successful, then he knows that the fields were parallel, if the projection was not successful then the fields were orthogonal. However, one may also demand that Alice and Bob only share some initial entanglement in the form of singlets and that no further quantum communication is possible. In that case Alice and Bob need altogether two pairs of singlet states to complete the task. The first pair is treated as outlined above, while the second one is kept isolated from Eve and will be needed to determine whether the first pair is in a singlet state or not. This can clearly be done, as one singlet state is enough to implement a controlled-NOT gate remotely [7]. The quantum circuit required for the (local) discrimination of the shared entangled state is shown in Figure 2. First a remote controlled-NOT gate with Alice's qubit acting the control bit is applied. This process consumes an e-bit of entanglement. Subsequently a Hadamard transformation onto Alices qubit takes the state $|0\rangle$ into $|0\rangle - |1\rangle$ and the state $|1\rangle$ into $|0\rangle + |1\rangle$. As a result of this protocol, the four Bell states are mapped into product states as follows

$$|00\rangle + |11\rangle \longrightarrow |00\rangle \tag{6}$$
$$|00\rangle - |11\rangle \longrightarrow |10\rangle \tag{7}$$
$$|01\rangle + |10\rangle \longrightarrow |01\rangle \tag{8}$$
$$|01\rangle - |10\rangle \longrightarrow |11\rangle. \tag{9}$$

The state discrimination is completed when Alice and

Bob measure their first pair. If they both find the state $|1\rangle$, then they initially shared a singlet state and the magnetic fields were parallel. Therefore, even without the use of quantum communication, Alice and Bob require only two singlets to accomplish the task of determining the relative orientation of the two fields without error.

We will now prove that the singlet is the *only* state which allows Alice and Bob to achieve error free discrimination. The state $|\psi\rangle$ which Alice and Bob must share in order to accomplish the required task needs to satisfy

$$\langle\psi|\vec{\sigma}^{(A)}.\hat{n} \otimes \vec{\sigma}^{(B)}.\hat{n} \, \vec{\sigma}^{(A)}.\hat{n}' \otimes \vec{\sigma}^{(B)}.\hat{n}'_{\perp}|\psi\rangle = 0, \quad (10)$$

where $\hat{n}$ and $\hat{n}'$ are two *completely arbitrary* directions. This is because Alice and Bob need to distinguish *all cases* of parallel field directions from *all cases* of perpendicular field directions. In other words, Eve could have picked one absolute direction $\hat{n}$ to apply parallel magnetic fields and a different absolute direction $\hat{n}'$ to apply the perpendicular fields. These two cases need to be perfectly distinguished. We first find out the restrictions on $|\psi\rangle$ which already arise from considering the special case $\hat{n} = \hat{n}'$. In that case Eq.(10) simplifies to

$$\langle\psi|I^A \otimes \vec{\sigma}^{(B)}.\hat{n}''|\psi\rangle = 0, \quad (11)$$

where I^A is the identity operator for qubit A and $\hat{n}'' = \hat{n} \times \hat{n}_{\perp}$ (i.e. $\hat{n}''$ is arbitrary as $\hat{n}$ itself is arbitrary). Eq.(11) restricts the class of allowed $|\psi\rangle$ to maximally entangled states of the qubits A and B. If we put a maximally entangled state $|\psi\rangle_{\max}$ in Eq.(10) and simplify, we get the condition

$$(\hat{n}.\hat{n}')(\hat{n}.\hat{n}'_{\perp}) - \langle\psi|\vec{\sigma}^{(A)}.(\hat{n} \times \hat{n}')$$
$$\otimes \vec{\sigma}^{(B)}.(\hat{n} \times \hat{n}'_{\perp})|\psi\rangle_{\max} = 0. \quad (12)$$

Substituting $|\psi\rangle_{\max}$ in the above equation by its expansion $c_1|\psi^+\rangle + c_2|\psi^-\rangle + c_3|\phi^+\rangle + c_4|\phi^-\rangle$ in terms of Bell states gives the unique solution $|c_2|^2 = 1$. This proves that the only state which satisfies Eq.(10) is the singlet state. Thus *the singlet becomes the only feasible state for error free discrimination*.

Let us now briefly point out how our scheme differs from those schemes which appear to be related. The fact that our scheme can only be carried out with singlets makes it different from quantum dense coding [4] and precision magnetic field determination [8], which would both work for *any* maximally entangled state. It is also different from the standard inability to distinguish specific entangled (and also some unentangled [9]) states locally by finite resources. Here, we are not given any prior sets of unknown states to distinguish, but some unknown relative orientations of fields to discriminate. We have identified the state which works best for this purpose (namely the singlet).

The task of determining the relative orientation of two magnetic fields can be generalized in many directions.

First of all one may allow for more possible relative directions, and ask Alice and Bob to determine the angle between the two directions. In this case both the entanglement based as well as the disentangled strategy are unable to deliver error free answers, but it can be expected that the entangled strategy will deliver the better overall precision or the lower error rate. A further generalization of the problem could also allow for a variable strength of the magnetic field. Again it can be expected that the entanglement based strategy will be superior. It should be noted that this problem is related to that of atomic frequency standards [10], which can be mapped onto a problem where a field of known orientation but unknown strength should be detected with the best possible resolution. It should also be noted that Eve could have given a different promise leading to a similar gap between the entanglement based and the disentangled strategy. If Eve promises that the fields are either parallel or anti-parallel, then either the singlet state remains invariant, or it is converted (after a suitable waiting time) into the triplet state $|01\rangle + |10\rangle$, which in turn allows to determine the relative orientation of the fields. Summarizing, we have presented a task that be solved efficiently using shared entanglement in the form of singlet states and demonstrated that the associated cost in resources represents an infinite gap as compared to applying a classical strategy. Error free performance requires that Alice and Bob hold an infinite number of disentangled particles while an entanglement-based strategy uses either one singlet pair, if subsequent quantum communication is allowed, or, at most, two singlet pairs if only local quantum operations and the exchange of classical communication is allowed. It is quite interesting to note that very recently yet another application of entanglement which uses the $U \otimes U$ invariance of a singlet in an essential way has been proposed [11].

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